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Ad-nilpotent elements in algebras and superalgebras

Autor:

Guillermo Vera de Salas

Directores:

Esther García González

Miguel Ángel Gómez Lozano

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Contents

Abstract

In this thesis we will deal with ad-nilpotent elements in associative algebras and superalgebras with involution and superinvolution, and ad-nilpotent elements in Lie superalgebras. The first aim of this work fits with Herstein's branch of theory that studies nilpotent inner derivations in algebras. There are many studies on this area, highlighting for our work the articles of W. S. Martindale and C. R. Miers [55], [56] and T. K. Lee [54]. Later, in the second part, we study how to associate some Jordan structures to a Lie superalgebra, motivated by the work of A. Fernández, E. García and M. Gómez Lozano [24].

Objectives

Three objectives are addressed throughout this thesis. In the first instance, we seek to describe in detail the ad-nilpotent elements in semiprime associative algebras with involution. The second aim of this thesis is to carry over the descriptions of ad-nilpotent elements in semiprime associative algebras to prime associative superalgebras, that is, to give a detailed description of homogeneous ad-nilpotent elements belonging to prime associative superalgebras. Finally, motivated by the work of A. Fernández, E. García and M. Gómez Lozano in [24], to associate a Jordan superstructure to a Lie superalgebra with an ad-nilpotent element of a certain index.

Methodology

To develop the first two goals we have worked within the framework of semiprime algebras with involution and prime associative superalgebras with superinvolution.

Moreover, the extended centroid will be an important tool in this thesis. For the last objective, we have worked with nonassociative superstructures such as Lie and Jordan superalgebras, defined by the Grassmann envelope, and Jordan superpairs. We can highlight the high combinatorial content throughout the entire thesis.

Results

We have successfully covered the three initial goals. First, we have described in detail ad-nilpotent elements belonging to a semiprime associative algebra. Moreover, we have succeeded in reducing the torsion in the classification of ad-nilpotent elements in semiprime associative algebras with involution due to the new concept of a pure ad-nilpotent element, introduced in this thesis in Chapter 2. The conditions on the scalar rings has been weakened to be free of $\binom{n}{s}$ and s torsion with $s := [\frac{n+1}{2}]$ instead of being free of $n!$ torsion. On the other hand, for the skew-symmetric ad-nilpotent elements of a semiprime associative algebra R with involution $*$, we have given a description that depends on their ad-nilpotent index modulo 4. In this description we can emphasize: If a skew-symmetric element a is ad-nilpotent such that its index of ad-nilpotence of $K := \mathrm{Skew}(R, *)$ and R do not coincide, that is, $\mathrm{ad}_a^n K = 0$ but $\mathrm{ad}_a^n R \neq 0$, (it can only occur for ad-nilpotent indices of K congruent to 0 or 3 modulo 4) then a certain corner of R satisfies a PI, hence R holds a GPI. These results have been developed throughout Chapter 2 which have originated an article that has been published in the journal *Bulletin of the Malaysian Mathematical Sciences Society* ([12]). The second aim, to describe in prime associative superalgebras with superinvolution nilpotent inner derivations, has also been positively solved during Chapter 3. This description depends on the parity of the homogeneous element: if the element is even, what has been developed in the previous chapter in algebra settings ([12]), is largely rescued. However, if the element is odd, we have worked on its square, which is an even ad-nilpotent element, and we have applied the descriptions for even ad-nilpotent elements studied above. These results has been published in the journal *Linear and Multilinear Algebra* ([28]). During Chapter 4, we have given examples for each of the cases appearing in the descriptions of the elements in both algebras and superalgebras, thus showing that these descriptions are not trivial. Finally, in Chapter 5, we have associated a Jordan superstructure to a Lie superalgebra L with a homogeneous ad-nilpotent element a of index 3 or 4, according to its parity. Furthermore, the Jordan superpair we have constructed following the spirit of the paper of A. Fernández, E. García and M. Gómez Lozano [24], coincides with the subquotient of a Lie superalgebra associated with an abelian inner ideal $[a, [a, L]]$. This last chapter has been published and can be consulted in the journal *Communications in Algebra* ([30]).

Introduction

The main topic of this thesis is the study of ad-nilpotent elements belonging to Lie algebras and superalgebras. This work could be splitted in two parts: the first part sticks to the branch of Herstein's theory which studies nilpotent inner derivations in algebras; at the same time, this part can be splitted again into two, the study of nilpotent inner derivations in associative algebras, and the study of nilpotent inner derivations in the super setting. The second part studies Jordan superstructures attached to an ad-nilpotent element of a Lie superalgebra and the subquotients associated to abelian inner ideals of Lie superalgebras.

On one hand, Herstein's theory, which started in 1954 in [40] (see also the influential works [41] and [63]), is the study of nonassociative objects in associative prime and semiprime rings perhaps with involution, or in rings with well-behaved idempotents that provide a context rich enough for the theory to be satisfactorily developed. Among the main contributors, apart from Herstein itself, we can also cite works of K. I. Beidar, M. Bresar, M. A. Chetobar and W. S. Martindale [6], P. Grzeszczuk [38], T. K. Lee, [54], W. S. Martindale and C. R. Miers [56] and E. C. Posner [63]. Herstein's theory developed into several similar but different branches: the study of sets with an additional nonassociative structure, as Lie and Jordan ideals (e.g. [58]), culminating in the development of GPI theory ([7]); the study of special conditions (e.g. commutativity) on certain maps (e.g. generalized derivations) over some sets (e.g. Jordan ideals), in which strong knowledge is gained through the a priori weaker properties (e.g. [9], [50], [21], [64]); and the determination of the structure of nonassociative maps, as Lie homomorphisms and derivations (e.g. [4], [5], [6]), culminating in the development of the theory of functional identities ([8]). It is to this last branch of Herstein's theory that the first part of this thesis is about, centering on the structure of nilpotent derivations, which have been broadly studied since the 1960'. In 1963, Herstein proved that for any ad-nilpotent element a of index n in a simple ring of characteristic zero or greater than n there exists some λ in the center of R such that $a-\lambda$ is nilpotent. Furthermore, he showed that the index of nilpotence of such element is not greater than $[\frac{n+1}{2}]$, see [42, Theorem page 84]. Herstein's result was extended by Martindale and Miers in 1983 ([55, Corollary 1]) to prime rings of characteristic greater than n making use of the extended centroid of R. In 1978, Kharchenko obtained in [48] an important result: all algebraic derivations of prime rings of characteristic zero are inner for certain elements in an overring; he extended this result to torsion-free semiprime rings in 1979, see [49]. In 1983, Chung and Luh stated that the index of nilpotence of a nilpotent derivation on a semiprime ring of zero characteristic is always odd (see [16] and [17]), and in 1984 Chung, Kobayashi and Luh ([18]) proved that if R is semiprime and $\operatorname{char} R = p > 2$ then the index of nilpotence of a nilpotent derivation is of the form $n = a_s p^s + a_{s+1} p^{s+1} + \cdots + a_l p^l$ where $0 \leq s \leq l$, the a_i are non negative integers less than p, a_s is odd, and $a_{s+1}, \ldots, a_l$ are even. Moreover, Chung in 1985 proved, for prime rings R of characteristic zero, that a nilpotent derivation is inner and induced by a nilpotent element of an overring, see [15]. In 1992, with different techniques, Grzeszczuk showed that any nilpotent derivation in a semiprime ring with minimal restrictions on its characteristic is an inner derivation in a semiprime subring of the right Martindale ring of quotients of R and is induced by a nilpotent element in such subring, see [38, Corollary 8] and its generalization by Chuang and T. K. Lee in [14, §3].

Some examples of Lie algebras appear when working with rings R with involution $*$: the Lie algebras of skew-symmetric elements $K := \operatorname{Skew}(R, *)$ and $K/Z(R)$ and the derived Lie algebras $[K, K]$ and $[K, K]/([K, K] \cap Z(R))$. The nilpotent derivations of the skew-symmetric elements of prime rings with involution were studied by Martindale and Miers in the 1990's. In this case, if R has zero characteristic and is not an order in a 4-dimensional central simple algebra, for every inner derivation ad_a with $\operatorname{ad}_a^n = 0$ there exists an element λ in the extended centroid of R such that either $(a-\lambda)^{[\frac{n+1}{2}]} = 0$ or the involution is the identity in the extended centroid of R and $a^{[\frac{n+1}{2}]+1} = 0$, see [56, Main Theorem]. This result was partially extended to semiprime rings by T.K. Lee in 2018. In his main result he proved that if R is semiprime with involution and has no $n!$-torsion, then for any $a \in K$ with $\mathrm{ad}_a^n(K) = 0$ there exist λ and a symmetric idempotent ϵ in the extended centroid of R such that $(\epsilon a - \lambda)^{[\frac{n+1}{2}]+1} = 0$, see [54, Theorem 1.5].

In chapter 2 we study ad-nilpotent elements in Lie algebras arising from semiprime associative algebras R free of 2-torsion. With the idea of keeping under control the torsion of R we introduce a more restrictive notion of ad-nilpotent element, pure ad-nilpotent element, which is a technical condition since every ad-nilpotent element can be expressed as an orthogonal sum of pure ad-nilpotent elements of decreasing indices. This allows us to be more precise when setting the torsion inside the algebra R in order to describe its ad-nilpotent elements. If R is a semiprime associative algebra, $C(R)$ its extented centroid and $a \in R$ is a pure ad-nilpotent element of R of index n with R free of t and $\binom{n}{t}$-torsion for $t = [\frac{n+1}{2}]$, then n is odd and there exists $\lambda \in C(R)$ such that $a - \lambda$ is nilpotent of index t. If R is a semiprime associative algebra with involution $*$ and a is a pure ad-nilpotent element of $\mathrm{Skew}(R,*)$ free of t and $\binom{n}{t}$-torsion for $t = [\frac{n+1}{2}]$, then either a is an ad-nilpotent element of R of the same index n (this may occur if $n \equiv_4 1, 3$) or R is a nilpotent element of R of index $t+1$ and R satisfies a nontrivial GPI (this may occur if $n \equiv_4 0, 3$). The case $n \equiv_4 2$ is not possible.

On the other hand, an associative superalgebra is a $\mathbb{Z}_2$-graded associative algebra $R = R_0 + R_1$. The elements of $R_0 \cup R_1$ are called homogeneous elements and we say that the degree of $a \in R_0 \cup R_1$ is i (denoted $|a| = i$) when $a \in R_i$, $i \in \{0, 1\}$. Given an associative superalgebra R, we obtain a Lie superalgebra if the associative product is replaced by the superbracket $[a, b] = ab - (-1)^{|a|\,|b|} ba$ for homogeneous $a, b \in R$. The Lie structure of prime/simple associative superalgebras was investigated by F. Montaner in [60] and S. Montgomery in [62].

We say that a $\mathbb{Z}_2$-linear map $* : R \to R$ is a superinvolution when $(a^*)^* = a$ and $(ab)^* = (-1)^{|a|\,|b|} b^* a^*$ for homogeneous $a, b \in R_0 \cup R_1$. The set of skew-symmetric elements of an associative superalgebra is a Lie superalgebra and it will be denoted by K throughout this paper. Moreover, the study of the Lie structure of K of a simple associative superalgebra with superinvolution was iniciated by C. Gómez-Ambrosi and I. Shestakov in 1997 in [37], and their results were extended to prime superalgebras in [35]. The study of superinvolutions in associative superalgebras has been of great interest. We highlight the work of J. Laliena [52] about the description of the derived superalgebra $[K, K]$ of a semiprime superalgebra with superinvolution, and the recent works of A. Giambruno, A. Ioppolo, D. La Mattina and F. Martino ([32], [33], [34], [45]) on superinvolutions in superalgebras related to polynomial identities and related to the growth of certain substructures of the superalgebras.

Another interesting and very active topic in superalgebras is the study of superderivations (see for example the works of A. Fošner and M. Fošner [26], H. Ghahramani, M. N. Ghosseiri and S. Safari [31] or Y. Wang [66]). A linear map $d = d_0 + d_1$ in R is called a superderivation if each d_i, $i \in \{0, 1\}$, satisfies $d_i(R_j) \subset R_{i+j}$ and $d_i(ab) = d_i(a)b + (-1)^{i|a|}ad_i(b)$, for homogeneous $a, b \in R_0 \cup R_1$. For instance, if $a \in R_0 \cup R_1$, the map $\mathrm{ad}_a : R \to R$ given by $\mathrm{ad}_a(x) = [a, x]$ is a superderivation (of degree $|a|$). Such a superderivation is called an inner derivation. In [31] the authors describe the structure of superderivations on some $\mathbb{Z}_2$-graded rings and study when superderivations are inner.

In chapter 3 we give an in-deph analysis of the nilpotency index of nilpotent homogeneous inner superderivations in associative prime superalgebras with and without superinvolution.

Chapter 4 is devoted to giving examples for all of the types of elements studied in the chapters 2 and 3. Since the even part of an associative superalgebra is an associative algebra and a superinvolution restricted to the even part of an associative superalgebra is an involution, the examples of even ad-nilpotent elements of an associative superalgebra with superinvolution will also provide examples of ad-nilpotent elements of an associative algebra with involution.

Finally, local algebras of Jordan systems were introduced by Meyberg [59], used by Zelmanov and revisited by D'Amour and McCrimmon in their classification of linear and quadratic Jordan systems [67], [19], [20]. Ever since their introduction, they have played a prominent role in the structure theory of Jordan systems, mainly due to the fact that nice properties flow between the system and their local algebras (see for example [1], [2] or [61]).

In [24] E. García, A. Fernández López and M. Gómez Lozano attached a Jordan algebra to any Lie algebra L with an ad-nilpotent element x of index less than or equal to three. Their construction extended the fact that every Lie algebra with an $\mathfrak{s}l_2$-triple $(e, [e, f], f)$ is automatically 5-graded relative to the eigenspaces of $\mathrm{ad}_{[e,f]}$ and $L_2 = \mathrm{ad}_e^2(L)$ is a unital Jordan algebra. Although their object imitates the construction of a "local" algebra of a Lie algebra, they did not get a Lie algebra again but a Jordan algebra, so this object was called the Jordan algebra of L at x. Furthermore, any $\mathbb{Z}$-graded Lie algebra $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ comes together with a Jordan pair $V = (L_{-n}, L_n)$ and any element x of L_n is ad-nilpotent of index less than or equal to three, so one can construct the local algebra of V at x (in the sense of Meyberg [59]) and this Jordan algebra coincides with the Jordan algebra of L at x.

The Jordan algebras of Lie algebras, together with their extension to subquotients (Jordan pairs) associated to abelian inner ideals of Lie algebras, have provided a new way of connecting the Lie and the Jordan settings. For example, they were used by E. Zelmanov in his proof of the Lie version of the Kurosh problem [68, §2], and by J. Hennig in her classification of ad-integrable simple, locally finite Lie algebras over algebraically closed fields of characteristic > 3 [39, Theorem 2]. This construction was also mimicked in [65] to construct a quasi-Jordan algebra from a Leibniz algebra and an ad-nilpotent element of index less than or equal to three.

In chapter 5, given a Lie superalgebra and an even ad-nilpotent element of index less or equal to 3, we can obtain a Jordan superalgebra attached to that element by using the Grassmann envelope; inspired by that construction we build a Jordan superpair attached to an odd ad-nilpotent element of index less or equal to 4. We introduce inner ideals for Lie superalgebras, and we prove that the associated subquotients are Jordan superpairs.

Chapter 1

Algebraic methodology

During all this work Φ is a unital commutative ring of scalars with $\frac{1}{2} \in \Phi$.

Previously, in the introduction, we have established the topics that are covered in this work. In this section we go a step further, laying the foundations of this thesis and establishing its main concepts. Firstly, we will define the basics fundamentals related to associative algebras and superalgebras. Next, we will review some relevant concepts and results to better understand the structure underlying such algebras and superalgebras such as, for example, the notions of primeness and semiprimeness. Afterwards, we will introduce the extended centroid and how it behaves in prime or semiprime associative algebras and superalgebras with involution and superinvolution. Finally, we will recall basic notions on nonassociative algebras and superalgebras, in particular, about Lie and Jordan superalgebras.

In Chapter 2, we will study ad-nilpotent elements belonging to semiprime associative algebras R over Φ with or without involution. In particular, the extended centroid will be a crucial tool (e.g., it allows us to define what is a pure ad-nilpotent element). In Chapter 3 we will work on the super setting, i.e, on ad-nilpotent elements in prime associative superalgebras R over Φ with or without superinvolution. Finally, throughout Chapter 5 we will deal with Lie superalgebras and Jordan superstructures.

1.1 Basic notions on associative algebras and superalgebras

1.1.1. Let R be an algebra over Φ. We say that R is a superalgebra if R is $\mathbb{Z}_2$-graded, i.e., $R = R_0 \oplus R_1$ such that $R_i \cdot R_j \subseteq R_{i+j}$ with $i, j \in \mathbb{Z}_2$. R_0 it is the even part and it is a subalgebra of R and R_1 is the odd part and it is a bimodule over R_0. Any element of $R_0 \cup R_1$ is called a homogeneous element and we define the parity of a homogeneous element as $|a| = 0$ if $a \in R_0$ and $|a| = 1$ if $a \in R_1$.

Let $f : R \to R'$ be a linear map where R and R' are both superalgebras. We say that f is homogeneous of degree $\gamma \in \mathbb{Z}_2$ if $f(R_i) \subset R'_{i+\gamma}$. In addition, we say that f is a superalgebra homomorphism if it is an algebra homomorphism and it is homogeneous of degree 0, i.e., $f(R_i) \subset R'_i$.

1.1.2. Let $X = \{x_1, x_2, ...\}$ be a countable infinite set of variables and let $\Phi\langle X\rangle$ be the free unital associative algebra generated by X over Φ. If I is the two-sided ideal of $\Phi\langle X\rangle$ generated by the set of elements $\{x_i x_j + x_j x_i \mid i, j \geq 1\}$, we set $G := \Phi\langle X\rangle / I$. We call G the (infinite dimensional) Grassmann algebra. We denote by $\xi_i := x_i + I$. With this notation G has the following presentation:

$$G = \langle 1, \xi_1, \xi_2, ... \mid \xi_i \xi_j + \xi_j \xi_i = 0, \text{ for all } i, j \geq 1\rangle.$$

Notice that $\xi_i^2 = 0$ since $\frac{1}{2} \in \Phi$. The set $B = \{1, \xi_{i_1}, ...\xi_{i_k} \mid 1 < i_1 < ... < i_k, \text{ for all } k \in \mathbb{N}\}$ is a basis of G over Φ. In addition, G is a $\mathbb{Z}_2$-graded module over Φ:

$$G_0 := \langle 1, \xi_{i_1} \cdots \xi_{i_{2k}} \mid 1 \leq i_1 < ... < i_{2k}, k \geq 1\rangle,$$

$$G_1 := \langle \xi_{i_1} \cdots \xi_{i_{2k+1}} \mid 1 \leq i_1 < ... < i_{2k+1}, k \geq 0\rangle.$$

Thus, G is an associative superalgebra. Moreover, if $R = R_0 \oplus R_1$ is a superalgebra over Φ, we can define the Grassmann envelope of R, $G(R)$, as the even part of the tensor product $G \otimes R$, i.e., $G(R) = (G \otimes R)_0 = G_0 \otimes R_0 + G_1 \otimes R_1$. Notice that $G(R)$ is an algebra.

The Grassmann envelope allows us to define varieties of superalgebras. Let $R = R_0 + R_1$ be a superalgebra. We say that R belongs to a certain variety of superalgebras (Lie, Jordan, associative,...) if $G(R)$ belongs to the same variety of algebras.

Notice that if $R = R_0 \oplus R_1$ is a superalgebra (i.e., $\mathbb{Z}_2$-graded) such that it is associative as algebra then it is easy to check that $G(R)$ is associative as well. Hence R is an associative superalgebra if and only if R is an associative $\mathbb{Z}_2$-graded algebra. But in general a Lie or Jordan superalgebra is not a Lie or Jordan $\mathbb{Z}_2$-graded algebra.

1.1.3. Let $R = R_0 \oplus R_1$ be an associative superalgebra over Φ. In these conditions the map $\sigma : R \to R$ defined by $\sigma(x_0 + x_1) = x_0 - x_1$, for every $x_0 \in R_0$, $x_1 \in R_1$, is an algebra automorphism with $\sigma^2 = \mathrm{id}$. Conversely, given an associative algebra R, every algebra automorphism $\sigma : R \to R$ with $\sigma^2 = \mathrm{id}$ defines a $\mathbb{Z}_2$-graduation on R given by $R_0 = \{a \in R \mid \sigma(a) = a\}$ and $R_1 = \{a \in R \mid \sigma(a) = -a\}$. Therefore, a $\mathbb{Z}_2$-graduation on R is equivalent to an algebra automorphism σ with $\sigma^2 = \mathrm{id}$.

Notice that a Φ-module S of R is graded if and only if $\sigma(S) \subset S$.

1.1.4. Let R be an associative algebra or superalgebra. We say that $*$ is an involution if it is a linear map $* : R \to R$ such that, for every $a, b \in R$, $(a^*)^* = a$ and $(ab)^* = b^*a^*$, and we say that $*$ is a superinvolution if it is a homogeneous, 0-degree, linear map such that for every homogeneous $a, b \in R$, $(a^*)^* = a$ and $(ab)^* = (-1)^{|a||b|}b^*a^*$. We denote the symmetric and skew-symmetric sets with respect an involution or a superinvolution $*$ as $H := \mathrm{Sym}(R, *) = \{a \in R \mid a^* = a\}$ and $K := \mathrm{Skew}(R, *) = \{a \in R \mid a^* = -a\}$ respectively.

1.1.5. An associative algebra R is semiprime (resp. $*$-semiprime) if for every nonzero ideal (resp. $*$-ideal) I of R, $I^2 := \{\sum_i x_i y_i \mid x_i, y_i \in I\} \neq 0$, and it is prime (resp. $*$-prime) if $IJ := \{\sum_i x_i y_i \mid x_i \in I, y_i \in J\} \neq 0$ for every pair of nonzero ideals (resp. $*$-ideals) I, J of R.

We recall that a $*$-ideal is an ideal I such that $I^* \subset I$.

It is easy to prove that R is semiprime if and only if is $*$-semiprime: It is clear that if R is semiprime then is $*$-semiprime. Conversely, let R be a $*$-semiprime algebra and let I be an ideal of R such that $I^2 = 0$. Notice that $I \cap I^*$ is a $*$-ideal whose square is zero. Then $I \cap I^* = 0$, hence $II^* = I^*I = 0$. Thus $(I + I^*)^2 = 0$, and since $I + I^*$ is a $*$-ideal, we have that $I = 0$. Therefore R is semiprime. However, an algebra can be $*$-prime but not prime: Let S be a prime associative algebra over Φ and let us consider $R = S \times S$ with involution $(a, b)^* = (b, a)$. Then R is a $*$-prime algebra but it is not prime. It is interesting to remark that the symmetric elements are of the form (a, a) and the skew-symmetric are of the form $(a, -a)$.

We can prove that an associative algebra R is prime if and only if $aRb \neq 0$ for arbitrary nonzero elements $a, b \in R$, and it is semiprime if and only if it is nondegenerate, i.e., $aRa \neq 0$ for every nonzero element $a \in R$ (see [53, §10]).

We are going to study these concepts in super setting. Let $R = R_0 \oplus R_1$ be an associative superalgebra and let σ be the automorphism associated to the $\mathbb{Z}_2$-graduation. We say that an ideal I is graded if $I = I_0 \oplus I_1$ where $I_0 = I \cap R_0$ and $I_1 = I \cap R_1$ or, as we remarked in 1.1.3, if $\sigma(I) \subset I$.

An associative superalgebra R is semiprime if for every nonzero graded ideal I of R, $I^2 \neq 0$. And it is prime (as a superalgebra) if it does not have nonzero orthogonal graded ideals.

Notice that a $*$-ideal or a graded ideal satisfies $I^* \subset I$ or $\sigma(I) \subset I$, respectively. Then, arguing as before, the concepts of semiprime associative superalgebra and semiprime associative algebra coincide. An associative superalgebra can be prime but not prime as an algebra: for instance, let S be a prime associative algebra over Φ. Then $R = S \times S$ with $R_0 = \{(a, a) \mid a \in S\}$ and $R_1 = \{(a, -a) \mid a \in S\}$ is a prime associative superalgebra, which is not prime as an algebra (see [25]). We can say more: If R is prime as a superalgebra but not as an algebra we can consider a nonzero ideal P of R with $P \cap \sigma(P) = 0$. Then $P \oplus \sigma(P)$ is an essential graded ideal of R, where $(P \oplus \sigma(P))_0 = \{x + \sigma(x) \mid x \in P\} \cong P$ as an algebra and $(P \oplus \sigma(P))_1 = \{x - \sigma(x) \mid x \in P\}$. Hence

$$P \oplus \sigma(P) \triangleleft_{ess} R \hookrightarrow R/P \oplus R/\sigma(P).$$

Primeness in associative superalgebras can be also characterized by elements: for any two elements a, b of a prime associative superalgebra R where a and b are homogeneous, the condition $aRb = 0$ implies that either a or b is zero (see [25, pag. 693]). As we said before, semiprime associative superalgebras are semiprime as algebras hence the property $aRa \neq 0$ for every nonzero homogeneous element $a \in R$ holds in semiprime superalgebras.

Moreover, when dealing with superalgebras we can always consider the algebra R_0. In the next two lemmas F. Montaner states what happens on the even part when the whole superalgebra is semiprime or prime:

Lemma 1.1.6. *[60, Lemma 1.2] If $R = R_0 \oplus R_1$ is a semiprime associative superalgebra, then R and R_0 are semiprime algebras.*

Lemma 1.1.7. *[60, Lemma 1.3] If $R = R_0 \oplus R_1$ is a prime associative superalgebra, then either R or R_0 are prime as algebras.*

1.1.8. An ideal I of an associative algebra R (resp., an associative algebra with involution $*$) is prime (resp., $*$-prime) if R/I is a prime (resp. $*$-prime) associative algebra. If R is a semiprime associative algebra then there exists a family of prime ideals $\{I_\alpha\}_{\alpha\in\Delta}$ such that $\bigcap_{\alpha\in\Delta} I_\alpha = \{0\}$ and therefore R can be seen as a subdirect product of prime associative algebras (see [53, §12]). Similarly, if R is a semiprime associative algebra with involution $*$ there exists a family of $*$-prime ideals $\{I_\alpha\}_{\alpha\in\Delta}$ such that $\bigcap_{\alpha\in\Delta} I_\alpha = \{0\}$ and therefore R can be seen as a subdirect product of $*$-prime associative algebras. This is also true for superalgebras.

Moreover, if R is semiprime and free of n-torsion then the intersection of all prime ideals I_α such that R/I_α is free of n-torsion is zero (notice that the intersection of all prime ideals I_α such that R/I_α has n-torsion contains the essential ideal nR) and therefore R is a subdirect product of prime associative algebras, all of them free of n-torsion.

1.2 The extended centroid of associative algebras and superalgebras

1.2.1. Given an ideal I of R, we can define the ideal $\mathrm{Ann}_R(I) := \{z \in R \mid zI = Iz = 0\}$, which is called the annihilator of I in R. Moreover, when R is semiprime, $\mathrm{Ann}_R(I) = \{z \in R \mid zIz = 0\}$. An ideal I of R is essential (for every nonzero ideal J of R, $I \cap J \neq 0$) if and only if $\mathrm{Ann}_R(I) = 0$ (see [23, Proposition 1.6(1)]).

1.2.2. Given an associative algebra R, we define a permissible map of R as a pair (I, f) where I is an essential ideal of R and $f : I \to R$ is a homomorphism of right R-modules. For permissible maps (I, f) and (J, g) of R, define a relation $\equiv$ by $(I, f) \equiv (J, g)$ if there exists an essential ideal K of R, contained in $I \cap J$, such that $f(x) = g(x)$ for all $x \in K$. It is easy to see that this is an equivalence relation. If R is a semiprime associative algebra then $Q_m^r(R)$ has an associative algebra structure coming from the addition of homomorphisms and from the composition of restrictions of homomorphisms, see [7, Chapter 2]:

- $[I, f] + [J, g] := [I \cap J, f + g]$,
- $[I, f] \cdot [J, g] := [(I \cap J)^2, f \circ g]$.

The quotient set $Q_m^r(R)$ with the operations defined above is called the Martindale algebra of quotients of R. Note that if R is a semiprime associative algebra then the map $f : R \to Q_m^r(R)$ defined by $f(r) := [R, \lambda_r]$, where $\lambda_r : R \to R$ is defined by $\lambda_r(x) := rx$, is a monomorphism of associative algebras, i.e., R can be considered as a subalgebra of its right Martindale algebra of quotients. The right Martindale algebra of quotients of R satisfies that for all $q \in Q_m^r(R)$ there exists an essential ideal I of R such that $qI \subseteq R$. This facts allow us to prove that every subalgebra S of $Q_m^r(R)$ which contains R is semiprime. Otherwise, if I is a nonzero nilpotent ideal of S and pick $0 \neq q \in I$. There exists an essential ideal J of R such that $qJ \subseteq R$, i.e., $qJ = qJ \cap I \subseteq R \cap I$ is a nonzero nilpotent ideal of R which is a contradiction with the semiprimeness of R.

The symmetric Martindale algebra of quotients of R is defined as

$$Q_m^s(R) := \{q \in Q_m^r(R) | \ \exists \text{ an essential ideal } I \text{ of } R \text{ such that } qI + Iq \subset R\}$$

(if R has an involution one can replace the filter of essential ideals by the filter of essential $*$-ideals in the definition of the symmetric Martindale algebra of quotients, see [3, p. 858-859]). If R is semiprime then $Q_m^s(R)$, which is a subalgebra of $Q_m^r(R)$ containing R, is also a semiprime algebra.

When R has an involution $*$, this involution can be extended to $Q_m^s(R)$ as follows: let us consider $q \in Q_m^s(R)$ and I an essential $*$-ideal such that $qI + Iq \subseteq R$. We define $f : I \to R$ by the rule $f(x) = (x^*q)^*$. We set $q^* := [I, f]$ and note that $q^*x^* = (xq)^*$ for all $x \in I$ (see [7, 2.5.4]).

The extended centroid $C(R)$ of a semiprime algebra R is defined as the center of $Q_m^s(R)$. The extended centroid of a prime algebra is a field (see [7, p. 70]), the set of symmetric elements of the extended centroid of a $*$-prime algebra is again a field (see [3, Theorem 4(a)]), and the extended centroid of a semiprime algebra is a commutative and unital von Neumann regular algebra (see [7, Theorem 2.3.9(iii)]). In particular, if R is semiprime, $C(R)$ is a semiprime algebra without nilpotent elements.

The central closure of R, denoted by $\hat{R}$, is defined as the unital subalgebra of $Q_m^s(R)$ generated by R and $C(R)$, i.e., $\hat{R} := C(R)R + C(R)$, and can be seen as a $C(R)$-algebra. Therefore we can consider R contained in $\hat{R}$. Moreover, since $\hat{R}$ contains R and it is contained in $Q_m^s(R)$, if R is semiprime then $\hat{R}$ is semiprime. The algebra $\hat{R}$ is centrally closed, i.e., it coincides with its central closure. In particular its center equals its extended centroid, $Z(\hat{R}) = C(\hat{R})$.

1.2.3. The notion of extended centroid for semiprime associative superalgebras was studied by M. Fošner, see [25]. Let R be a semiprime associative superalgebra. Since R is semiprime as algebra we can consider the symmetric Martindale algebra of quotients $Q_m^s(R)$. Let $\sigma : R \to R$ be the automorphism associated to the $\mathbb{Z}_2$-grading of R ($\sigma^2 =$ id). This automorphism, by [7, Proposition 2.5.3], can be extended to $Q_m^s(R)$ and we denote this extension by $\hat{\sigma}$. Therefore $Q_m^s(R)$ is an associative superalgebra such that $R_i \subset (Q_m^s(R))_i$ with $i = 0, 1$. Moreover, if R is endowed with a superinvolution $*$, this can be also extended to $Q_m^s(R)$ as follows: let us consider $q \in Q_m^s(R)_i$, with $i = 0, 1$, and I an essential graded $*$-ideal such that $qI + Iq \subseteq R$. We define $f : I \to R$ by $f(x) = (-1)^{|q||x|}(x^*q)^*$. We set $q^* := [I, f]$ and note that $q^*x^* = (-1)^{|x||q|}(xq)^*$ for all $x \in I$ homogeneous. Indeed, $*$ is a superinvolution on $Q_m^s(R)$: Let us consider $q_i \in (Q_m^s(R))_i$ and $q_j \in (Q_m^s(R))_j$ with $i, j = 0, 1$. Choose an essential graded $*$-ideal J of R such that Jq_i, q_iJ, Jq_j, q_jJ, Jq_iq_j, q_iq_jJ are all contained in R and let $I = J^2$. Then Iq_i, q_iI, Iq_j, $Iq_i \subseteq J$. For every homogeneous $x \in I$ we have

$$\begin{aligned}(q_iq_j)^*x &= (-1)^{|x|(i+j)}(x^*q_iq_j)^* = (-1)^{|x|(i+j)+(|x|+i)j}q_j^*(x^*q_i)^* \\ &= (-1)^{|x|(i+j)+(|x|+i)j+|x|i}q_j^*q_i^*x = (-1)^{ij}q_j^*q_i^*x.\end{aligned}$$

Hence $(q_iq_j)^* = (-1)^{ij}q_j^*q_i^*$ for all $q_i \in (Q_m^s(R))_i$ and $q_j \in (Q_m^s(R))_j$ with $i, j = 0, 1$.

On the other hand, since R is semiprime as an algebra, we can consider the extended centroid $C(R)$ of R, which it is also $\mathbb{Z}_2$-graded because $C(R) = Z(Q_m^r(R))$. Let $\hat{R} = C(R)R + C(R)$ be the central closure of R. We will say that R is centrally closed if $R = \hat{R}$.

1.2.4. Let R be a prime associative superalgebra such that R is not prime as an algebra. Let σ denote the automorphism associated to the $\mathbb{Z}_2$-grading of R and consider a nonzero ideal P of R with $P \cap \sigma(P) = 0$. Then $P \oplus \sigma(P)$ is a graded essential ideal of R, where $(P \oplus \sigma(P))_0 = \{x + \sigma(x) \mid x \in P\} \cong P$ as an algebra and $(P \oplus \sigma(P))_1 = \{x - \sigma(x) \mid x \in P\}$. Since $P \oplus \sigma(P)$ is essential in R,

$$C(R) \cong C(P \oplus \sigma(P)) = C(P) \oplus \hat{\sigma}(C(P)),$$

where the isomorphism is given by the restriction of permissible maps (for any $\lambda = [I, f] \in C(R)$ we define $\hat{\lambda} = [(I \cap (P \oplus \sigma(P)))^2, g]$ where $g : (I \cap (P \oplus \sigma(P))^2 \to P \oplus \sigma(P)$ is the restriction of f to the essential ideal $(I \cap (P \oplus \sigma(P)))^2$ of $P \oplus \sigma(P)$). Notice that the $\mathbb{Z}_2$-grading of $C(P) \oplus \hat{\sigma}(C(P))$ comes from the $\mathbb{Z}_2$-grading of $P \oplus \sigma(P)$: $(C(P) \oplus \hat{\sigma}(C(P)))_0 = \{\lambda + \hat{\sigma}(\lambda) \mid \lambda \in C(P)\}$ and $(C(P) \oplus \hat{\sigma}(C(P)))_1 = \{\lambda - \hat{\sigma}(\lambda) \mid \lambda \in$

$C(P)\}$. In particular,

$$C(R)_0 \cong \{\lambda + \hat{\sigma}(\lambda) \mid \lambda \in C(P)\} \cong C(P).$$

On the other hand, by Lemma 1.1.7, R_0 is prime as an algebra, and therefore its nonzero ideals are essential. By restricting permissible maps from R_0 to $(P \oplus \sigma(P))_0$ we get $C(R_0) \cong C((P \oplus \sigma(P))_0) \cong C(P)$.

We have obtained that $C(R)_0 \cong C(R_0)$.

Lemma 1.2.5. *Let $R = R_0 \oplus R_1$ be a prime associative superalgebra, and let $a \in R_0$. If there exists $\lambda \in C(R)$ such that $a - \lambda$ is nilpotent of index n and R has no n-torsion then $\lambda \in C(R)_0$.*

Proof. Let us consider $a \in R_0$ and suppose that there exists $\lambda = \lambda_0 + \lambda_1 \in C(R)$ such that $a - \lambda$ is nilpotent of index n. If $\lambda_1 \neq 0$, it is invertible by Lemma 1.2.6 and there exists $\mu_1 \in C(R)_1$ such that $\lambda_1 \mu_1 = 1$. From the nilpotency of $a - \lambda_0 - \lambda_1$ we get that $\mu_1 a - \mu_1 \lambda_0 - 1$ is again nilpotent of index n, i.e., the element $b = \mu_1 a - \mu_1 \lambda_0 \in R_1$ satisfies a polynomial of the form $p(X) = (X-1)^n \in C(R)_0[X]$. Since $C(R)_0$ is a field, $p(X) \in C(R)_0[X]$ is the minimal polynomial of b over $C(R)_0$. In particular

$$b^n - \binom{n}{1} b^{n-1} + \binom{n}{2} b^{n-2} + \cdots = 0$$

and by homogeneity

$$\binom{n}{1} b^{n-1} + \binom{n}{3} b^{n-3} + \cdots = 0$$

i.e., b satisfies the polynomial $q(X) = \sum_{i=1}^{[\frac{n+1}{2}]} \binom{n}{2i-1} X^{n-2i+1}$. But $n - 1 = \deg q(X) < \deg p(X) = n$, a contradiction with the minimality of $p(X)$. Therefore $\lambda_1 = 0$ and $\lambda \in C(R)_0$.

$\square$

Lemma 1.2.6. *[25, Lemma 3.1] Let R be a semiprime associative superalgebra. Then the following assertions are equivalent:*

(i) R is a prime superalgebra.

(ii) all nonzero homogeneus elements on $C(R)$ are invertible.

(iii) $C(R)_0$ is a field.

1.3 Basic notions on Lie and Jordan algebras and superalgebras

1.3.1. We will work with Lie algebras and superalgebras arising from associative algebras and superalgebras. A Lie algebra L over a ring of scalars Φ is a Φ-module with a bilinear product $[\ ,\ ]$ satisfying, for every $x, y, z \in L$, the anticommutativity property and the Jacobi identity:

(i) $[x, y] = -[y, x]$,

(ii) $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ (Jacobi identity).

Let $L = L_0 + L_1$ be a superalgebra over Φ with bilinear product denoted by $[\ ,\ ]_s$. By using the Grassmann envelope, L is a Lie superalgebra if $G(L)$ is a Lie algebra. Let us suppose that $L = L_0 + L_1$ is a Lie superalgebra, i.e., $G(L) = L_0 \otimes G_0 + L_1 \otimes G_1$ is a Lie algebra. We can deepen into which identities L satisfies: let us pick $x \otimes \xi_i, y \otimes \xi_j \in (L_0 \otimes G_0) \cup (L_1 \otimes G_1)$, then

$$[x, y]_s \otimes \xi_i \xi_j = [x \otimes \xi_i, y \otimes \xi_j] = -[y \otimes \xi_j, x \otimes \xi_i] = -[y, x]_s \otimes \xi_j \xi_i = -(-1)^{|x||y|}[y, x]_s \otimes \xi_i \xi_j$$

so we can assure, by linearity, that $[x, y]_s = -(-1)^{|x||y|}[y, x]_s$ for every $x, y \in L_0 \cup L_1$. Notice that the factor (-1) in the identity naturally arises from the property $\xi_i \xi_j + \xi_j \xi_i = 0$, i.e, $\xi_i \xi_j = -\xi_j \xi_i$ of the generators of the Grassman algebra. Therefore, the identities (i) and (ii) can be translated to super setting as follows: Let L be a $\mathbb{Z}_2$-graded module over Φ with a bilinear product $[\ ,\ ]_s$ such that for every homogeneous $x, y, z \in L$:

(i) $[x, y]_s = -(-1)^{|x||y|}[y, x]_s$ (super-anticommutativity),

(ii) $[x,[y,z]_s]_s+(-1)^{|x|(|y|+|z|)}[z,[x,y]_s]_s+(-1)^{|z|(|x|+|y|)}[y,[z,x]_s]_s=0$ (Jacobi super-identity).

Conversely, a superalgebra is a Lie superalgebra if both identities above are satisfied (see [**?**, Section 1]).

Recall that the adjoint map determined by any $a \in L$ (resp. any homogeneous $a \in L$) is $\mathrm{ad}_a(x) := [a,x]$ (resp. $\mathrm{ad}_a(x) := [a,x]_s$ in super setting) for every $x \in L$. We say that an element $a \in L$ is ad-nilpotent of index $n \geq 1$ if $\mathrm{ad}_a^n L = 0$ and $\mathrm{ad}_a^{n-1} L \neq 0$. We say that an element a in L is a Jordan element if $\mathrm{ad}_a^3 L = 0$ (see [23, Chapter 4]). Since in superalgebras we will always consider homogeneous elements, we will define Jordan element in superalgebras for even elements as an even element which is ad-nilpotent of index less or equal to 3 of the whole Lie superalgebra. For odd elements we will work with ad-nilpotency of index less or equal to 4.

Typical examples of Lie algebras and superalgebras come from the associative setting: if R is an associative algebra (resp. superalgebra) over a ring of scalars Φ, then R with product, called bracket, $[x,y] := xy - yx$ for every $x, y \in R$ (resp. $[x,y]_s = xy - (-1)^{|x||y|}yx$, called super-bracket, for every homogeneous $x, y \in R$) is a Lie algebra (resp. a Lie superalgebra) denoted by R^-. When dealing with R^- as a superalgebra, if $a \in R_0$ then ad_a behaves as the usual adjoint map in the non-super setting; when $a \in R_1$, $\mathrm{ad}_a^2 = \mathrm{ad}_{a^2}$.

We will deal with Jordan algebras and superalgebras in Chapter 5. A linear Jordan algebra J over a ring of scalars Φ, with $\frac{1}{2} \in \Phi$, is a Φ-module with a bilinear product $\bullet$ satisfying, for every $x, y \in J$, the commutativity property and Jordan identity:

(i) $x \bullet y = y \bullet x$,

(ii) $((x \bullet x) \bullet y) \bullet x = (x \bullet x) \bullet (y \bullet x)$ (Jordan identity).

We already know that a superalgebra is a Jordan superalgebra if its Grassmann envelope is a Jordan algebra. But to translate the Jordan identity to super setting first we need to linearize it because the generatos in the Grassman algebra satisfy $\xi_i^2 = 0$. We can prove that a $\mathbb{Z}_2$-graded module J over Φ with a bilinear product $\bullet_s$ is a Jordan superalgebra if it satisfies

(i) $x \bullet_s y = (-1)^{|x||y|} y \bullet_s x$ (super-commutativity),

(ii) $(x \bullet_s y) \bullet_s (z \bullet_s t) + (-1)^{|y||z|}(x \bullet_s z) \bullet_s (y \bullet_s t) + (-1)^{|y||t|+|z||t|}(x \bullet_s t) \bullet_s (y \bullet_s z) =$
$= ((x\bullet_s y)\bullet_s z)\bullet_s t + (-1)^{|y||z|+|y||t|+|z||t|}((x\bullet_s t)\bullet_s z)\bullet_s y + (-1)^{|x||y|+|x||z|+|x||t|+|z||t|}((y\bullet_s t) \bullet_s z) \bullet_s x$ (Jordan super-identity)

for every homogeneous $x, y, z, t \in J$. As above, if R is an associative algebra (resp. superalgebra) over a ring of scalars Φ, then R with product, called bullet, $x \bullet y = xy + yx$ for every $x, y \in R$ (resp. $x \bullet_s y = xy + (-1)^{|x||y|}yx$, called super-bullet, for every homogeneous $x, y \in R$) is a Jordan algebra (resp. Jordan superalgebra) denoted by R^+.

The algebras R^- and R^+ are well-known and it was I.N. Herstein the first one to study the relations between R and both of them in the non-super case (see for example [43]). Moreover, K is a Lie subalgebra (resp. subsuperalgebra) of R^- and H is a Jordan subalgebra (resp. subsuperalgebra) of R^+. We refer the reader to [25], [35], [36], [37], [52], [60] and [62] for further information on associative superalgebras and on the Herstein theory on superalgebras. Although we have denoted super bracket as $[\ ,\ ]_s$, in Chapter 3, in order to simplify the notation, we will denote it as $[\ ,\ ]$ (we will just work with the super bracket and there will not be any confusion).

1.3.2. If R is a centrally closed $*$-prime algebra and $\mathrm{Skew}(C(R), *) \neq 0$ then for any $0 \neq \lambda \in \mathrm{Skew}(C(R), *)$ we have $R = H + K = \lambda^2 H + K \subseteq \lambda K + K \subseteq R$ because $0 \neq \lambda^2$ is invertible, so $R = \lambda K + K$ for every $0 \neq \lambda \in \mathrm{Skew}(C(R), *)$. This occurs in particular when R is $*$-prime but not prime, because in this situation there exists a nonzero ideal I of R such that $I \cap I^* = 0$, and so we can define a nonzero skew element $\lambda : I \oplus I^* \to R$ in $C(R)$ given by $\lambda(x + y) := x - y$.

If R is a centrally closed semiprime ring then R^- is a Lie algebra over the ring of scalars $C(R)$; if in addition R has an involution $*$, then K is a Lie algebra over $H(C(R), *)$.

Lemma 1.3.3. ([13, Lemma 2.11]) *Let $(R, *)$ be a semiprime associative algebra with involution and let $a \in R$. If there exist $\lambda \in C(R)$ such that $a - \lambda$ is nilpotent then λ*

is the unique element of $C(R)$ such that $a - \lambda$ is nilpotent. Moreover, if $a \in K$ then $\lambda \in \mathrm{Skew}(C(R), *)$.

Proof. If $a-\lambda$ and $a-\mu$ are nilpotent elements of the central closure $\hat{R}$ of R, $a-\lambda-(a-\mu) = \mu-\lambda$ is a nilpotent element in the semiprime commutative ring $C(R)$. Therefore $\lambda = \mu$. Now, if $a \in K$ and $a - \lambda$ is nilpotent then $(a - \lambda)^* = -(a + \lambda^*)$ is nilpotent and therefore $a + \lambda^*$ is nilpotent, which implies that $\lambda = -\lambda^* \in \mathrm{Skew}(C(R), *)$. $\square$

We will need also this result in superalgebras. With the same argument as in the above lemma we have:

Lemma 1.3.4. *Let $R = R_0 \oplus R_1$ be a semiprime associative superalgebra with super-involution $*$, and let $a \in R_0 \cup R_1$. If there exists $\lambda \in C(R)$ such that $a-\lambda$ is nilpotent then λ is the unique element of $C(R)$ such that $a-\lambda$ is nilpotent. Moreover, if $a \in K$ then $\lambda \in \mathrm{Skew}(C(R), *)$.*

Chapter 2

Ad-nilpotent elements in an associative algebra

This chapter has been published in the journal *Bulletin of the Malaysian Mathematical Sciences Society* and can be found in [12].

Throughout all this chapter R is an associative algebra over Φ with $\frac{1}{2} \in \Phi$.

The main goal of this chapter is to deepen into the description of ad-nilpotent elements of R and K where R is a semiprime associative algebra with involution. In the spirit of Martindale and Miers' result [56, Main Theorem], we will obtain different types of ad-nilpotent elements of K of index n depending on the equivalence class of n modulo 4. In this chapter we will also study ad-nilpotent elements in semiprime associative algebras, as T.K. Lee did in [54], but we introduce a new concept called pure ad-nilpotent, that it will allow us to weaken torsion conditions and to obtain a more detailed classification. We say that an ad-nilpotent element a of index n in R^- is pure if λa remains ad-nilpotent of the same index for every λ in the extended centroid such that $\lambda a \neq 0$. An ad-nilpotent element a of index n in K is pure if for every symmetric λ in the extended centroid such that $\lambda a \neq 0$, λa is ad-nilpotent of the same index n. This is just a technical condition, since every ad-nilpotent element of R^- can be expressed as an orthogonal sum of pure ad-nilpotent elements of the central closure $\hat{R}$ of R with decreasing indices of ad-nilpotency.

As a first step we focus on ad-nilpotent elements of R^-. In this case, under the hypothesis of pure ad-nilpotence, the condition on the torsion of the algebra can be weakened when compared with the result of T.K. Lee in [54, Theorem 1.3].

From Theorems 2.2.4 and 2.3.6 we easily recover Lee's results [54, Theorem 1.3 and Theorem 1.5]. Furthermore, we also describe ad-nilpotent elements of Lie algebras of the form $R/Z(R)$ and $K/(K\cap Z(R))$, and of their derived Lie algebras $[R,R]/([R,R]\cap Z(R))$ and $[K,K]/([K,K]\cap Z(R))$.

Let us write down some useful results where the extended centroid $C(R)$ plays a really important role. We will use the following results due to Beidar, Martindale and Mikhalev.

Theorem 2.0.1. ([57, Theorem 2(a)]) *Let R be a prime associative algebra. Let $a_i, b_i \in R$ for $i = 1, 2, \ldots, n$ with $b_1 \neq 0$ be such that $\sum_{i=1}^{n} a_i x b_i = 0$ for every $x \in R$. Then there exist $\lambda_i \in C(R)$ for $i = 2, \ldots, n$ such that $a_1 = \sum_{i=2}^{n} \lambda_i a_i$ in $\hat{R}$.*

Theorem 2.0.2. ([7, Theorem 2.3.3]) *Let R be a semiprime associative algebra and let $a_1, a_2, \ldots, a_n \in R$. If $a_1 \notin \sum_{i=2}^{n} C(R)a_i$ in $\hat{R}$ then there exist $r_j, s_j \in R$ for $j = 1, 2, \ldots, m$ such that $\sum_{j=1}^{m} r_j a_1 s_j \neq 0$ and $\sum_{j=1}^{m} r_j a_k s_j = 0$ for $k = 2, \ldots, n$.*

The next corollary can be found in [13]. For the sake of completeness we include its proof here.

Corollary 2.0.3. *Let R be a semiprime associative algebra. Let $a_i, b_i \in R$ for $i = 1, 2, \ldots, n$ be such that $\mathrm{Id}_R(a_1) \subset \mathrm{Id}_R(b_1)$ and $\sum_{i=1}^{n} a_i x b_i = 0$ for every $x \in R$. Then there exist $\lambda_i \in C(R)$ for $i = 2, \ldots, n$ such that $a_1 = \sum_{i=2}^{n} \lambda_i a_i$ in $\hat{R}$.*

Proof. By Theorem 2.0.2, if $a_1 \notin \sum_{i=2}^{n} C(R)a_i$ there exist $r_j, s_j \in R$, $j = 1, \ldots, m$, such that $\sum_{j=1}^{m} r_j a_1 s_j \neq 0$ and $\sum_{j=1}^{m} r_j a_k s_j = 0$ for $k = 2, 3, \ldots, n$. Replace x by $s_j x$ and multiply $\sum_{i=1}^{n} a_i x b_i = 0$ on the left by r_j. We have

$$0 = \sum_{i=1}^{n} \sum_{j=1}^{m} r_j a_i s_j x b_i = \sum_{j=1}^{m} r_j a_1 s_j x b_1,$$

which implies that the ideal generated by $\sum_{j=1}^{m} r_j a_1 s_j$ is orthogonal to the ideal generated by b_1 and therefore, since $\mathrm{Id}_R(a_1) \subset \mathrm{Id}_R(b_1)$, the ideal generated by $\sum_{j=1}^{m} r_j a_1 s_j$ has zero square, a contradiction because R is semiprime. $\square$

The following proposition is an easy generalization of [7, Theorem 2.3.9(i)].

Proposition 2.0.4. *Let R be a centrally closed semiprime associative algebra. For any subset $V \subset R$ there exists a unique idempotent $\epsilon \in C(R)$ such that $\epsilon v = v$ for all $v \in V$, the annihilator in $C(R)$ of V is $\mathrm{Ann}_{C(R)}(V) = (1-\epsilon)C(R)$, the annihilator in R of the ideal generated by V is $\mathrm{Ann}_R(\mathrm{Id}_R(V)) = (1-\epsilon)R$, and the ideal generated by V is essential in ϵR. Moreover, when R has an involution $*$ and $V \subset H$ or $V \subset K$, then $\epsilon \in H(C(R), *)$.*

Proof. The first part of the proof follows as in [7, Theorem 2.3.9(i)] with the obvious changes. Let $V \subset H$ or $V \subset K$, and consider the unique idempotent $\epsilon \in C(R)$ such that $\epsilon v = v$ for all $v \in V$, the annihilator in $C(R)$ of V is $\mathrm{Ann}_{C(R)}(V) = (1-\epsilon)C(R)$ and the annihilator in R of the ideal generated by V is $\mathrm{Ann}_R(\mathrm{Id}_R(V)) = (1-\epsilon)R$. When R has an involution we can decompose $\epsilon = \epsilon_k + \epsilon_h$ with $\epsilon_k \in \mathrm{Skew}(C(R), *)$ and $\epsilon_h \in H(C(R), *)$. We have that $\epsilon v = v$ implies $\epsilon_k v = 0$. Therefore, $\epsilon_k \in \mathrm{Ann}_{C(R)}(V) = (1-\epsilon)C(R)$, i.e., $\epsilon_k \epsilon = 0$ and $\epsilon_k^2 = \epsilon_k \epsilon_h = 0$ and therefore $\epsilon = \epsilon^2 = (\epsilon_k + \epsilon_h)^2 = \epsilon_h^2 \in H(C(R), *)$. □

Lemma 2.0.5. *Let R be a centrally closed semiprime associative algebra and let $\{\nu_i\}_{i\in I}$ be a family of idempotent elements in $C(R)$. Suppose there exists a family $\{\lambda_i\}_{i\in I}$ of elements in $C(R)$ such that for every $i, j \in I$, $\lambda_i \nu_i \nu_j = \lambda_j \nu_i \nu_j$. Then there exists $\lambda \in C(R)$ such that $\lambda \nu_i = \lambda_i \nu_i$ for every $i \in I$. Moreover, if the ideal generated by the family $\{\nu_i\}_{i\in I}$ is essential in R, such λ is unique.*

Proof. Let us consider the ideal $S = \sum R\nu_i$ generated by the family of idempotents $\{\nu_i\}_{i\in I}$ and the essential ideal $T = S \oplus \mathrm{Ann}_R(S)$. Define $\lambda : T \to R$ by

$$\lambda(\sum x_i \nu_i + z) := \sum \lambda_i x_i \nu_i.$$

Let us prove that λ is well defined and an element in $C(R)$. If $\sum x_i \nu_i + z = 0$ then $\sum x_i \nu_i = 0 = z$ and for every ν_k we have

$$\left(\sum \lambda_i x_i \nu_i\right) \nu_k = \sum \lambda_k x_i \nu_i \nu_k = \lambda_k \left(\sum x_i \nu_i\right) \nu_k = 0.$$

Therefore $\sum \lambda_i x_i \nu_i \in S \cap \mathrm{Ann}_R(S) = 0$ which proves that λ is well defined. By construction $[T, \lambda] \in C(R)$. Moreover, if the ideal S generated by the family $\{\nu_i\}_{i\in I}$ is essential, $\mathrm{Ann}_R(S) = 0$ and $[S, \lambda] \in C(R)$ is uniquely defined. □

2.1 Pure ad-nilpotent elements

Recall that an element a in a Lie algebra L is ad-nilpotent of index n if $\mathrm{ad}_a^n L = 0$ and $\mathrm{ad}_a^{n-1} L \neq 0$.

2.1.1. (i) Let us consider R^-: we say that an element a is a pure ad-nilpotent element of R^- of index n if for every $\lambda \in C(R)$ with $\lambda a \neq 0$, λa is ad-nilpotent in $\hat{R}^-$ of index n, where $\hat{R}$ is the central closure of R.
(ii) Let us consider K: we say that an element a is a pure ad-nilpotent element of K of index n if for every $\lambda \in H(C(R)), *)$ with $\lambda a \neq 0$, λa is ad-nilpotent in $\mathrm{Skew}(\hat{R}, *)$ of index n, where $\hat{R}$ is the central closure of R.

Lemma 2.1.2. *If R is a semiprime associative algebra and a is an ad-nilpotent element of R of index n, the following conditions are equivalent:*

(i) a is a pure ad-nilpotent element of R^-.

(ii) $\mathrm{Id}_R(\mathrm{ad}_a^{n-1}(R))$ is an essential ideal of $\mathrm{Id}_R(a)$.

(iii) $\mathrm{Ann}_R(\mathrm{Id}_R(\mathrm{ad}_a^{n-1}(R))) = \mathrm{Ann}_R(\mathrm{Id}_R(a))$.

Proof. Suppose that R is semiprime and centrally closed (otherwise, substitute R by its central closure $\hat{R}$).

(i) ⇒ (ii). Let us consider $V = \{\mathrm{ad}_a^{n-1} x \mid x \in R\}$. By Proposition 2.0.4 there exists $e \in C(R)$ such that $ev = v$ for every $v \in V$ and $\mathrm{Ann}_R(\mathrm{Id}_R(V)) = (1-e)R$. Suppose that $(1-e)a \neq 0$. By hypothesis $(1-e)a$ is ad-nilpotent of index n, hence $0 \neq \mathrm{ad}_{(1-e)a}^{n-1}(R) = (1-e)\mathrm{ad}_a^{n-1}(R) = 0$, a contradiction. So $ea = a$ and $\mathrm{Ann}_{\mathrm{Id}_R(ea)}(\mathrm{Id}_R(\mathrm{ad}_a^{n-1}(R))) \subset \mathrm{Ann}_R(\mathrm{Id}_R(\mathrm{ad}_a^{n-1}(R))) = (1-e)R$ must be zero, i.e., $\mathrm{Id}_R(\mathrm{ad}_a^{n-1}(R))$ is essential in $\mathrm{Id}_R(ea)$.

(ii)$\Rightarrow$ (iii). This holds in general if I and J are ideals of R with I essential in J: $0 = \mathrm{Ann}_J(I) = \mathrm{Ann}_R(I) \cap J$ implies $\mathrm{Ann}_R(I)J = 0$, so $\mathrm{Ann}_R(I) \subset \mathrm{Ann}_R(J)$.

(iii) $\Rightarrow$ (i). Let $\lambda \in C(R)$ be such that $\lambda a \neq 0$. Clearly $\mathrm{ad}^n_{\lambda a}(R) = 0$. Suppose that $\mathrm{ad}^{n-1}_{\lambda a}(R) = 0$: then $\lambda^{n-1}\mathrm{ad}^{n-1}_a(R) = 0$, so $\lambda^{n-1} \in \mathrm{Ann}_R(\mathrm{Id}_R(\mathrm{ad}^{n-1}_a(R))) = \mathrm{Ann}_R(\mathrm{Id}_R(a))$, which is not possible because R is semiprime and $\lambda a \neq 0$. $\square$

Lemma 2.1.3. *Let R be a centrally closed semiprime associative algebra with involution $*$, and let $a \in K$ be a pure ad-nilpotent element of K of index n. If there exists $\lambda \in H(C(R), *)$ such that λa is ad-nilpotent of R of index n, then λa is a pure ad-nilpotent element of R of index n.*

Proof. Let us see that for every $\mu \in C(R)$ with $\mu\lambda a \neq 0$, the element $\mu\lambda a$ has index of ad-nilpotency in R equal to n. Suppose that there exists $\mu \in C(R)$ with $\mathrm{ad}^{n-1}_{\mu\lambda a} R = 0$, and let us prove that $\mu\lambda a = 0$:

We have that $\mu^{n-1}\mathrm{ad}^{n-1}_{\lambda a} R = \mathrm{ad}^{n-1}_{\mu\lambda a} R = 0$, so $\mu\mathrm{ad}^{n-1}_{\lambda a} R = 0$ because $C(R)$ is regular von Neumann. In particular, $\mu\mathrm{ad}^{n-1}_{\lambda a} H = \mu\mathrm{ad}^{n-1}_{\lambda a} K = 0$. Since $\mu = \mu_h + \mu_k$, we have that $\mu_h\mathrm{ad}^{n-1}_{\lambda a} R = \mu_k\mathrm{ad}^{n-1}_{\lambda a} R = 0$.

From $0 = \mu_h^{n-1}\mathrm{ad}^{n-1}_{\lambda a} R = \mathrm{ad}^{n-1}_{\mu_h\lambda a} R$ we get that $\mu_h\lambda a$ index of ad-nilpotency in K lower than n, implying $\mu_h\lambda a = 0$ because a is a pure ad-nilpotent element of K.

From $0 = (\mu_k^2)^{n-1}\mathrm{ad}^{n-1}_{\lambda a} R = \mathrm{ad}^{n-1}_{\mu_k^2\lambda a} R$ we get that $\mu_k^2\lambda a$ has index of ad-nilpotency in K lower than n, so again $\mu_k^2\lambda a = 0$ (because a is a pure ad-nilpotent element of K), and by regularity of $C(R)$, $\mu_k\lambda a = 0$.

This implies $\mu\lambda a = 0$. $\square$

The next proposition shows that every ad-nilpotent of R^- or of K can be expressed as an orthogonal sum of pure ad-nilpotent elements of decreasing indices.

Proposition 2.1.4. *Let R be a centrally closed semiprime associative algebra and let $a \in R$ be an ad-nilpotent element of R^- of index n. There exists a family of orthogonal idempotents $\{\epsilon_i\}_{i=1}^k \subset C(R)$ such that $a = \sum_{i=1}^k \epsilon_i a$ with $\epsilon_i a$ a pure ad-nilpotent element of index n_i in $\epsilon_i R$ for $n = n_1 > n_2 > \cdots > n_k$.*

Similarly, if R has an involution $*$ and a is an ad-nilpotent element of K of index n, then there exists a family of orthogonal idempotents $\{\epsilon_i\}_{i=1}^k \subset H(C(R), *)$ such *that* $a = \sum_{i=1}^{k} \epsilon_i a_i$ *with* $\epsilon_i a$ *a pure ad-nilpotent element of index* n_i *in* $\mathrm{Skew}(\epsilon_i R, *)$ *for* $n = n_1 > n_2 > \cdots > n_k$.

Proof. Let us prove the result for Lie algebras of skew-symmetric elements. We will proceed by induction on n. If $n = 1$ there is nothing to prove. Let us suppose that the result is true for every ad-nilpotent element of index less than n and let $a \in K$ be an ad-nilpotent element of index $n \geq 2$. Let us consider $V = \{\mathrm{ad}_a^{n-1} x \mid x \in K\}$. By Proposition 2.0.4 there exists $\epsilon \in H(C(R), *)$ such that $\epsilon v = v$ for every $v \in V$ and $\mathrm{Ann}_R(\mathrm{Id}_R(V)) = (1 - \epsilon)R$. Then $a = \epsilon a + (1 - \epsilon)a$.

Clearly, by construction $(1 - \epsilon)a$ is ad-nilpotent of index less than n in K: for every $x \in K$, $\mathrm{ad}_{(1-\epsilon)a}^{n-1} x = (1 - \epsilon)\mathrm{ad}_a^{n-1} x = \mathrm{ad}_a^{n-1} x - \epsilon \mathrm{ad}_a^{n-1} x = 0$.

Let us prove that ϵa is pure ad-nilpotent of index n in $\mathrm{Skew}(\epsilon R, *)$. For any $\lambda \in H(C(R), *)$ such that $\lambda \epsilon a \neq 0$, $\lambda \epsilon a$ is ad-nilpotent of index n: clearly $\mathrm{ad}_{\lambda\epsilon a}^{n}(\mathrm{Skew}(\epsilon R, *)) = 0$ and if $\mathrm{ad}_{\lambda\epsilon a}^{n-1}(\mathrm{Skew}(\epsilon R, *)) = 0$ then $\lambda^{n-1}\epsilon \in \mathrm{Ann}_R(\mathrm{Id}_R(V)) = (1 - \epsilon)R$, which leads to a nilpotent ideal generated by the nonzero element $\lambda \epsilon a$, a contradiction with the semiprimeness of R.

Apply now the induction hypothesis to $(1 - \epsilon)a$ and the Lie algebra of skew-symmetric elements $\mathrm{Skew}((1 - \epsilon)R, *)$. □

2.2 Ad-nilpotent elements of R^-

In this section we are going to prove that every nilpotent inner derivation is induced by a nilpotent element, generalizing to semiprime algebras Herstein's result [42, Theorem in p. 84] for simple algebras. This result was already proved by Grzeszczuk ([38, Corollary 8]). Our techniques are rather elementary and, by adding the hypothesis of pure ad-nilpotence, we can describe such elements with less restrictions on the torsion of the algebra.

Lemma 2.2.1. *Let* R *be a semiprime associative algebra and let* $a \in R$ *be a nilpotent* *element. Suppose that there exist some $\lambda_i \in \mathbb{Z}$, $i = 0, \ldots, n$, such that*

$$\sum_{i=0}^{n} \lambda_i a^i [x, y] a^{n-i} = 0$$

for all $x, y \in R$. Then for every $i = 0, \ldots, n$ we have $\lambda_i a^{\max(i,n-i)} = 0$. In particular, each term in the identity above is zero.

Proof. First, let us suppose that R is prime and suppose that $a \neq 0$ has index of nilpotence s. If the lemma is not satisfied, there exists some k with $\lambda_k a^{\max(k,n-k)} \neq 0$. In particular, $\max(k, n-k) < s$. Let us multiply the expression $\sum_{i=0}^{n} \lambda_i a^i [x, y] a^{n-i}$ by a^{s-1-k} on the left and by $a^{s-1-(n-k)}$ on the right, so that

$$0 = a^{s-1-k} \left(\sum_{i=0}^{n} \lambda_i a^i [x, y] a^{n-i} \right) a^{s-1-(n-k)} = \lambda_k a^{s-1} [x, y] a^{s-1}$$

for every $x, y \in R$. Hence $\lambda_k a^{s-1} xy a^{s-1} = \lambda_k a^{s-1} yx a^{s-1}$ for every $x, y \in R$. Since $a^{s-1} \neq 0$ for every $x \in R$ we have by Theorem 2.0.1 that there exists $\alpha_x \in C(R)$ such that $\lambda_k a^{s-1} x = \alpha_x \lambda_k a^{s-1}$. Multiplying this last expression by a on the right we get $\lambda_k a^{s-1} x a = 0$ for every $x \in R$. By primeness of R we get that either $a^{s-1} = 0$ or $\lambda_k a = 0$, leading to a contradiction.

If R is semiprime then R is a subdirect product of prime quotients R/I_α with $\bigcap_\alpha I_\alpha = 0$. For any α and any i, by the prime case $\lambda_i a^{\max(i,n-i)} \in I_\alpha$, so $\lambda_i a^{\max(i,n-i)} = 0$. □

Lemma 2.2.2. *Every nilpotent element of an associative algebra R is ad-nilpotent. If a has index of nilpotence t and index of ad-nilpotence n then $n \leq 2t - 1$. If R is semiprime then $n \geq t$, and if in addition R is free of $\binom{n}{s}$-torsion for $s := [\frac{n+1}{2}]$, then $t = s$ and $n = 2t - 1$.*

Proof. Since $a^t = 0$, for every $x \in R$ we have

$$\mathrm{ad}_a^{2t-1} x = \sum_{i=0}^{2t-1} \binom{2t-1}{i} (-1)^{2t-1-i} a^i x a^{2t-1-i} = 0$$

because if $i < t$ then $2t - 1 - i \geq s$. Therefore $n \leq 2t - 1$.

Suppose now that R is semiprime and let us see that $n \geq t$: if on the contrary

$$\mathrm{ad}_a^{t-1} x = \sum_{i=0}^{t-1} \binom{t-1}{i} (-1)^{t-1-i} a^i x a^{t-1-i} = 0$$

for every $x \in R$, focusing on the first summand of this expression $((-1)^{t-1} x a^{t-1})$ we get that $a^{t-1} = 0$ by Lemma 2.2.1, a contradiction.

Moreover, since for every $x \in R$ we have $0 = \mathrm{ad}_a^n(x) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} a^i x a^{n-i}$, again by Lemma 2.2.1 $\binom{n}{s} a^s = 0$ for $s := [\frac{n+1}{2}]$. If R is free of $\binom{n}{s}$-torsion then $a^s = 0$ so $s \geq t$, i.e., $n \geq 2t - 1$, and therefore $n = 2t - 1$ (equivalently, $t = s$). $\square$

The next example shows that all possible cases in the lemma above can be realized: Let p be an odd prime number and R a prime associative algebra with characteristic p. If $a \in R$ is a nilpotent element of index $t \in \{\frac{p+1}{2}, \ldots, p\}$ then a is ad-nilpotent of index p. In particular there are no ad-nilpotent elements of index between $p + 1$ and $2p - 1$, and a nilpotent element of index p is ad-nilpotent of the same index p.

Proposition 2.2.3. *Let R be a prime associative algebra and let $a \in R$ be an ad-nilpotent element of R^- of index n. Let $\overline{\mathbb{F}}$ denote the algebraic closure of the field $\mathbb{F} := C(R)$ and $\overline{R} := \hat{R} \otimes \overline{\mathbb{F}}$. Then:*

1. *There exists $\mu \in \overline{\mathbb{F}}$ such that $a - \mu$ is a nilpotent element of $\overline{R}$.*

2. *If R is free of $\binom{n}{s}$-torsion for $s := [\frac{n+1}{2}]$ then n is odd and the index of nilpotence of $a - \mu$ is $\frac{n+1}{2}$. If in addition R is free of s-torsion then $\mu \in C(R)$.*

Proof. (1) Since R is prime, $\mathbb{F} = C(R)$ is a field and $\overline{R}$ is a centrally closed prime algebra (see [7, pp. 445–446]). From

$$0 = \mathrm{ad}_a^n x = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} a^i x a^{n-i}$$

for every $x \in R$ we have, by Theorem 2.0.1, that a seen as an element of $\hat{R}$ is an algebraic element over $\mathbb{F}$ of degree not greater than n. Let us consider the minimal polynomial $p(X) \in \mathbb{F}[X]$ of a. Let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$ and let $\mu_1, \ldots, \mu_r \in \overline{\mathbb{F}}$ be the roots of $p(X)$ in $\overline{\mathbb{F}}$, i.e., $p(X) = (X-\mu_1)^{k_1} \cdots (X-\mu_r)^{k_r} \in \overline{\mathbb{F}}[X]$.

Let us prove that $p(X)$ has only one root in $\overline{\mathbb{F}}$ and therefore $p(X) = (X-\mu)^k \in \mathbb{F}[X]$, whence $a-\mu$ is nilpotent in $\overline{R}$: Suppose on the contrary that $p(X)$ has different roots $\mu_1, \ldots, \mu_r$, $r > 1$, and define $q_i(X) := p(X)/(X-\mu_i)$ for every i. Since $p(X)$ is the minimal polynomial of a, $q_i(a) \neq 0$ in $\overline{R}$. Note that $(a-\mu_i)q_i(a) = p(a) = 0$ and therefore $aq_i(a) = \mu_i q_i(a)$. Now, since we are in the prime case, there exists $y \in R$ such that $q_1(a)yq_2(a) \neq 0$ and therefore $\mathrm{ad}_a(q_1(a)yq_2(a)) = aq_1(a)yq_2(a) - q_1(a)yq_2(a)a = (\mu_1-\mu_2)q_1(a)yq_2(a) \neq 0$. This means that $q_1(a)yq_2(a)$ is an eigenvector of the linear map ad_a associated to the eigenvalue $\mu_1 - \mu_2$, hence it is an eigenvector of ad_a^2 associated to $(\mu_1-\mu_2)^2$, etc. This is a contradiction because both $q_1(a)yq_2(a)$ and each power of $(\mu_1-\mu_2)$ are nonzero, while ad_a is nilpotent. Therefore $r = 1$, $p(X) = (X-\mu)^k \in \mathbb{F}[X]$ and $(a-\mu)^k = 0$.

(2) Let us consider $b := a - \mu \in \overline{R}$, which is ad-nilpotent of index n. Let us see that n is odd: Suppose on the contrary that $n = 2m$. Then

$$0 = \mathrm{ad}_a^n x = \mathrm{ad}_b^n x = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} b^i x b^{n-i}$$

implies by Lemma 2.2.1 that $\binom{n}{m} b^m = 0$ and, since $\overline{R}$ is free of $\binom{n}{m}$-torsion, that $b^m = 0$. Substituting in $\mathrm{ad}_b^{n-1}x = \sum_{i=0}^{n-1} \binom{n-1}{i}(-1)^{n-1-i} b^i x b^{n-1-i}$ we get that $\mathrm{ad}_b^{n-1}x = 0$ for every $x \in R$, a contradiction.

Therefore n is odd and $a - \mu$ is nilpotent of $\overline{R}$ of index $s := \frac{n+1}{2}$ by Lemma 2.2.2. Moreover, since the coefficient of degree $s-1$ of $p(X) = (X-\mu)^s \in \mathbb{F}[X]$ is $-s\mu \in \mathbb{F}$, if R is free of s-torsion then $\mu \in \mathbb{F}$, i.e., there exists $\mu \in C(R)$ such that $a - \mu$ is nilpotent of index $s = \frac{n+1}{2}$. □

In the following theorem we get the description of the pure ad-nilpotent elements of R^-. In its proof, Proposition 2.2.3 is primarily used to find that any ad-nilpotent element $a \in R$ of index n forces $[a, [\mathrm{ad}_a^{n-1}x, [\mathrm{ad}_a^{n-1}x, y]]] = 0$ for every $x, y \in R$. If $2, 3, \ldots, r$ were invertible in R for $r \geq n + [\frac{n}{2}] + 1$, this identity would directly follow from the proof of [29, Theorem 2.3].

Theorem 2.2.4. *Let R be a semiprime associative algebra, let $\hat{R}$ be its central closure, and let $a \in R$ be a pure ad-nilpotent element of R^- of index n. Put $s := [\frac{n+1}{2}]$, and suppose that R is free of $\binom{n}{s}$-torsion and s-torsion. Then n is odd and there exists $\lambda \in C(R)$ such that $a - \lambda \in \hat{R}$ is nilpotent of index $\frac{n+1}{2}$.*

Proof. Let us suppose that R is a prime associative algebra and, without loss of generality, that it is centrally closed. Consider $\mu \in C(R)$ as given by Proposition 2.2.3. Putting $b := a - \mu$, we know that $b^s = 0$ for $s := \frac{n+1}{2}$, hence for every $x, y \in R$ we have

$$(\mathrm{ad}_a^{n-1}x)(\mathrm{ad}_a^{n-1}x) = (\mathrm{ad}_b^{n-1}x)(\mathrm{ad}_b^{n-1}x) = 0, \text{ and}$$

$$\begin{aligned}&[a, [\mathrm{ad}_a^{n-1}x, [\mathrm{ad}_a^{n-1}x, y]]] = [b, [\mathrm{ad}_b^{n-1}x, [\mathrm{ad}_b^{n-1}x, y]]]\\ &= -2\binom{n-1}{s-1}\binom{n-1}{s-1}[b, b^{s-1}xb^{s-1}yb^{s-1}xb^{s-1}] = 0.\end{aligned}$$

If R is semiprime, R is a subdirect product of prime associative algebras (without $\binom{n}{s}$ and s-torsion) and in any of these prime quotients

$$\overline{(\mathrm{ad}_a^{n-1}x)(\mathrm{ad}_a^{n-1}x)} = \overline{0} \text{ and } \overline{[a, [\mathrm{ad}_a^{n-1}x, [\mathrm{ad}_a^{n-1}x, y]]]} = \overline{0},$$

which imply that

$$(\mathrm{ad}_a^{n-1}x)(\mathrm{ad}_a^{n-1}x) = 0, \text{ and } [a, [\mathrm{ad}_a^{n-1}x, [\mathrm{ad}_a^{n-1}x, y]]] = 0$$

for every $x, y \in R$. For every $x \in R$, let $z_x := \mathrm{ad}_a^{n-1}x$. By the identity above,

$$0 = \frac{1}{2}[a, [z_x, [z_x, y]]] = -az_xyz_x + z_xyz_xa.$$

Therefore, since $\mathrm{Id}_R(z_xa) \subset \mathrm{Id}_R(z_x)$, by Corollary 2.0.3 there exists $\lambda_x \in C(R)$ such that $z_xa = \lambda_xz_x$ and by Proposition 2.0.4 there exists $\epsilon_x \in C(R)$ such that $\epsilon_xz_x = z_x$ and $\mathrm{Ann}_R(\mathrm{Id}_R(z_x)) = (1 - \epsilon_x)R$. Therefore

$$0 = z_x \mathrm{ad}_a^n y = z_x \left(\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} a^i y a^{n-i} \right) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} z_x a^i y a^{n-i}$$
$$= \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} z_x \lambda_x^i y a^{n-i} = z_x y \left(\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \lambda_x^i a^{n-i} \right) = z_x y (a - \lambda_x)^n$$

for every $y \in R$, whence $(a - \lambda_x)^n \in \mathrm{Ann}_R(\mathrm{Id}_R(z_x))$. So $\epsilon_x (a - \lambda_x)^n = 0$. Now, for every $x, x' \in R$ there exist $\lambda_x, \lambda_{x'} \in C(R)$ and idempotents $\epsilon_x, \epsilon_{x'} \in C(R)$ such that $0 = (\epsilon_x \epsilon_{x'} a - \epsilon_x \epsilon_{x'} \lambda_x)^n = (\epsilon_x \epsilon_{x'} a - \epsilon_x \epsilon_{x'} \lambda_{x'})^n$, so $\epsilon_x \epsilon_{x'} \lambda_x = \epsilon_x \epsilon_{x'} \lambda_{x'}$ by Lemma 1.3.3. By Lemma 2.0.5 there exists $\lambda \in C(R)$ such that $\epsilon_x \lambda = \epsilon_x \lambda_x$ for every $x \in R$. Then for every $x \in R$ we have $z_x (a - \lambda)^n = \epsilon_x z_x (a - \lambda_x)^n = 0$, so $0 = \epsilon_x z_x \mathrm{ad}_a^n y = z_x y (a - \lambda)^n$ for every $y \in R$ thus $(a - \lambda)^n \in \mathrm{Ann}_R(\mathrm{Id}_R(z_x))$ (see 1.2.1). Moreover $\bigcap_{x \in R} \mathrm{Ann}_R(\mathrm{Id}_R(z_x)) = \mathrm{Ann}_R(\mathrm{Id}_R(\mathrm{ad}_a^{n-1}(R)))$ by definition of z_x, and $\mathrm{Ann}_R(\mathrm{Id}_R(\mathrm{ad}_a^{n-1}(R))) = \mathrm{Ann}_R(\mathrm{Id}_R(a))$ because a is pure (Lemma 2.1.2(iii)). Finally, let $\epsilon \in C(R)$ be such that $\epsilon a = a$ and $\mathrm{Ann}_R(\mathrm{Id}_R(a)) = (1 - \epsilon)R$. Then $\epsilon (a - \lambda)^n = (a - \epsilon \lambda)^n = 0$ because it is contained in $(1 - \epsilon)R$.

Hence $a - \epsilon\lambda$ is nilpotent in addition to being ad-nilpotent of index n. Put $s := [\frac{n+1}{2}]$ and take any prime quotient without s and $\binom{n}{s}$-torsion in which $\overline{a - \epsilon\lambda}$ is still ad-nilpotent of index n. By Proposition 2.2.3(2) we get that n must be odd and $\overline{a - \epsilon\lambda}$ is nilpotent of index s. Since in any prime quotient $(\overline{a - \epsilon\lambda})^s = \bar{0}$ by Proposition 2.2.3(2), we have that s is the index of nilpotence of $a - \epsilon\lambda$. □

Lee's description of ad-nilpotent elements of R^- is recovered when the hypothesis of being pure is removed.

Corollary 2.2.5. ([54, Theorem 1.3]) *Let R be a semiprime associative algebra, let $\hat{R}$ be its central closure, let $a \in R$ be an ad-nilpotent element of R^- of index n, and suppose that R is free of $n!$-torsion. Then n is odd and there exists $\lambda \in C(R)$ such that $a - \lambda \in \hat{R}$ is nilpotent of index $\frac{n+1}{2}$.*

Proof. Suppose without loss of generality that R is centrally closed, i.e., $R = \hat{R}$.

By Proposition 2.1.4 there exists a family of orthogonal idempotents $\{\epsilon_i\}_{i=1}^k \subset C(R)$ such that $a = \sum_{i=1}^k \epsilon_i a$ with $\epsilon_i a$ a pure ad-nilpotent element of index n_i ($n =$

$n_1 > n_2 > \cdots$) of $R\epsilon_i$. Then by Theorem 2.2.4 there exists $\lambda_i \subset C(R\epsilon_i) \subset C(R)$ such that $(\epsilon_i a - \lambda_i)^{s_i} = 0$ for $s_i := [\frac{n_i+1}{2}]$ and for all $i = 1, ..., k$. Hence $\lambda = \sum_{i=1}^n \epsilon_i \lambda_i$ satisfies the claim. □

Interesting Lie algebras associated to simple associative algebras R are the quotient algebras $[R,R]/([R,R] \cap Z(R))$, which are simple unless R has 2-torsion and is 4-dimensional over its center ([44, Theorem 1.13]). Let us study ad-nilpotent elements in these associative algebras.

Lemma 2.2.6. ([23, Lemma 4.6]) *Let R be a semiprime associative algebra and let $a \in R$ be such that $\mathrm{ad}_a^n(R) \subset Z(R)$. Then $\mathrm{ad}_a^n(R) = 0$.*

Proof. For every $x \in R$ we have

$$0 = [\mathrm{ad}_a^n(xa), x] = [(\mathrm{ad}_a^n x)a, x] = (\mathrm{ad}_a^n x)[a, x].$$

Therefore $0 = \mathrm{ad}_a^{n-1}((\mathrm{ad}_a^n x)[a,x]) = (\mathrm{ad}_a^n x)^2$ which implies, since R is semiprime and $\mathrm{ad}_a^n x \in Z(R)$, that $\mathrm{ad}_a^n x = 0$. □

Lemma 2.2.7. *Let R be a semiprime associative algebra, let $L := [R,R]/([R,R] \cap Z(R))$ and let $\overline{a} := a + ([R,R] \cap Z(R)) \in L$ be an ad-nilpotent element of L of index n. Then a is an ad-nilpotent element of index n in R^-.*

Proof. For every $x \in R$, $\mathrm{ad}_a^{n+1} x = \mathrm{ad}_a^n([a,x]) \in \mathrm{ad}_a^n([R,R]) \subset Z(R)$ so, by Lemma 2.2.6, $\mathrm{ad}_a^{n+1} x = 0$ for every $x \in R$, i.e., a is ad-nilpotent in R^- of index n or $n+1$.

Let us suppose that R is prime. Then, by Proposition 2.2.3, there exists $\mu \in \overline{\mathbb{F}}$, the algebraic closure of $\mathbb{F} := C(R)$, such that $a - \mu$ is nilpotent in $R \otimes \overline{\mathbb{F}}$ of some index s. Moreover, by Lemma 2.2.2, $s \leq n+1$. Put $b := a - \mu$. Then

$$0 = \mathrm{ad}_a^n([x,y]) = \mathrm{ad}_b^n([x,y]) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} b^i [x,y] b^{n-i}$$

for every $x, y \in R$. By Lemma 2.2.1, for every $k \in \{0, 1, \ldots, [\frac{n+1}{2}]\}$ we have $\binom{n}{k} b^{\max(k,n-k)} = 0$, so

$$\mathrm{ad}_a^n x = \mathrm{ad}_b^n x = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} b^i x b^{n-i} = 0,$$

i.e., a is an ad-nilpotent element of R^- of index n.

Finally, since $\overline{a}$ is ad-nilpotent of index not greater than n in any prime quotient, a is an ad-nilpotent element of R^- of index n when R is semiprime. $\square$

In particular, from these last two lemmas we get that if R is semiprime then $[R,R]/([R,R]\cap Z(R))$ and $R/Z(R)$ are nondegenerate Lie algebras (see [44, Sublemma in p. 5]).

Corollary 2.2.8. *Let R be a semiprime associative algebra, let $\hat{R}$ be its central closure, and let $L := [R,R]/([R,R]\cap Z(R))$ or $L := R/Z(R)$. If $\overline{a} \in L$ is an ad-nilpotent element of L of index n and R is free of $n!$-torsion, then n is odd and there exists $\lambda \in C(R)$ such that $a - \lambda \in \hat{R}$ is nilpotent of index $\frac{n+1}{2}$.*

Proof. If $L = [R,R]/([R,R]\cap Z(R))$ the result follows by Lemma 2.2.7 and Corollary 2.2.5. If $L = R/Z(R)$ the result follows by Lemma 2.2.6 and Corollary 2.2.5. $\square$

2.3 Ad-nilpotent elements of K

In this section we focus on semiprime algebras R with involution $*$ and their set of skew-symmetric elements K. As in the previous section, we will first describe the pure ad-nilpotent elements of K, and then remove the hypothesis of being pure by decomposing each ad-nilpotent element into a sum of pure ad-nilpotent elements of decreasing indices.

The following lemma collects some results about $*$-identities. Item (1) is [44, Remark on p. 43] (with a different proof), item (2) is a generalization of [56, Lemma 5], and item (3) is a generalization of [13, Lemma 5.2].

Lemma 2.3.1. *Let R be a semiprime associative algebra with involution $*$. Let $k \in K$ and $h \in H$. Then:*

1. *$kKk = 0$ implies $k = 0$.*

2. *$hKh = 0$ implies $hRh \subset H(C(R),*)h$. In particular, R satisfies*

$$hxhyh = hyhxh \quad \textit{for every } x, y \in R,$$

and if $\mathrm{Id}_R(h)$ *is essential then* $\mathrm{Skew}(C(R),*) = 0$.

3. $hKh = 0$ *and* $hKk = 0$ *imply* $hRk = 0$. *In particular, if* $\mathrm{Id}_R(h)$ *is essential then* $k = 0$, *while if* $h \in \mathrm{Id}_R(k)$ *then* $h = 0$ *(resp. if* $k \in \mathrm{Id}_R(h)$ *then* $k = 0$*).*

4. $k[K,K]k = 0$ *and* $k^2 = 0$ *imply* $k = 0$.

Proof. We can suppose without loss of generality that $R = \hat{R}$, i.e., R is centrally closed.

(1) Take $x \in R$. Note that $k(x - x^*)k = 0$, so that $kxk = kx^*k$. Then

$$\begin{aligned} k(xkx)k = k(xkx)^*k = -kx^*kx^*k = -(kx^*k)x^*k = -kxkx^*k \\ = -kx(kx^*k) = -kxkxk \end{aligned}$$

and so we have $kxkxk = 0$ since R is free of 2-torsion. Therefore $kxkxkyk = 0$ for every $y \in R$, hence

$$0 = -kxk(xky)k = -kxk(xky)^*k = kxky^*kx^*k = kxkykxk,$$

so $(kxk)R(kxk) = 0$ and $kxk = 0$ since R is semiprime. Now $kRk = 0$ implies, again by semiprimeness, that $k = 0$.

(2) If $h = 0$ then the claim is trivially fulfilled, so assume $h \neq 0$. Take $x, y \in R$. Note that $h(x - x^*)h = 0$ and therefore $hxh = hx^*h$. Then

$$0 = h(xhy - (xhy)^*)h = hxhyh - hy^*hx^*h = hxhyh - (hy^*h)x^*h =$$

$$= hxhyh - hy(hx^*h) = hxhyh - hyhxh = (hxh)yh - hy(hxh),$$

i.e., $hxhyh = hyhxh$. By Corollary 2.0.3, since $h \neq 0$ and $\mathrm{Id}_R(hxh) \subseteq \mathrm{Id}_R(h)$, for each $x \in R$ there exists $\mu_x \in C(R)$ such that $hxh = \mu_x h$. Hence $0 \neq hRh \subset C(R)h$. Moreover, since $hx^*h = hxh$, $2hxh = hxh + hx^*h = (\mu_x + \mu_x^*)h \in H(C(R),*)h$, so $hRh \subseteq H(C(R),*)h$.

Let us suppose that $\mathrm{Id}_R(h)$ is essential in R and let us show that $\mathrm{Skew}(C(R),*) =$

0: Take $\lambda \in \mathrm{Skew}(C(R),*)$ and $y \in R$. Then $(\lambda h)y(\lambda h) = \lambda h(y\lambda)h = \lambda\mu_{\lambda y}h \in K$ for some $\mu_{\lambda y} \in H(C(R),*)$. On the other hand $(\lambda h)y(\lambda h) = \lambda^2 hyh = \lambda^2\mu_y h \in H$ for some $\mu_y \in H(C(R),*)$. Therefore $(\lambda h)y(\lambda h) = 0$ for every $y \in R$, and by semiprimeness of R, $\lambda h = 0$, so $\lambda = 0$ because $\mathrm{Id}_R(h)$ is essential.

(3) Suppose first that R is $*$-prime and, without loss of generality, that it is centrally closed. If R is not prime then there is $\lambda \in \mathrm{Skew}(C(R),*)$ such that $R = K + \lambda K$ (see 1.3.2), hence $hKh = 0$ implies $hRh = 0$ and $h = 0$ since R is semiprime, so trivially $hRk = 0$. Now assume R is prime. Since $R = H + K$ we only need to show that $hHk = 0$. Let $x \in H$ and $y \in R$. Then

$$0 = h(xky - (xky)^*)h = hxkyh + hy^*kxh = hxkyh + hykxh$$

since $h(y^* - y)k = 0$ for every $y \in R$. By Corollary 2.0.3, since $\mathrm{Id}_R(hxk) \subset \mathrm{Id}_R(h)$, for each $x \in R$ there exists $\mu_x \in C(R)$ such that $hxk = \mu_x h$. If $\mu_x = 0$ then $hxk = 0$ and we are done. Otherwise, $0 = hxkxk = \mu_x hxk = \mu_x^2 h$, hence $h = 0$ and we are also done.

Suppose now that R is semiprime. Then there exists a family of $*$-prime ideals $\{I_\alpha\}_{\alpha\in\Delta}$ such that $\bigcap_{\alpha\in\Delta} I_\alpha = 0$. In each $*$-prime quotient R/I_α we have $\bar{h}R/I_\alpha\bar{k} = \bar{0}$, so $hRk \subset I_\alpha$ for all α, hence $hRk = 0$.

(4) Since $k^2 = 0$ and $k[K,K]k = 0$, for all $x, y \in K$ we get

$$0 = k[[x,k],y]k = kxkyk + kykxk, \qquad \textbf{(a)}$$

thus $kxkyk = -kykxk$ and $2kxkxk = 0$ for all $x \in K$, hence $kxkxk = 0$ since R is free of 2-torsion. Now, by **(a)**,

$$0 = (kxkxk)yk = kx(kxkyk) = -kxkykxk$$

for all $x, y \in K$. Thus $(kxk)K(kxk) = 0$ for all $x \in K$, $kKk = 0$ and $k = 0$ by item (1) applied twice. □

Remark 2.3.2. Let R be a semiprime associative algebra with involution. If $a \in K$ is an ad-nilpotent element of K of index n, then for every $x = x_h + x_k \in R$ with $x_h \in H$ and $x_k \in K$:

$$\begin{aligned}\mathrm{ad}_a^n(ax + xa) &= \mathrm{ad}_a^n(ax_k + x_k a) + \mathrm{ad}_a^n(ax_h + x_h a) \\ &= a\mathrm{ad}_a^n(x_k) + \mathrm{ad}_a^n(x_k)a + \mathrm{ad}_a^n(ax_h + x_h a) = 0,\end{aligned}$$

since $ax_h + x_h a \in K$. On the other hand, expanding this expression,

$$0 = \mathrm{ad}_a^n(ax + xa) = (-1)^n x a^{n+1} + \sum_{i=1}^{n} \left(\binom{n}{i} - \binom{n}{i-1} \right) (-1)^{n-i} a^i x a^{n+1-i} + a^{n+1} x.$$

Observe that a nilpotent element in K is ad-nilpotent of both K and R, but its index of ad-nilpotence in R may be higher than the one found in K.

In the following proposition we describe the ad-nilpotent elements of K of index n that are already nilpotent of certain index s. The description depends on the equivalence class of the index of ad-nilpotence modulo 4 and relates the index of nilpotence to the index of ad-nilpotence.

Proposition 2.3.3. *Let R be a semiprime associative algebra with involution $*$, let $\hat{R}$ be its central closure, and let $a \in K$ be a nilpotent element of index of nilpotence t. Then a is ad-nilpotent in R. If the index of ad-nilpotence of a in K is n and R is free of $\binom{n}{s}$-torsion for $s := [\frac{n+1}{2}]$, then:*

1. *If $n \equiv_4 0$ then $t = s + 1$ and $a^s K a^s = 0$.*

2. *If $n \equiv_4 1$ then $t = s$ and the index of ad-nilpotence of a in R is also n.*

3. *The case $n \equiv_4 2$ is not possible.*

4. *If $n \equiv_4 3$ then there exists an idempotent $\epsilon \in C(R)$ such that $\epsilon a^s = a^s$. Moreover, when we write $a = \epsilon a + (1 - \epsilon)a$, we have:*

   (4.1) If $0 \neq \epsilon a \in \hat{R}$ then ϵa is nilpotent of index $s + 1$, $\epsilon a^s = a^s$ generates an essential ideal in $\epsilon\hat{R}$ and $(\epsilon a)^{s-1} k (\epsilon a)^s = (\epsilon a)^s k (\epsilon a)^{s-1}$ for every $k \in$ $\mathrm{Skew}(\hat{R}, *)$.

(4.2) If $0 \neq (1-\epsilon)a \in \hat{R}$, then the index of ad-nilpotence of $(1-\epsilon)a$ in $\hat{R}$ is not greater than n, and $(1-\epsilon)a^s = 0$.

Furthermore, if a is a pure ad-nilpotent element of K then in (2) and in (4.2) we obtain pure ad-nilpotent elements of R (respectively of $\hat{R}$) of index n.

Proof. Let us suppose without loss of generality that $R = \hat{R}$, i.e., R is centrally closed.

Let $a \in K$ be a nilpotent element of index of nilpotence t. Then a is ad-nilpotent of K of a certain index n. If we apply Lemma 2.2.1 to the second formula obtained in Remark 2.3.2 we get that all the monomials appearing in it are zero. We will now focus on certain monomials depending on the parity of n.

• If n is even, $n = 2s$. Let us see that $t = s+1$: on the one hand, for any $x \in R$ we know that

$$\left(\binom{n}{s} - \binom{n}{s-1}\right)(-1)^s a^s x a^{s+1} = 0$$

and, since $\binom{n}{s} - \binom{n}{s-1}$ is a divisor of $2\binom{n}{s}$ and R is free of $2\binom{n}{s}$-torsion, we have that $a^s x a^{s+1} = 0$ for all x. Therefore $a^{s+1} = 0$ by semiprimeness, hence $t \leq s+1$. On the other hand, if $t = s$ then $a^s = 0$ and $\mathrm{ad}_a^{2s-1}(R) = 0$, a contradiction.

Let us see that $n \equiv_4 0$: For any $k \in K$,

$$0 = \mathrm{ad}_a^{2s}(k) = \sum_{i=1}^{2s} \binom{2s}{i}(-1)^{2s-i} a^i k a^{2s-i} = \binom{2s}{s}(-1)^s a^s k a^s,$$

so $a^s k a^s = 0$ for every $k \in K$, which implies that s has to be even, since otherwise $a^s \in K$ and $a^s K a^s = 0$ imply $a^s = 0$ by Lemma 2.3.1(1), a contradiction. We have shown that, if n is even, $n \equiv_4 2$ is not possible.

• If n is odd, $n = 2s-1$, and for any $x \in R$,

$$\left(\binom{n}{s-1} - \binom{n}{s-2}\right) a^{s-1} x a^{s+1} = 0.$$

Since $\binom{n}{s-1} - \binom{n}{s-2}$ is a divisor of $2\binom{n}{s}$ and R is free of $2\binom{n}{s}$-torsion, we have that $a^{s-1} x a^{s+1} = 0$ for all x. Therefore $a^{s+1} = 0$ by semiprimeness, hence $t \leq s+1$. On the other hand $t > s-1$ since otherwise $\mathrm{ad}_a^{2s-2}(R) = 0$, a contradiction.

If $a^s = 0$ then a is already an ad-nilpotent element of R of index n. In this case $n \equiv_4 1$ or $n \equiv_4 3$ by Proposition 2.2.3(2). Furthermore, if a is pure in K then a is pure in R by Lemma 2.1.3.

Suppose from now on that $a^s \neq 0$. Let us show that $n \equiv_4 3$. By Proposition 2.0.4 there exists an idempotent $\epsilon \in H(C(R), *)$ such that $\epsilon a^s = a^s$ and $\mathrm{Ann}_R(\mathrm{Id}_R(a^s)) = (1-\epsilon)R$ (so $a^s = \epsilon a^s$ generates an essential ideal in ϵR). Notice that $\epsilon a \neq 0$ (otherwise $0 = (\epsilon a)^s = \epsilon a^s = a^s$, a contradiction). For every $k \in K$ we have

$$
\begin{aligned}
0 = \mathrm{ad}_{\epsilon a}^n k &= \sum_{i=1}^{n} \binom{n}{i} (-1)^{n-i} \epsilon a^i k a^{n-i} = \\
&= \binom{n}{s-1} (-1)^s \epsilon a^{s-1} k a^s + \binom{n}{s} (-1)^{s-1} \epsilon a^s k a^{s-1} = \\
&= \binom{n}{s} (-1)^{s-1} (-\epsilon a^{s-1} k a^s + \epsilon a^s k a^{s-1}).
\end{aligned}
$$

Since R has no $\binom{n}{s}$-torsion, $\epsilon a^{s-1} k a^s = \epsilon a^s k a^{s-1}$ for every $k \in K$. Moreover, multiplying by a on the right we get $\epsilon a^s k a^s = a^s k a^s = 0$, so $a^s K a^s = 0$, which by Lemma 2.3.1(1) is only possible if $a^s \neq 0$ is symmetric, hence s is even and $n \equiv_4 3$.

If $(1-\epsilon)a \neq 0$ then $\mathrm{ad}_{(1-\epsilon)a}^{2s-1}(R) = 0$ and $(1-\epsilon)a$ is an ad-nilpotent element of R of index not greater than $2s-1$.

If a is a pure ad-nilpotent element of index n in K then $(1-\epsilon)a$ is ad-nilpotent of K of index n and therefore $(1-\epsilon)a^{s-1} \neq 0$. From this the index of ad-nilpotence of $(1-\epsilon)a$ in R must be $n = 2s-1$. Then by Lemma 2.1.3 $(1-\epsilon)a$ is a pure ad-nilpotent element of R of index n. □

Remark 2.3.4. Let $a \in K$ be a nilpotent element of index t. If we denote its index of ad-nilpotence in K by n, we obtain from Proposition 2.3.3 that, under the right torsion hypothesis, $2t-3 \leq n \leq 2t-1$ and $\frac{n+1}{2} \leq t \leq \frac{n+3}{2}$.

The next two results can be joint in one, but in order to clarify our proof we have decided split them in two. Firstly, in the next proposition we prove that a pure ad-nilpotent element of K can be descomposed into two parts, where one part is ad-nilpotent of R and the other part is nilpotent. After that, in Theorem 2.3.6, we apply Proposition 2.3.3 to obtain the classification of a pure ad-nilpotent element of K depending on its index of ad-nilpotence modulo 4.

Proposition 2.3.5. *Let R be a semiprime associative algebra with involution $*$, let $\hat{R}$ be its central closure, and let $a \in K$ be a pure ad-nilpotent element of K of index $n > 1$. Then:*

(1) There exists an idempotent $\epsilon \in H(C(R), *)$ such that $(1-\epsilon)a$ is an ad-nilpotent element of $\hat{R}$ of index $\leq n$ and ϵa is nilpotent with $\mathrm{ad}^n_{\mu\epsilon a}(\hat{R}) \neq 0$ for every $\mu \in C(R)$ such that $\mu\epsilon a \neq 0$.

(2) Moreover, if a is pure ad-nilpotent in K and R is free of $\binom{n}{s}$-torsion and t-torsion for $s := [\frac{n+1}{2}]$, when we write $a = \epsilon a + (1-\epsilon)a$ we have:

(2.1) If $\epsilon a \neq 0$ then ϵa is nilpotent of index $s+1$.

(2.2) If $(1-\epsilon)a \neq 0$ then $(1-\epsilon)a$ is pure ad-nilpotent in $\hat{R}$ of index n. In this case n is odd and there exists $\lambda \in \mathrm{Skew}(C(R), *)$ such that $((1-\epsilon)a - \lambda)^s = 0$.

Proof. Notice that $n \geq 3$ since $\mathrm{ad}^2_a(K) = 0$ implies $a \in Z(R)$ by [27, Corollary 4.8] and so $\mathrm{ad}_a(K) = 0$, which is not possible because $n > 1$ by hypothesis.

(1) Let us suppose first that R is a $*$-prime associative algebra and, without loss of generality, that it is centrally closed.

(1.a) Case 1: $\mathrm{ad}^n_a(R) = 0$ and we get the claim for the idempotent $\epsilon = 0$.

(1.b) Case 2: $\mathrm{ad}^n_a(R) \neq 0$ implies that there are no nonzero skew elements λ in $C(R)$, since otherwise (by 1.3.2) $R = K + \lambda K$ would imply $\mathrm{ad}^n_a(R) = 0$; in particular R is prime. Since $\mathrm{ad}^n_a(K) = 0$, by the second formula of Remark 2.3.2 and Corollary 2.0.3, a is an algebraic element of R over the field $\mathbb{F} := C(R)$. Let us consider the minimal polynomial $p(X) \in \mathbb{F}[X]$ of a. Let $\overline{\mathbb{F}}$ be the algebraic closure of $C(R)$ and let $\mu_1, \ldots, \mu_t \in \overline{\mathbb{F}}$ such that $p(X) = (X - \mu_1)^{k_1} \cdots (X - \mu_s)^{k_s}$. Let $q_1(X) :=$

$p(X)/(X-\mu_1)$, so $q_1(a)a = \mu_1 q_1(a)$. Now, for any $x \in R \otimes \overline{\mathbb{F}}$,

$$\begin{aligned}
0 &= \mathrm{ad}_a^n(ax+xa)q_1(a) \\
&= a\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}a^i x a^{n-i} q_1(a) + \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}a^i x a^{n-i} a q_1(a) \\
&= a\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}a^i x \mu_1^{n-i} q_1(a) + \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}a^i x \mu_1^{n-i} \mu_1 q_1(a) \\
&= a\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}a^i \mu_1^{n-i} x q_1(a) + \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}a^i \mu_1^{n-i} \mu_1 x q_1(a) \\
&= a(a-\mu_1)^n x q_1(a) + (a-\mu_1)^n \mu_1 x q_1(a) = (a-\mu_1)^n (a+\mu_1) x q_1(a)
\end{aligned}$$

and therefore, since $R \otimes \overline{\mathbb{F}}$ is a centrally closed prime algebra (see [7, pp. 445–446]), $(a-\mu_1)^n(a+\mu_1) = 0$. If $\mu_1 = 0$ then a is nilpotent of index at most $n+1$. If $\mu_1 \neq 0$, since the involution is the identity over $C(R)$ because $\mathrm{Skew}(C(R),*) = 0$, it extends to $R \otimes \overline{\mathbb{F}}$ via $(r \otimes \lambda)^* := r^* \otimes \lambda$, hence $0 = ((a-\mu_1)^n)^*(a+\mu_1)^* = (a^*-\mu_1)^n(a^*+\mu_1) = (-a-\mu_1)^n(-a+\mu_1)$ implies $(a+\mu_1)^n(a-\mu_1) = 0$. From the conditions $(a-\mu_1)^n(a+\mu_1) = 0$ and $(a+\mu_1)^n(a-\mu_1) = 0$ we obtain $p(X) = (X-\mu_1)(X+\mu_1)$. Thus $a^2 = \mu_1^2$, but then $\mathrm{ad}_a^3(k) = 4\mu_1^2[a,k]$ for every $k \in K$, a contradiction with $n \geq 3$.

Let us study the semiprime case, and suppose without loss of generality that R is centrally closed: If a is already ad-nilpotent in R of index n, take $\epsilon = 0$ and the claim holds. Suppose from now on that $\mathrm{ad}_a^n(R) \neq 0$. By Proposition 2.0.4 let $\epsilon \in H(C(R),*)$ be an idempotent such that $\epsilon \mathrm{ad}_a^n(x) = \mathrm{ad}_a^n(x)$ for every $x \in R$, $\mathrm{Ann}_R(\mathrm{Id}_R(\mathrm{ad}_a^n(R))) = (1-\epsilon)R$ and $\mathrm{Ann}_{C(R)}(\mathrm{ad}_a^n(R)) = (1-\epsilon)C(R)$. Then $\mathrm{ad}_{(1-\epsilon)a}^n(R) = (1-\epsilon)\mathrm{ad}_a^n(R) = 0$.
Let us study the element ϵa: First notice that $\mathrm{ad}_{\mu\epsilon a}^n R \neq 0$ for every μ such that $\mu\epsilon a \neq 0$, since otherwise $\mu\epsilon\mathrm{ad}_a^n(R) = \mathrm{ad}_{\mu\epsilon a}^n R = 0$ implies $\mu\epsilon \in \mathrm{Ann}_{C(R)}(\mathrm{ad}_a^n(R)) = (1-\epsilon)C(R)$ and hence $\mu\epsilon = 0$, a contradiction. Let us see that ϵa is nilpotent. Since R is semiprime, the intersection of all $*$-prime ideals of R is zero. Consider the essential $*$-ideal $S := \mathrm{Id}_R(\mathrm{ad}_a^n(R)) \oplus \mathrm{Ann}_R(\mathrm{Id}_R(\mathrm{ad}_a^n(R))) = \mathrm{Id}_R(\mathrm{ad}_a^n(R)) \oplus (1-\epsilon)R$. Let us

consider the families

$$\Delta_1 := \{I \triangleleft^* R \mid R/I \text{ is } *\text{-prime and } S \not\subset I\}$$

and

$$\Delta_2 := \{I \triangleleft^* R \mid R/I \text{ is } *\text{-prime and } S \subset I\}.$$

Since $S \subset \bigcap_{I \in \Delta_2} I$ and S is essential, $\bigcap_{I\in\Delta_1} I = 0$ and R is a subdirect product of R/I with $I \in \Delta_1$. Let us see that in any $*$-prime quotient ϵa is nilpotent of index not greater than $n+1$. Take $I \in \Delta_1$ and consider $\bar{R} := R/I$. We may have two cases:

- If $\overline{\epsilon} = \overline{0}$ then $\overline{\epsilon a} = \overline{0}$.
- If $\overline{\epsilon} \neq \overline{0}$ then $\overline{\epsilon} = \overline{1} \in R/I$ and $\overline{1-\epsilon} = \overline{0}$, so $(1-\epsilon)R \subset I$. Moreover, $\mathrm{ad}^n_{\overline{\epsilon a}}(R/I) \neq \overline{0}$ since otherwise $\mathrm{ad}^n_{\overline{\epsilon a}}(R/I) = \overline{0}$ would imply $S \subset I$, a contradiction. Let us see that R/I is prime: if R/I is $*$-prime and not prime there would exist a nonzero skew element λ in $C(R/I)$, which implies that $R/I = \mathrm{Skew}(R/I,*) \oplus \lambda\mathrm{Skew}(R/I,*)$ (see 1.3.2), so $\mathrm{ad}^n_{\overline{\epsilon a}}(R/I) = \mathrm{ad}^n_{\overline{\epsilon a}}(\mathrm{Skew}(R/I,*) \oplus \lambda\mathrm{Skew}(R/I,*)) = 0$, a contradiction. So R/I is a prime algebra with involution and $\mathrm{ad}^n_{\overline{\epsilon a}}(R/I)) \neq \overline{0}$ which implies, by the case (1.b), that $\overline{\epsilon a}$ is nilpotent of index not greater than $n+1$.

In conclusion, for any $I \in \Delta_1$ we have $\epsilon a^{n+1} \in I$ and therefore $\epsilon a^{n+1} = 0$.

(2) Suppose now that a is a pure element of K of index n and R is free of $2\binom{n}{s}$-torsion and free of s-torsion for $s := [\frac{n+1}{2}]$. If a is already ad-nilpotent of R of index n then a is pure in R by Lemma 2.1.3 and we can use Theorem 2.2.4 to find that n is odd and there exists $\lambda \in \mathrm{Skew}(C(R),*)$ such that $(a-\lambda)^s = 0$. Otherwise write $a = \epsilon a + (1-\epsilon)a$ as before. Since ϵa is nilpotent and ad-nilpotent of K of index n (because we are assuming that a is pure in K), ϵa is nilpotent of index $s+1$ (it has index s or $s+1$ by Proposition 2.3.3, but $\mathrm{ad}^n_{\epsilon a}(R) \neq 0$). Moreover, $(1-\epsilon)a$ is a pure ad-nilpotent element of R of index n (if it is nonzero, its index of ad-nilpotence cannot be lower than n since $(1-\epsilon)a$ is ad-nilpotent in K of index n), and we can apply Theorem 2.2.4 and Lemma 1.3.3 to get that n is odd and there

exists $\lambda \in \mathrm{Skew}(C(R), *)$ such that $((1-\epsilon)a - \lambda)^s = 0$. $\square$

Theorem 2.3.6. *Let R be a semiprime associative algebra with involution $*$, let $\hat{R}$ be its central closure, and let $a \in K$ be a pure ad-nilpotent element of K of index $n > 1$. If R is free of $\binom{n}{s}$-torsion and s-torsion for $s := [\frac{n+1}{2}]$ then:*

1. *If $n \equiv_4 0$ then $a^{s+1} = 0$, $a^s \neq 0$ and $a^s K a^s = 0$. Moreover, there exists an idempotent $\epsilon \in H(C(R), *)$ such that $\epsilon a = a$ and the ideal generated by a^s is essential in $\epsilon \hat{R}$. In addition $\epsilon \hat{R}$ satisfies the GPI $a^s x a^s y a^s = a^s y a^s x a^s$ for every $x, y \in \epsilon \hat{R}$.*

2. *If $n \equiv_4 1$ then there exists $\lambda \in \mathrm{Skew}(C(R), *)$ such that $(a - \lambda)^s = 0$ (a is an ad-nilpotent element of R of index n).*

3. *It is not possible that $n \equiv_4 2$.*

4. *If $n \equiv_4 3$ then there exists an idempotent $\epsilon \in H(C(R), *)$ making $a = \epsilon a + (1-\epsilon)a \in \hat{R}$ such that:*

   (4.1) If $\epsilon a \neq 0$ then $\epsilon a^{s+1} = 0$, $\epsilon a^s \neq 0$ and $\epsilon a^s k \epsilon a^{s-1} = \epsilon a^{s-1} k \epsilon a^s$ for every $k \in \mathrm{Skew}(\hat{R}, *)$. The ideal generated by ϵa^s is essential in $\epsilon \hat{R}$ and $\epsilon \hat{R}$ satisfies the GPI $a^s x a^s y a^s = a^s y a^s x a^s$ for every $x, y \in \epsilon \hat{R}$.

   (4.2) If $(1-\epsilon)a \neq 0$ then there exists $\lambda \in \mathrm{Skew}(C(R), *)$ such that $((1-\epsilon)a - \lambda)^s = 0$ ($(1-\epsilon)a$ is a pure ad-nilpotent element of $\hat{R}$ of index n).

In particular, for all $n > 1$ there exists $\lambda \in \mathrm{Skew}(C(R), *)$ such that $(a-\lambda)^{s+1} = 0$, $(a-\lambda)^{s-1} \neq 0$.

Proof. We can suppose without loss of generality that $R = \hat{R}$, i.e., R is centrally closed. By Proposition 2.3.5 there exists an idempotent $\epsilon \in H(C(R), *)$ such that $\epsilon \mathrm{ad}_a^n x = \mathrm{ad}_a^n x$ for every $x \in R$ and $\mathrm{Ann}_R(\mathrm{Id}_R(\mathrm{ad}_a^n(R))) = (1-\epsilon)R$, and moreover:

- If $\epsilon a \neq 0$, it is nilpotent of index $s+1$ and ad-nilpotent of K of index n. By Proposition 2.3.3 this may happen if either $n \equiv_4 0$, in which case $a^{s+1} = 0$, $a^s \neq 0$, $a^s K a^s = 0$ and $(1-\epsilon)a = 0$ (because $(1-\epsilon)a$ is ad-nilpotent of R and its index cannot be even), or $n \equiv_4 3$. The case $n \equiv_4 1$ is not possible because $\epsilon a^s \neq 0$.

- If $(1-\epsilon)a \neq 0$ then $(1-\epsilon)a$ is a pure ad-nilpotent element of R, n is odd and there exists $\lambda \in \mathrm{Skew}(R,*)$ with $((1-\epsilon)a-\lambda)^s = 0$. By Proposition 2.3.3 this may happen if either $n \equiv_4 1$ (in this case $\epsilon a = 0$) or $n \equiv_4 3$. The decomposition $(1-\epsilon)a - \lambda = a_1 + a_2$ given by Proposition 2.3.3(4) occurs with $a_1 = 0$ since otherwise the index $s+1$ of a_1 would contradict $((1-\epsilon)a-\lambda)^s = 0$.

In the particular case of $n \equiv_4 3$ with $\epsilon a \neq 0$, the idempotent ϵ_1 produced in Proposition 2.3.3(4) for the nilpotent element ϵa satisfies $\epsilon_1 \epsilon a^s = \epsilon a^s$, so $(1-\epsilon_1)\epsilon \in \mathrm{Ann}_R(\mathrm{Id}_R(\mathrm{ad}_a^n(R))) = (1-\epsilon)R$, thus $\epsilon_1\epsilon = \epsilon$ and $\epsilon a^s = \epsilon_1 \epsilon a^s$ generates an essential ideal in ϵR. On the other hand, we know from Proposition 2.3.5 that $(\epsilon a)^{s-1}k(\epsilon a)^s = (\epsilon a)^s k (\epsilon a)^{s-1}$ for every $k \in K$; in particular $(\epsilon a)^s K (\epsilon a)^s = 0$. Therefore, by Lemma 2.3.1(2) the identity

$$a^s x a^s y a^s = a^s y a^s x a^s$$

holds in ϵR.

In the particular case of $n \equiv_4 0$ the idempotent ϵ produced in Proposition 2.3.5 satisfies $\epsilon a^s x a^s = \epsilon a^s$ for every $x \in R$ and $\mathrm{Ann}_R \mathrm{Id}_R(a^s R a^s) = (1-\epsilon)R$. On the other hand, $(1-\epsilon)a$ must be zero because $\mathrm{ad}^n_{(1-\epsilon)a}(R) = 0$ and a is a pure ad-nilpotent element (so $a = \epsilon a$). Therefore, the ideal generated by a^s in ϵR is essential in ϵR and the identity $a^s x a^s y a^s = a^s y a^s x a^s$ holds in ϵR by Lemma 2.3.1(2). $\square$

Remark 2.3.7. It is worth noting that in the semiprime case, when $n \equiv_4 3$ there can exist elements a with two nonzero parts ϵa and $(1-\epsilon)a$ behaving as in Theorem 2.3.6(4.1) and Theorem 2.3.6(4.2). This is no longer true in the prime case, see [56, Main Theorem].

In the next corollary we recover T.K. Lee's main result by taking into account that every ad-nilpotent element can be expressed as a sum of pure ad-nilpotent elements of decreasing indices.

Corollary 2.3.8. ([54, Theorem 1.5]) *Let R be a semiprime associative algebra with involution $*$ and free of $n!$-torsion, let $\hat{R}$ be its central closure, and let $a \in K$ be an ad-nilpotent element of K of index n. Then there exist $\lambda \in \mathrm{Skew}(C(R), *)$ and an idempotent $\epsilon \in H(C(R), *)$ such that $(\epsilon a - \lambda)^{s+1} = 0$ and $(\epsilon a - \lambda)^{s-1} \neq 0$ for $s := [\frac{n+1}{2}]$, and $(1-\epsilon)\hat{R}$ is a PI-algebra satisfying the standard identity S_4.*

Proof. We can suppose without loss of generality that $R = \hat{R}$, i.e., R is centrally closed. By Proposition 2.1.4 there exists a family of orthogonal symmetric idempotents $\{\epsilon_i\}_{i=1}^k$ of the extended centroid such that $a = \sum_{i=1}^k \epsilon_i a$, with $\epsilon_i a$ a pure ad-nilpotent element of index n_i ($n = n_1 > n_2 > \dots$) of $\mathrm{Skew}(\epsilon_i R, *)$. If $n_k = 1$ then $\epsilon_k a$ can be decomposed as $\epsilon_k a = \epsilon_{k1} a + (1-\epsilon_{k1})a$, where $\epsilon_{k1} a \in Z(R)$ and $(1-\epsilon_{k1})R$ is a PI-algebra satisfying the standard identity S_4 by [13, Theorem 4.2(i),(ii) and (*)]. The claim follows now from Theorem 2.3.6. $\square$

Let us extend this last result to Lie algebras of the form $K/(K \cap Z(R))$ and $[K,K]/([K,K] \cap Z(R))$.

Corollary 2.3.9. *Let R be a semiprime associative algebra with involution free of $n!$-torsion, let $\hat{R}$ be its central closure, and consider the Lie algebra $L := K/(K \cap Z(R))$. If $\bar{a}$ is an ad-nilpotent element of L of index n then there exist $\lambda \in \mathrm{Skew}(C(R), *)$ and an idempotent $\epsilon \in H(C(R), *)$ such that $(\epsilon a - \lambda)^{s+1} = 0$ and $(\epsilon a - \lambda)^{s-1} \neq 0$ for $s := [\frac{n+1}{2}]$, and $(1-\epsilon)\hat{R}$ is a PI-algebra that satisfying the standard identity S_4.*

Proof. Let us prove that $\mathrm{ad}_a^n(K) \subset Z(R)$ implies $\mathrm{ad}_a^n(K) = 0$: Suppose first that R is $*$-prime and, without loss of generality, centrally closed. If $\mathrm{ad}_a^n(K) \neq 0$, there would exist $0 \neq \lambda \in \mathrm{ad}_a^n(K) \cap Z(R)$, so $R = K + \lambda K$ by 1.3.2 and hence $\mathrm{ad}_a^n(R) \subset Z(R)$, which implies by Lemma 2.2.6 that $\mathrm{ad}_a^n(R) = 0$, a contradiction. The same result follows for semiprime algebras because they can be expressed as subdirects product of $*$-prime quotients.

The claim follows now from Corollary 2.3.8. $\square$

Now we turn to Lie algebras of the form $[K,K]/([K,K] \cap Z(R))$. We first need a technical lemma.

Lemma 2.3.10. *Let R be a semiprime associative algebra with involution $*$ and $a \in K$ be such that $\mathrm{ad}_a^n([K,K]) \subset Z(R)$, $n > 1$. If R is free of $(n+1)!$-torsion then $\mathrm{ad}_a^n(K) = 0$.*

Proof. Let us first suppose that R is a $*$-prime associative algebra and, without loss of generality, that it is centrally closed. If $\mathrm{Skew}(C(R),*) \neq 0$ then $R = K + \lambda K$ for any $0 \neq \lambda \in \mathrm{Skew}(C(R),*)$ (see 1.3.2); thus $\mathrm{ad}_a^n([R,R]) \subset Z(R)$, and by Lemma 2.2.7 a is an ad-nilpotent element of R of index n. Otherwise $\mathrm{Skew}(C(R),*) = 0$, in which case R must be prime and $K \cap Z(R) = 0$, so $\mathrm{ad}_a^n([K,K]) = 0$. From $\mathrm{ad}_a^{n+1}K \subset \mathrm{ad}_a^n([K,K]) = 0$ and $\mathrm{Skew}(C(R),*) = 0$ we get from Proposition 2.3.5 that a is a nilpotent element of R. Let t be its index of nilpotence. If $\mathrm{ad}_a^n K = 0$ we are done; suppose it is not and let us compare the index of ad-nilpotence of a in K with its index of nilpotence t (see Proposition 2.3.3) to get a contradiction:

(a) If $n+1 \equiv_4 0$ then $t = \frac{n+3}{2}$ and $a^{t-1}Ka^{t-1} = 0$. From $\binom{n}{t-2} = \binom{n}{t-1}$ we get, for every $x \in R$, that $\mathrm{ad}_a^n x = (-1)^{t-1}\binom{n}{t-2}(a^{t-2}xa^{t-1} - a^{t-1}xa^{t-2})$. Then, for every $k, k' \in K$,

$$
\begin{aligned}
&2(\mathrm{ad}_a^n k)k'(\mathrm{ad}_a^n k) = \\
&= 2\binom{n}{t-2}\binom{n}{t-2}\left(a^{t-2}ka^{t-1}k'a^{t-2}ka^{t-1} + a^{t-1}ka^{t-2}k'a^{t-1}ka^{t-2}\right) \\
&= 2\binom{n}{t-2}\binom{n}{t-2}a^{t-2}k(a^{t-1}k'a^{t-2} - a^{t-2}k'a^{t-1})ka^{t-1} + \\
&+ 2\binom{n}{t-2}\binom{n}{t-2}a^{t-1}k(a^{t-2}k'a^{t-1} - a^{t-1}k'a^{t-2})ka^{t-2} = \\
&= 2(-1)^{t-2}\binom{n}{t-2}(a^{t-2}k(\mathrm{ad}_a^n k')ka^{t-1} - a^{t-1}k(\mathrm{ad}_a^n k')ka^{t-2}) = \\
&= (-1)^{t-2}\binom{n}{t-2}(a^{t-2}\mathrm{ad}_k^2(\mathrm{ad}_a^n k')a^{t-1} - a^{t-1}\mathrm{ad}_k^2(\mathrm{ad}_a^n k')a^{t-2}) = \\
&= \mathrm{ad}_a^n(\mathrm{ad}_k^2(\mathrm{ad}_a^n k')) \in \mathrm{ad}_a^n([K,K]) = 0
\end{aligned}
$$

because $a\mathrm{ad}_a^n k = 0 = (\mathrm{ad}_a^n k)a$, $a^{t-1}Ka^{t-1} = 0$ and $t \geq 3$ implies $a^{t-1}a^{t-2} = 0$. Therefore $(\mathrm{ad}_a^n k)K(\mathrm{ad}_a^n k) = 0$ and hence $\mathrm{ad}_a^n k = 0$ for every $k \in K$ by Lemma 2.3.1(1).

(b) If $n+1 \equiv_4 1$ then $t = \frac{n}{2}+1$. For every $x \in R$, $\mathrm{ad}_a^n x = (-1)^{t-1}\binom{n}{t-1} a^{t-1} x a^{t-1}$. Then, for every $k, k' \in K$,

$$\begin{aligned} 2(\mathrm{ad}_a^n k)k'(\mathrm{ad}_a^n k) &= 2\binom{n}{t-1}\binom{n}{t-1} a^{t-1}ka^{t-1}k'a^{t-1}ka^{t-1} = \\ &= \binom{n}{t-1}\binom{n}{t-1} a^{t-1}\mathrm{ad}_k^2(a^{t-1}k'a^{t-1})a^{t-1} = \\ &= \mathrm{ad}_a^n(\mathrm{ad}_k^2(\mathrm{ad}_a^n k')) \in \mathrm{ad}_a^n([K,K]) = 0 \end{aligned}$$

because $a^{t-1}a^{t-1} = 0$. Therefore $(\mathrm{ad}_a^n k)K(\mathrm{ad}_a^n k) = 0$ and hence $\mathrm{ad}_a^n k = 0$ for every $k \in K$ by Lemma 2.3.1(1).

(c) The case $n+1 \equiv_4 2$ is not possible.

(d) If $n+1 \equiv_4 3$ then, by primeness of R, either $t = \frac{n}{2}+2$ and $a^{t-2}ka^{t-1} = a^{t-1}ka^{t-2}$ for every $k \in K$ (case (4.1) in Theorem 2.3.6) or $t \le \frac{n}{2}+1$ (case (4.2) in Theorem 2.3.6).

(d.1) Suppose $t = \frac{n}{2}+2$ and $a^{t-2}ka^{t-1} = a^{t-1}ka^{t-2}$ **(1)** for every $k \in K$. For convenience write $\alpha := \binom{n}{t-3}$, $\beta := \binom{n}{t-2}$ and observe that $\alpha \neq \beta$ (since $n \neq 2t-5$). For every $k, k' \in K$ we have

$$0 = \mathrm{ad}_a^n([k,k']) = \alpha a^{t-3}[k,k']a^{t-1} - \beta a^{t-2}[k,k']a^{t-2} + \alpha a^{t-1}[k,k']a^{t-3}. \qquad \mathbf{(2)}$$

Multiplying on the left by a and applying **(1)** to the second term afterwards,

$$\begin{aligned} 0 = a\mathrm{ad}_a^n([k,k']) &= \alpha a^{t-2}[k,k']a^{t-1} - \beta a^{t-1}[k,k']a^{t-2} = \\ &= \alpha a^{t-2}[k,k']a^{t-1} - \beta a^{t-2}[k,k']a^{t-1} = (\alpha-\beta)a^{t-2}[k,k']a^{t-1}, \end{aligned}$$

which gives $a^{t-2}[k,k']a^{t-1} = 0$ **(3)** since R is free of $(\alpha-\beta)$-torsion. Now we study two separate cases:

If $n = 2$ then $t = 3$ and $a \in K$ satisfies $\mathrm{ad}_a^3(K) = 0$ and $a^2 \neq 0, a^3 = 0$, so it is a Clifford element (see [10]). Since R is free of $2,3$-torsion there is a *twin* element $b \in K$ of a such that $aba = a$ and $a^2b^2a^2 = a^2$ ([10, p. 289 and Proposition 3.7(6)]).

Then, by **(3)**,

$$0 = a[[b,a],b]a^2 = 2(aba)ba^2 - a^2b^2a^2 - ab^2a^3 = 2aba^2 - a^2 = a^2,$$

a contradiction.

If $n > 2$ then $n \geq 6$ and $t \geq 5$, so $2t-4 > t$ and $(a^{t-2})^2 = 0$. We see that

$$a^{t-2}[k_1,k_1']a^{t-2}[k_2,k_2']a^{t-2}[k_1,k_1']a^{t-2} = 0 \qquad \textbf{(4)}$$

for every $k_1, k_1', k_2, k_2' \in K$: from **(2)** we can write $\beta a^{t-2}[k_2,k_2']a^{t-2}$ as a linear combination of $a^{t-1}[k,k']a^{t-3}$ and $a^{t-3}[k,k']a^{t-1}$, so **(4)** follows since R is free of β-torsion and $a^{t-2}[k_1,k_1']a^{t-1} = 0 = a^{t-1}[k_1,k_1']a^{t-2}$ by **(3)** and **(1)**. Since for each $k_1, k_1' \in K$ we have that $b := a^{t-2}[k_1,k_1']a^{t-2} \in K$ is such that $b^2 = 0$ and $b[K,K]b = 0$ by **(4)**, by Lemma 2.3.1(4) we get $b = 0$ for each $k_1, k_1' \in K$, so $a^{t-2}[K,K]a^{t-2} = 0$, and $a^{t-2} = 0$ again by Lemma 2.3.1(4), a contradiction.

(d.2) Suppose $t \leq \frac{n}{2}+1$. In this case, the proof follows as in (b): $\mathrm{ad}_a^n x = (-1)^{\frac{n}{2}}\binom{n}{\frac{n}{2}}a^{\frac{n}{2}}xa^{\frac{n}{2}}$ for every $x \in R$, $(\mathrm{ad}_a^n k)K(\mathrm{ad}_a^n k) = 0$ and hence $\mathrm{ad}_a^n k = 0$ for every $k \in K$ by Lemma 2.3.1(1).

In any case $\mathrm{ad}_a^n(K) = 0$. Finally, the semiprime case follows because R is a subdirect product of $*$-prime associative algebras. □

From this lemma and Corollary 2.3.8 we get:

Corollary 2.3.11. *Let R be a semiprime associative algebra with involution $*$, let $\hat{R}$ be its central closure, and consider the Lie algebra $L := [K,K]/(Z(R) \cap [K,K])$. If $\bar{a}$ is an ad-nilpotent element of L of index $n > 1$ and R is free of $(n+1)!$-torsion then there exist $\lambda \in \mathrm{Skew}(C(R),*)$ and an idempotent $\epsilon \in H(C(R),*)$ such that $(\epsilon a - \lambda)^{s+1} = 0$ and $(\epsilon a - \lambda)^{s-1} \neq 0$ for $s := [\frac{n+1}{2}]$, and $(1-\epsilon)\hat{R}$ is a PI-algebra satisfying the standard identity S_4.*

Chapter 3

Ad-nilpotent elements in a prime associative superalgebra

This chapter is part of an article that has been published in the journal *Linear and Multilinear Algebra* [28].

In this chapter we are going to study nilpotent inner superderivations in prime associative superalgebras with and without involution.

The goal is to extend the results of the previous chapter to the prime super setting. In the first section we will give a detailed description of a homogeneous ad-nilpotent element a of index n in a prime associative superalgebra R free of $\binom{n}{s}$ and s-torsion, where $s = [\frac{n+1}{2}]$, depending on the degree of the element and the equivalence class of n modulo 4. If a belongs to R_0 we can adjust the techniques and use the results from the previous chapter because R_0 is an algebra. On the other hand, if $a \in R_1$ we will work with $a^2 \in R_0$ and we will show that the only possible indexes of ad-nilpotency of a are $n \equiv_4 1, 2$. These two cases correspond to a nilpotent element of index $\frac{n+1}{2}$, when $n \equiv_4 1$, or to an element a for which there exists $\lambda \in C(R)_0$ with $(a^2 - \lambda)^{\frac{n+2}{4}} = 0$, when $n \equiv_4 2$.

In the second section we will study ad-nilpotent elements of the skew-symmetric elements K of a prime superalgebra with superinvolution and characteristic $p > n$, i.e., elements $a \in K_0 \cup K_1$ such that $\mathrm{ad}_a^n K = 0$ and $\mathrm{ad}_a^{n-1} K \neq 0$. The key point is the fact proven in Proposition 3.2.3 that any ad-nilpotent element a of K of index

n is either nilpotent or ad-nilpotent of the whole R with the same index n. When $a \in K$ is an ad-nilpotent homogeneous even element, it will be classified depending on its index of ad-nilpotency modulo 4 (see Theorem 3.2.4), and when $a \in K_1$ is ad-nilpotent of index n, its description will depend on the congruence class of n modulo 8 (see Theorem 3.2.5): if $n \equiv_8 1, 2, 5, 6$ then a behaves as an ad-nilpotent element of R and if $n \equiv_8 0, 7$ then a is nilpotent of index $s+1$ for $s = [\frac{n+1}{2}]$, and $a^s K a^s = 0$, implying that $a^s R a^s$ is commutative as a local superalgebra at a^s. We will also show that the indexes of ad-nilpotency $n \equiv_8 3, 4$ are not possible.

3.0.1. Let R be an associative superalgebra. We recall that a homogeneous 0-degree linear map $* : R \to R$ is a superinvolution in R if $(a^*)^* = a$ and $(ab)^* = (-1)^{|a||b|} b^* a^*$ for every homogeneous $a, b \in R_0 \cup R_1$. In particular

$$(abc)^* = (-1)^{|a||b|+|a||c|+|b||c|} c^* b^* a^*$$

for homogeneous $a, b, c \in R_0 \cup R_1$ and and

$$(aba)^* = (-1)^{|a||b|+|a||a|+|b||a|} a^* b^* a^* = (-1)^{|a|} a^* b^* a^*.$$

the set of skew-symmetric elements $K := \{a \in R \mid a^* = -a\}$ and the set of symmetric elements $H := \{a \in R \mid a^* = a\}$ are graded submodules of R. Since $\frac{1}{2} \in \Phi$, $R = H \oplus K$. We will denote $H_i = H \cap R_i$ and $K_i = K \cap R_i$, $i = 0, 1$. Notice that

$$a \in K_0 \implies \begin{cases} a^s \in H_0, & \text{when } s \text{ is even,} \\ a^s \in K_0, & \text{when } s \text{ is odd,} \end{cases}$$

$$a \in K_1 \implies \begin{cases} a^s \in H_0, & \text{when } s \equiv_4 0, \\ a^s \in K_1, & \text{when } s \equiv_4 1, \\ a^s \in K_0, & \text{when } s \equiv_4 2, \\ a^s \in H_1, & \text{when } s \equiv_4 3. \end{cases}$$

Moreover, if R is a prime superalgebra and $\mathrm{Skew}(C(R), *) \neq 0$, then $R = K + \mu K$ for any nonzero homogeneous $\mu \in \mathrm{Skew}(C(R), *)$ (indeed, $\mu^2 \in C(R)_0$ is invertible because $C(R)_0$ is field, and therefore $R \subseteq K + \mu^2 H \subseteq K + \mu K \subseteq R$).

3.0.2. Let $a \in R_1$. Taking into account that $\mathrm{ad}_a^2 = \mathrm{ad}_{a^2}$, it is convenient to compute the adjoint map depending on n modulo 4 and focus in the central terms because if a is nilpotent these will remain:

$n \equiv_4 0$

$$\mathrm{ad}_a^{2s} x = \mathrm{ad}_{a^2}^s x = \sum_{i=0}^{s} \binom{s}{i} (-1)^{s-i} a^{2i} x a^{2s-2i} =$$
$$= \ldots + \binom{s}{\frac{s}{2}-1} (-1)^{\frac{s}{2}-1} a^{s-2} x a^{s+2} + \binom{s}{\frac{s}{2}} (-1)^{\frac{s}{2}} a^s x a^s + \binom{s}{\frac{s}{2}+1} (-1)^{\frac{s}{2}+1} a^{s+2} x a^{s-2} + \ldots$$

$n \equiv_4 1$

$$\mathrm{ad}_a^{2s-1} x = \mathrm{ad}_a \mathrm{ad}_a^{2s-2} x = \mathrm{ad}_a \left(\mathrm{ad}_{a^2}^{s-1} x\right) = \mathrm{ad}_a \left(\sum_{i=0}^{s-1} \binom{s-1}{i} (-1)^{s-i} a^{2i} x a^{2s-2i-2} \right) =$$
$$= \mathrm{ad}_a \left(\ldots + \binom{s-1}{\frac{s-1}{2}} (-1)^{\frac{s-1}{2}} a^{s-1} x a^{s-1} + \ldots \right) =$$
$$= \ldots + \binom{s-1}{\frac{s-1}{2}} (-1)^{\frac{s-1}{2}} a^s x a^{s-1} - \binom{s-1}{\frac{s-1}{2}} (-1)^{\frac{s-1}{2}+|x|} a^{s-1} x a^s + \ldots$$

$n \equiv_4 2$

$$\mathrm{ad}_a^{2s} x = \mathrm{ad}_{a^2}^s x = \sum_{i=0}^{s} \binom{s}{i} (-1)^{s-i} a^{2i} x a^{2s-2i} =$$
$$= \ldots + \binom{s}{\frac{s-1}{2}} (-1)^{\frac{s-1}{2}} a^{s-1} x a^{s+1} + \binom{s}{\frac{s+1}{2}} (-1)^{\frac{s+1}{2}} a^{s+1} x a^{s-1} + \ldots$$

$n \equiv_4 3$

$$\mathrm{ad}_a^{2s-1} x = \mathrm{ad}_a \mathrm{ad}_a^{2s-2} x = \mathrm{ad}_a \left(\mathrm{ad}_{a^2}^{s-1} x\right) = \mathrm{ad}_a \left(\sum_{i=0}^{s-1} \binom{s-1}{i} (-1)^{s-i} a^{2i} x a^{2s-2i-2} \right) =$$
$$= \mathrm{ad}_a \left(\ldots + \binom{s-1}{\frac{s-2}{2}} (-1)^{\frac{s-2}{2}} a^{s-2} x a^s + \binom{s-1}{\frac{s}{2}} (-1)^{\frac{s}{2}} a^s x a^{s-2} + \ldots \right) =$$
$$= \ldots + \binom{s-1}{\frac{s}{2}-1} (-1)^{\frac{s}{2}-1} a^{s-1} x a^s + \binom{s-1}{\frac{s}{2}} (-1)^{\frac{s}{2}} a^{s+1} x a^{s-2} -$$
$$- \binom{s-1}{\frac{s}{2}-1} (-1)^{\frac{s}{2}-1+|x|} a^{s-2} x a^{s+1} - \binom{s-1}{\frac{s}{2}} (-1)^{\frac{s}{2}+|x|} a^s x a^{s-1} + \ldots$$

Throughout all this chapter we will use these calculations without mentioning them.

3.1 Ad-nilpotent elements of R^-

In the following result we will relate the index of nilpotence of a homogeneous element of R with its index of ad-nilpotence in R. It will be useful in our study of ad-nilpotent elements of K.

Proposition 3.1.1. *Let $R = R_0 \oplus R_1$ be a semiprime associative superalgebra. If $a \in R$ is a homogeneous nilpotent element of index s and*

(1) $a \in R_0$ and R is free of $\binom{2s-2}{s-1}$-torsion, then a is ad-nilpotent of R (and of R_0) of index $n = 2s - 1$,

(2a) $a \in R_1$, s is even and R is free of $\binom{s-2}{\frac{s-2}{2}}$-torsion, then a is ad-nilpotent of R of index $n = 2s - 2$ ($n \equiv_4 2$),

(2b) $a \in R_1$, s is odd and R is free of $\binom{s-1}{\frac{s-1}{2}}$-torsion, then a is ad-nilpotent of R of index $n = 2s - 1$ ($n \equiv_4 1$).

Proof. (1) Since $a \in R_0$, the operator ad_a behaves as the adjoint map in the non-super setting. From $a^s = 0$ we get that $\mathrm{ad}_a^{2s-1}(R) = 0$. On the other hand, $a^{s-1} \neq 0$, so by semiprimeness of R (and of R_0) (see Lemma 1.1.6) there exists $x \in R$ (respectively, $x \in R_0$) such that $a^{s-1}xa^{s-1} \neq 0$ and, since R has no $\binom{2s-2}{s-1}$-torsion, $\binom{2s-2}{s-1}a^{s-1}xa^{s-1} \neq 0$. Thus

$$\mathrm{ad}_a^{2s-2}(x) = \binom{2s-2}{s-1}(-1)^{s-1}a^{s-1}xa^{s-1} \neq 0.$$

We have shown that a is ad-nilpotent of R (and of R_0) of index $n = 2s - 1$.

(2a) Suppose that $a \in R_1$ is a nilpotent element of even index s. Since $\mathrm{ad}_a^2 = \mathrm{ad}_{a^2}$ and $a^2 \in R_0$ is nilpotent of index $\frac{s}{2}$, we have by (1) that a^2 is ad-nilpotent of R of index $2(\frac{s}{2}) - 1 = s - 1$. Hence the index of ad-nilpotence of a is less or equal to $2s - 2$.

Let x be any element in $R_0 \cup R_1$:

$$\begin{aligned}
\mathrm{ad}_a^{2s-3}(x) &= \mathrm{ad}_a^{2s-4}\mathrm{ad}_a(x) = \mathrm{ad}_{a^2}^{s-2}\mathrm{ad}_a(x) = \\
&= \binom{s-2}{\frac{s-2}{2}}(-1)^{\frac{s-2}{2}} a^{s-2}(ax-(-1)^{|x|}xa)a^{s-2} = \\
&= \binom{s-2}{\frac{s-2}{2}}(-1)^{\frac{s-2}{2}} a^{s-1}xa^{s-2} - \binom{s-2}{\frac{s-2}{2}}(-1)^{\frac{s-2}{2}+|x|}a^{s-2}xa^{s-1}, \text{ hence} \\
\mathrm{ad}_a^{2s-3}(x)a &= \binom{s-2}{\frac{s-2}{2}}(-1)^{\frac{s-2}{2}} a^{s-1}xa^{s-1}.
\end{aligned}$$

Therefore $\mathrm{ad}_a^{2s-3}(R)$ cannot be zero, since otherwise $a^{s-1}=0$ because R is free of $\binom{s-2}{\frac{s-2}{2}}$-torsion and semiprime, a contradiction. We have shown that a is ad-nilpotent of index $n = 2s-2$.

(2b) Suppose that $a \in R_1$ is a nilpotent element of odd index s. For any homogeneous $x \in R_0 \cup R_1$:

$$\begin{aligned}
\mathrm{ad}_a^{2s-1}(x) &= \mathrm{ad}_a\mathrm{ad}_a^{2s-2}(x) = \mathrm{ad}_a\mathrm{ad}_{a^2}^{s-1}(x) = \mathrm{ad}_a\left(\binom{s-1}{\frac{s-1}{2}}(-1)^{\frac{s-1}{2}}a^{s-1}xa^{s-1}\right) = \\
&= \binom{s-1}{\frac{s-1}{2}}(-1)^{\frac{s-1}{2}}(a^sxa^{s-1}-(-1)^{|x|}a^{s-1}xa^s) = 0
\end{aligned}$$

so $\mathrm{ad}_a^{2s-1}(R)=0$. Let us see that $\mathrm{ad}_a^{2s-2}(R) \neq 0$: $a^{s-1} \neq 0$, so there exists $x \in R$ such that

$$\mathrm{ad}_a^{2s-2}(x) = \mathrm{ad}_{a^2}^{s-1}(x) = \binom{s-1}{\frac{s-1}{2}}(-1)^{\frac{s-1}{2}}a^{s-1}xa^{s-1} \neq 0$$

because R is semiprime and free of $\binom{s-1}{\frac{s-1}{2}}$-torsion. We have shown that a is ad-nilpotent of index $n = 2s-1$. □

In the following theorem we describe the homogeneous ad-nilpotent elements of R, depending on the equivalence class of their indexes of ad-nilpotence modulo 4.

Theorem 3.1.2. *Let us consider a prime associative superalgebra $R = R_0 \oplus R_1$, let $\hat{R}$ denote the central closure of R, and let $a \in R_0 \cup R_1$ be a homogeneous ad-nilpotent element of index n. If R is free of $\binom{n}{s}$-torsion and free of s-torsion, for $s = [\frac{n+1}{2}]$, then:*

1. *If $a \in R_0$, n is odd and exists $\lambda \in C(R)_0$ such that $a - \lambda \in \hat{R}$ is nilpotent of index $\frac{n+1}{2}$.*

2. *If $a \in R_1$, then*

   (a) *if $n \equiv_4 1$ and R is free of $\binom{\frac{n-1}{2}}{\frac{s-1}{2}}$-torsion, then a is nilpotent of index $\frac{n+1}{2}$.*

   (b) *if $n \equiv_4 2$ then there is $\lambda \in C(R)_0$ such that $(a^2 - \lambda) \in \hat{R}$ is nilpotent of index $\frac{n+2}{4}$.*

   (c) *the cases $n \equiv_4 0$ and $n \equiv_4 3$ do not occur.*

Proof. We will suppose without loss of generality that R is centrally closed.
(1) Let $a \in R_0$ be an ad-nilpotent element of index n. By Lemma 1.1.6, R is semiprime as an algebra. Moreover, the element a is a pure ad-nilpotent element of R because every graded ideal of R is essential (see 2.1.2). Therefore, we can use Theorem 2.2.4 to obtain that n is odd and there exists $\lambda \in C(R)$ such that $a - \lambda$ is nilpotent of index $\frac{n+1}{2}$. Moreover, $a \in R_0$, R is prime and has no $\frac{n+1}{2}$-torsion, so $\lambda \in C(R)_0$ by Lemma 1.2.5.
(2) Let $a \in R_1$ be an ad-nilpotent element of index n. Let us split our argument in two cases:

(2a) If n is odd, $n = 2s - 1$ for some s. Then $0 = \mathrm{ad}_a^{n+1}(R) = \mathrm{ad}_a^{2s}(R) = \mathrm{ad}_{a^2}^{s}(R)$, and $a^2 \in R_0$ is ad-nilpotent of index s (notice that $\mathrm{ad}_{a^2}^{s-1}(R) = \mathrm{ad}_a^{2s-2}(R) = \mathrm{ad}_a^{n-1}(R) \neq 0$). Therefore, by (1), s is odd (equivalently, $n \equiv_4 1$) and there exists $\lambda \in C(R)_0$ such that $a^2 - \lambda$ is nilpotent of index $\frac{s+1}{2}$. Let us see prove that $\lambda = 0$: Let us denote $b = (a^2 - \lambda)^{\frac{s-1}{2}}$. Then, for every $x \in R_0 \cup R_1$,

$$
\begin{aligned}
0 &= \mathrm{ad}_a^n(x) = \mathrm{ad}_a(\mathrm{ad}_{a^2}^{\frac{n-1}{2}}(x)) = \mathrm{ad}_a(\mathrm{ad}_{a^2-\lambda}^{\frac{n-1}{2}}(x)) = \\
&= [a, \sum_{i=0}^{\frac{n-1}{2}} \binom{\frac{n-1}{2}}{i} (-1)^{\frac{n-1}{2}-i} (a^2-\lambda)^i x (a^2-\lambda)^{\frac{n-1}{2}-i}] = \\
&= [a, \binom{\frac{n-1}{2}}{\frac{s-1}{2}} (-1)^{\frac{s-1}{2}} (a^2-\lambda)^{\frac{s-1}{2}} x (a^2-\lambda)^{\frac{s-1}{2}}] = \\
&= [a, \binom{\frac{n-1}{2}}{\frac{s-1}{2}} (-1)^{\frac{s-1}{2}} bxb] = \binom{\frac{n-1}{2}}{\frac{s-1}{2}} (-1)^{\frac{s-1}{2}} \left(abxb - (-1)^{|x|} bxba\right).
\end{aligned}
$$

Since R is free of $\binom{\frac{n-1}{2}}{\frac{s-1}{2}}$-torsion, we get that

$$abxb = (-1)^{|x|}bxba, \qquad \text{for every } x \in R_0 \cup R_1.$$

Take any $x \in R_0$. Multiplying this last equality by a on the left and taking into account that $ab = ba$ we have $a^2bxb = a(abxb) = a(bxba) = abxab$; but $a^2bxb = ab(ax)b = -b(ax)ba = -abxab$ because $ax \in R_1$. Then $a^2bR_0b = abR_0ab = 0$. Similarly, for any $x \in R_1$ we have that $a^2bxb = a(abxb) = -a(bxab)$, and we also have that $a^2bxb = ab(ax)b = b(ax)ba = abxab$ because $ax \in R_0$. Then $a^2bR_1b = abR_1ab = 0$. We have obtained

$$a^2bRb = abRab = 0.$$

From the definition of b we have that $(a^2 - \lambda)b = 0$, i.e., $a^2b = \lambda b$, so $0 = a^2bRb = \lambda bRb$. If $\lambda \neq 0$, we would have that $bRb = 0$ (notice that $\lambda \in C(R)_0$ and $C(R)_0$ is a field (Lemma 1.2.6)), leading to a contradiction with the semiprimeness of R and $b \neq 0$.

Thus $\lambda = 0$, so $0 \neq b = a^{s-1}$, $ab = a^s$ and $0 = abRab = a^sRa^s$ implies $a^s = 0$ by semiprimeness of R.

($2b$) If n is even, then $n = 2s$ for some s, so $a^2 \in R_0$ is ad-nilpotent of index s ($\mathrm{ad}_{a^2}^s(R) = \mathrm{ad}_a^n(R) = 0$ and $\mathrm{ad}_{a^2}^{s-1}(R) = \mathrm{ad}_a^{2s-2}(R) = \mathrm{ad}_a^{n-2}(R) \neq 0$). Then by (1) we obtain that s is odd (equivalently, $n \equiv_4 2$) and there exists $\lambda \in C(R)_0$ such that $(a^2 - \lambda)^{\frac{s+1}{2}} = 0$.

Notice that the cases $n \equiv_4 0$ and $n \equiv_4 3$ do not occur. $\square$

3.2 Ad-nilpotent elements of K

As in the non-super setting, the associative local superalgebra at the ad-nilpotent element give us extra information about the structure. In non-super setting for example we get that the GPI $a^txa^tya^t = a^tya^txa^t$ holds for an ad-nilpotent element a of index $n \equiv_4 0$ of K for every $x, y \in K$.

3.2.1. Let R be an associative superalgebra over Φ and take an element $a \in R_0 \cup R_1$. Then $R_a := aRa$ with $(aRa)_i := aR_{i+|a|}a$, $i \in \{0,1\}$, is a $\mathbb{Z}_2$-graded Φ-module. Moreover, the product $(axa)(aya) := axaya$ for any $x, y \in R$ induces an associative superalgebra structure in R_a, which is called the local superalgebra of R at a. When R is an associative superalgebra with superinvolution $*$, the superinvolution induces a superinvolution $\star$ in R_a given by $(axa)^\star := (-1)^{|a|}ax^*a$, for every $x \in R$.

We start with a technical lemma, which is also interesting by itself. For example, it claims that every semiprime superalgebra with superinvolution and no nonzero skew even elements is a trivial superalgebra, i.e., the odd part is zero.

Lemma 3.2.2. *Let $R = R_0 \oplus R_1$ be a semiprime associative superalgebra with superinvolution $*$.*

(i) If $K_0 = 0$ then $R_1 = 0$ and $R = R_0 = H_0$ is commutative.

(ii) Let us consider $h_0 \in H_0$. If $h_0K_0h_0 = 0$ then $h_0R_1h_0 = 0$ and $h_0Rh_0 = h_0R_0h_0 = h_0H_0h_0$ is commutative as the (trivial) local superalgebra of R at h_0.

Proof. (i) Take any $k_1, k_1' \in K_1$ and $h_1, h_1' \in H_1$. Then, since $R_0 = H_0$, we have that

$$k_1h_1 = (k_1h_1)^* = h_1k_1, \qquad k_1k_1' = (k_1k_1')^* = -k_1'k_1, \qquad h_1h_1' = (h_1h_1')^* = -h_1'h_1.$$

In particular, $k_1^2 = h_1^2 = 0$.

We claim that $K_1 = 0$. Take any $k_1 \in K_1$. Then for every $h_0 \in H_0$, $k_1h_0k_1 = (k_1h_0k_1)^* = -k_1h_0k_1$ implies $k_1h_0k_1 = 0$, so $k_1H_0k_1 = 0$; similarly, for every $h_1 \in H_1$, $(k_1h_1)k_1 = h_1k_1^2 = 0$, so $k_1H_1k_1 = 0$, and, for every $k_1' \in K_1$, $(k_1k_1')k_1 = -k_1'k_1^2 = 0$, so $k_1K_1k_1 = 0$. We have shown that $k_1Rk_1 = 0$, so by semiprimeness of R, $k_1 = 0$.

Let us show that $H_1 = 0$. Take any $h_1 \in H_1$. For every $h_0 \in H_0$, since $h_1h_0h_1 = (h_1h_0h_1)^* = -h_1h_0h_1$, we have that $h_1h_0h_1 = 0$, so $h_1H_0h_1 = 0$. Similarly, for every $h_1' \in H_1$, $h_1h_1'h_1 = -h_1'h_1^2 = 0$, so $h_1H_1h_1 = 0$, and, finally, for every $k_1 \in K_1$, $h_1k_1h_1 = k_1h_1^2 = 0$, so $h_1K_1h_1 = 0$. We have shown that $h_1Rh_1 = 0$, so by semiprimeness of R, $h_1 = 0$.

Therefore, $R_1 = H_1 + K_1 = 0$.

Finally, H_0 is commutative because for every $h_0, h_0' \in H_0$,

$$h_0 h_0' = (h_0 h_0')^* = h_0' h_0.$$

(ii) Take $h_0 \in H_0$ and let us consider the local algebra $R_{h_0} = h_0 R h_0$ as defined in 3.2.1, which is an associative superalgebra with induced superinvolution $(h_0 x h_0)^\star := h_0 x^* h_0$, for every $x \in R$. Clearly $\mathrm{Skew}(h_0 R h_0, \star) = h_0 K h_0$ and $\mathrm{Sym}(h_0 R h_0, \star) = h_0 H h_0$. If we suppose that $h_0 K_0 h_0 = 0$ then $\mathrm{Skew}(h_0 R h_0, \star)_0 = 0$ and by (i) we have

$$(R_{h_0})_1 = h_0 R_1 h_0 = 0 \quad \text{and} \quad R_{h_0} = h_0 R h_0 = (R_{h_0})_0 = h_0 R_0 h_0 = h_0 H_0 h_0.$$

$\square$

Proposition 3.2.3. *Let R be a prime associative superalgebra with superinvolution $*$ and let $a \in K$ be a homogeneous ad-nilpotent element of K of index $n > 2$. Suppose that R is free of $\binom{n}{s}$-torsion and free of s-torsion, for $s = [\frac{n+1}{2}]$. If $\mathrm{Skew}(C(R), *) \neq 0$ then a is ad-nilpotent of R of index n. Otherwise, a is nilpotent.*

Proof. If there exists a homogeneous $0 \neq \lambda \in \mathrm{Skew}(C(R), *)$ then λ^2 is invertible in the field $C(R)_0$, and $R = K + \lambda^2 H \subseteq K + \lambda K$ so $\mathrm{ad}_a^n(R) = 0$. Suppose from now on that $\mathrm{Skew}(C(R), *) = 0$. We split our proof in two cases, depending on the parity of a:

(I) Suppose that $a \in K_0$. Let us see that a is nilpotent. Every $x \in R$ can be expressed as $x = x_h + x_k$, so for every $x \in R$

$$\begin{aligned}\mathrm{ad}_a^n(ax + xa) &= \mathrm{ad}_a^n(ax_k + x_k a) + \mathrm{ad}_a^n(ax_h + x_h a) = a\mathrm{ad}_a^n(x_k) + \mathrm{ad}_a^n(x_k)a \\ &\quad + \mathrm{ad}_a^n(ax_h + x_h a) = 0\end{aligned}$$

because $ax_h + x_h a \in K$ and $a\mathrm{ad}_a^i(x) = \mathrm{ad}_a^i(ax)$ for every $x \in R$ and any $i \in \mathbb{N}$.

Expanding this expression

$$0 = \mathrm{ad}_a^n(ax + xa) = (-1)^n xa^{n+1} + \sum_{i=1}^{n} \left(\binom{n}{i} - \binom{n}{i-1} \right) (-1)^{n-i} a^i x a^{n+1-i} + a^{n+1}x.$$

Since R is semiprime as an algebra, by Lemma 2.0.3, a is an algebraic element of R over $C(R)$.

(I.a) Let us suppose that R is prime as an algebra. The calculations of (1.b) in the proof of Proposition 2.3.5 [12, Proposition 5.5] show that a is nilpotent.

(I.b) If R is prime as a superalgebra but not prime as an algebra, R_0 is prime by 1.1.7, $C(R)_0 \cong C(R_0)$ by 1.2.4, the superinvolution $*$ restricted to R_0 is an involution and $\mathrm{Skew}(C(R_0), *) = 0$ because we are assuming that $\mathrm{Skew}(C(R), *) = 0$. The element a is a pure ad-nilpotent element of K_0 because $C(R_0)$ is a field, so we can apply Proposition 2.3.5(2) to the prime associative algebra R_0 to obtain that a is nilpotent.

(II) If $a \in K_1$, consider $a^2 \in K_0$ and by (I), a^2 is nilpotent, i.e., a is nilpotent. $\square$

In the following two theorems we will describe the homogeneous ad-nilpotent elements of K. Our goal is to relate the index of ad-nilpotence of a homogeneous element of K with its index of ad-nilpotence in R (and in R_0 and in K_0 when the element is even). Moreover, when these indexes in K and in R do not coincide, we will show that the element is nilpotent of an explicit index.

We begin with the description of even ad-nilpotent elements of K.

Theorem 3.2.4. *Let R be a prime associative superalgebra of characteristic $p > n$ with superinvolution $*$, let $\hat{R}$ be its central closure, let $a \in K_0 := \mathrm{Skew}(R, *)_0$ be an ad-nilpotent element of K of index $n > 1$ and let $s = [\frac{n+1}{2}]$. Then*

(1) If $n \equiv_4 0$ then a is nilpotent of index $s+1$, ad-nilpotent of R and of R_0 of index $n+1$ and satisfies $a^s K a^s = 0$. Moreover, the index of ad-nilpotence of a in K_0 can be $n-1$ or n.

(2) If $n \equiv_4 1$ then there exists $\lambda \in \mathrm{Skew}(C(R), *)_0$ such that $a - \lambda \in \hat{R}$ is nilpotent of index s and a is ad-nilpotent of R, of R_0 and of K_0 of index n.

(3) The case $n \equiv_4 2$ is not possible.

(4) If $n \equiv_4 3$ then either:

(4.1) a is nilpotent of index $s+1$, ad-nilpotent of K_0 of index n, ad-nilpotent of R and of R_0 of index $n+2$ and satisfies $a^s k a^{s-1} - a^{s-1} k a^s = 0$ for every $k \in K$. In particular R satisfies $a^s K a^s = 0$, or

(4.2) there exists $\lambda \in \mathrm{Skew}(C(R), *)_0$ such that $a - \lambda \in \hat{R}$ is nilpotent of index s and a is ad-nilpotent of R, of R_0 and of K_0 of index n.

Proof. Suppose without loss of generality that R is centrally closed. Let $a \in K_0$ be an ad-nilpotent element of K of index n.

– If $\mathrm{Skew}(C(R), *) \neq 0$, by Proposition 3.2.3, a is ad-nilpotent of index n of R and by Theorem 3.1.2 n has to be odd ($n \equiv_4 1$ or $n \equiv_4 3$) and there exists $\lambda \in C(R)_0$ such that $a - \lambda$ is nilpotent of index s, so a is ad-nilpotent of R and of R_0 of the same index $n = 2s-1$, see Proposition 3.1.1(1). Moreover, $\lambda \in \mathrm{Skew}(C(R), *)_0$ by Lemma 1.3.4 and since $\mathrm{Skew}(C(R), *)_0 \subset \mathrm{Skew}(C(R_0), *)$, the index of ad-nilpotence of $a - \lambda$ in K_0 is again $n = 2s - 1$ (notice that, by Lemma 1.3.4, λ is the unique element of $C(R_0)$ such that $a - \lambda$ is nilpotent). These are the cases (2) and (4.1).

– If $\mathrm{Skew}(C(R), *) = 0$, by Proposition 3.2.3, a is nilpotent. We are going to approach this case considering the index of ad-nilpotence of a in K_0 and comparing it with its index of ad-nilpotence in K and in R. Let us suppose that a is ad-nilpotent of K_0 of index $m \leq n$ and let $r = [\frac{m+1}{2}]$. Since R_0 is a semiprime algebra and the superinvolution $*$ restricted to R_0 is an involution, by Proposition 2.3.3 we have four possibilities:

- $m \equiv_4 0$ then a is nilpotent of index $r+1$ and $a^r K_0 a^r = 0$, which, by Lemma 3.2.2(ii), implies that $a^r R_1 a^r = 0$, so a is also ad-nilpotent of index m of K, i.e., $m = n$ and a is nilpotent of index $s+1$ with $s = \frac{n}{2} = r$. Now, since $s+1$ is the index of nilpotence of a, by Proposition 3.1.1(1) a is ad-nilpotent of index $n+1$ of R and of R_0. This is the case (1) ($n \equiv_4 0$) with the index of ad-nilpotence of a in K_0 equal to the index of ad-nilpotence of a in K.

- $m \equiv_4 1$ then a is nilpotent of index r. This implies, by Proposition 3.1.1(1), that a is ad-nilpotent of R and of R_0 of index m. So n has to be equal to m and therefore the index of nilpotence of a is $s = \frac{n+1}{2} = r$. This is the case (2), i.e., $n \equiv_4 1$.
- $m \equiv_4 2$ does not occur.
- $m \equiv_4 3$ then there exists an idempotent $\epsilon \in C(R_0)$ such that $\epsilon a^r = a^r$ and a decomposes as $a = \epsilon a + (1-\epsilon)a$ (although the elements ϵa and $(1-\epsilon)a$ do not belong to R but in central closure of R_0, this decomposition will be useful for our purposes):

  ◇ If $\epsilon a = 0$ then $a = (1-\epsilon)a$ is nilpotent of index r. By Proposition 3.1.1(1), this implies that a is ad-nilpotent of R and of R_0 of index m, so $n = m$ and the index of nilpotence of a is $s = \frac{n+1}{2} = r$. This is the case (4.2), i.e., $n \equiv_4 3$.

  ◇ If $\epsilon a \neq 0$ then a is nilpotent of index $r+1$ and $a^r k_0 a^{r-1} - a^{r-1} k_0 a^r = (\epsilon a)^r k_0 (\epsilon a)^{r-1} - (\epsilon a)^{r-1} k_0 (\epsilon a)^r = 0$ for every $k_0 \in K_0$. Since $a^{r+1} = 0$, $a^r K_0 a^r = 0$ and, by Lemma 3.2.2(ii), $a^r R_1 a^r = 0$, so $a^r K a^r = 0$ and therefore $\mathrm{ad}_a^{m+1} K = 0$. There are two possibilities:

    - Either $a^r k a^{r-1} - a^{r-1} k a^r = 0$ for every homogeneous $k \in K$ and therefore a is ad-nilpotent of index m of K. Then $n = m$, $r = \frac{n+1}{2} = s$, so $a^s k a^{s-1} - a^{s-1} k a^s = 0$ and a is nilpotent of index $s+1$ which, by Proposition 3.1.1(1), implies that a is ad-nilpotent of R and of R_0 of index $n+2$ and fits with the case (4.1), i.e., $n \equiv_4 3$,
    - or there exists $k \in K$ such that $a^r k a^{r-1} - a^{r-1} k a^r \neq 0$, so a is ad-nilpotent of K of index $m+1$. Hence $n = m+1$, $r = \frac{n}{2} = s$, and a is nilpotent of index $s+1$. Therefore, by Proposition 3.1.1(1), a is ad-nilpotent of R and of R_0 of index $n+1$. This is again case (1) with the index of ad-nilpotence of a in K_0 equal to $n-1$ and $n \equiv_4 0$.

□

In the following theorem we describe the odd ad-nilpotent elements of K. We will first distinguish whether $C(R)$ has skew-symmetric elements, in which case a is ad-nilpotent of R of the same index, or $\mathrm{Skew}(C(R), *) = 0$, which implies by Proposition

3.2.3 that a is nilpotent. In this second case, we will consider $a^2 \in K_0$ and use Theorem 3.2.4 applied to a^2 to obtain the description of a.

Theorem 3.2.5. *Let R be a prime associative superalgebra of characteristic $p > n$ with superinvolution $*$, let $\hat{R}$ be its central closure, let $a \in K_1 := \mathrm{Skew}(R, *)_1$ be an ad-nilpotent element of K of index $n > 1$ and let $s = [\frac{n+1}{2}]$.*

(1) If $n \equiv_8 0$ then a is nilpotent of index $s+1$, ad-nilpotent of R of index $n+1$ and $a^s K a^s = 0$ (so $a^s R a^s$ is a commutative trivial local superalgebra).

(2) If $n \equiv_8 1$ then $a^{s-1} \in H_0$, and a is nilpotent of index s and ad-nilpotent of R of index n.

(3) If $n \equiv_8 2$ then there exists $\lambda \in \mathrm{Skew}(C(R), *)_0$ such that $a^2 - \lambda \in \hat{R}$ is nilpotent of index $\frac{s+1}{2}$ and a is ad-nilpotent of R of index n.

(4) If $n \equiv_8 5$ then $a^{s-1} \in K_0$, and a is nilpotent of index s and ad-nilpotent of R of index n.

(5) If $n \equiv_8 6$ then there exists $\lambda \in \mathrm{Skew}(C(R), *)_0$ such that $a^2 - \lambda \in \hat{R}$ is nilpotent of index $\frac{s+1}{2}$ and a is ad-nilpotent of R of index n.

(6) If $n \equiv_8 7$ then a is nilpotent of index $s+1$, ad-nilpotent of R of index $n+2$ and $a^s k a^{s-1} + (-1)^{|k|} a^{s-1} k a^s = 0$ for every homogeneous $k \in K$ (so $a^s R a^s$ is a commutative trivial local superalgebra).

(7) The cases $n \equiv_8 3$ and $n \equiv_8 4$ do not occur.

Proof. Suppose without loss of generality that R is centrally closed.

Let $a \in K_1$ be an ad-nilpotent element of K of index n. If $\mathrm{Skew}(C(R), *) \neq 0$, by Proposition 3.2.3, a is ad-nilpotent of R of index n. By Theorem 3.1.2 n can be:

- $n \equiv_4 1$ and therefore a is nilpotent of index s (cases (2) and (4)), or
- $n \equiv_4 2$ and therefore there exists $\lambda \in \mathrm{Skew}(C(R)_0, *)$ such that $a^2 - \lambda$ is nilpotent of index $\frac{s+1}{2}$ (cases (3) and (5)).

Let us suppose that $\mathrm{Skew}(C(R),*) = 0$. By Proposition 3.2.3, a is nilpotent. Then, since $a^2 \in K_0$ and $\mathrm{ad}_a^2(x) = \mathrm{ad}_{a^2}(x)$, a^2 is an ad-nilpotent element of K. Let us denote by m the index of ad-nilpotence of a^2 in K and let $r = \left[\frac{m+1}{2}\right]$. By Theorem 3.2.4 applied to the element a^2 we have:

• If $m \equiv_4 0$ and $r = \frac{m}{2}$, $(a^2)^r \neq 0$, $(a^2)^{r+1} = 0$ and $a^{2r}Ka^{2r} = 0$. We are going to show that $a^{2r+1} = 0$: let x be any homogeneous element in R, so $ax + (-1)^{|x|}x^*a \in K_{1+|x|}$,

$$0 = \mathrm{ad}_{a^2}^m(ax + (-1)^{|x|}x^*a)a = \binom{m}{\frac{m}{2}}(-1)^{\frac{m}{2}}(a^m(ax + (-1)^{|x|}x^*a)a^m)a =$$
$$= \binom{m}{r}(-1)^r a^{2r}(ax + (-1)^{|x|}x^*a)a^{2r+1} = \binom{m}{r}(-1)^r a^{2r+1}xa^{2r+1}$$
$$+ \binom{m}{r}(-1)^r(-1)^{|x|}a^{2r}x^*a^{2r+2} = \binom{m}{r}(-1)^r a^{2r+1}xa^{2r+1}.$$

Since R is semiprime and free of $\binom{m}{r}$-torsion, $a^{2r+1} = 0$. Moreover, since $ad_{a^2}^{m-1}(K) \neq 0$, we have two possibilities:

• If $\mathrm{ad}_a^{2m-1}(K) \neq 0$, then a is an ad-nilpotent element of K of index $n = 2m$. In this case $n \equiv_8 0$ and for $s = \frac{n}{2}$ we have that $a^{s+1} = 0$, $a^s \neq 0$ and $a^sKa^s = 0$. Moreover, by Proposition 3.1.1, a is ad-nilpotent of R of index $n+1$, case (1).

• If $\mathrm{ad}_a^{2m-1}(K) = 0$, then a is an ad-nilpotent element of K of index $n = 2m-1$. So in this case we have got $n \equiv_8 7$ and for $s = \frac{n+1}{2}$ we have that $a^{s+1} = 0$, $a^s \neq 0$. Moreover, for every homogeneous $k \in K$,

$$0 = \mathrm{ad}_a^{2m-1}(k) = \binom{m-1}{\frac{m}{2}}(-1)^{\frac{m}{2}}(a^mka^{m-1} + (-1)^{|k|}a^{m-1}ka^m) =$$
$$= \binom{m-1}{\frac{s}{2}}(-1)^{\frac{s}{2}}(a^ska^{s-1} + (-1)^{|k|}a^{s-1}ka^s)$$

and since R is free of $\binom{m-1}{\frac{s}{2}}$-torsion we have that $a^ska^{s-1} + (-1)^{|k|}a^{s-1}ka^s = 0$. In addition, by Proposition 3.1.1, a is ad-nilpotent element of R of index $n+2$, case (6).

• If $m \equiv_4 1$ and $r = \frac{m+1}{2}$ we have that $(a^2)^r = 0$, $(a^2)^{r-1} \neq 0$ and $\mathrm{ad}_{a^2}^m(R) = 0$. Since $ad_{a^2}^{m-1}(K) \neq 0$, we have two possibilities:

• If $\mathrm{ad}_a^{2m-1}(K) \neq 0$, then a is an ad-nilpotent element of K of index $n = 2m$

and there exists a homogeneous k in K such that:

$$\begin{aligned}
0 \neq \operatorname{ad}_a^{2m-1}(k) &= \operatorname{ad}_{a^2}^{m-1}\operatorname{ad}_a(k) = \\
&= \binom{m-1}{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}}(a^m k a^{m-1} - (-1)^{|k|}a^{m-1}ka^m) = \\
&= \binom{m-1}{r}(-1)^r(a^{2r-1}ka^{2r-2} - (-1)^{|k|}a^{2r-2}ka^{2r-1}).
\end{aligned}$$

Therefore, since R is free of $\binom{m-1}{r}$-torsion, $a^{2r-1} \neq 0$. In this case $n \equiv_8 2$ and for $s = \frac{n}{2}$ we have that $a^{s+1} = 0$, $a^s \neq 0$. By Proposition 3.1.1, a is ad-nilpotent of index n, case (3).

• If $\operatorname{ad}_a^{2m-1}(K) = 0$, then a is ad-nilpotent of K of index $n = 2m-1$. Let x be any homogeneous element in R and let us consider $ax + (-1)^{|x|}x^*a \in K_{1+|x|}$:

$$\begin{aligned}
0 &= \operatorname{ad}_a^{2m-1}(ax + (-1)^{|x|}x^*a) = \operatorname{ad}_a^{2m-2}\operatorname{ad}_a(ax + (-1)^{|x|}x^*a) = \\
&= \operatorname{ad}_{a^2}^{m-1}\operatorname{ad}_a(ax + (-1)^{|x|}x^*a) = \\
&= \binom{m-1}{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}}a^{m-1}(a^2x + (-1)^{|x|}ax^*a - (-1)^{1+|x|}(axa + (-1)^{|x|}x^*a^2))a^{m-1} = \\
&= \binom{m-1}{r-1}(-1)^{r-1}a^{2r-2}(a^2x + (-1)^{|x|}ax^*a - (-1)^{1+|x|}(axa + (-1)^{|x|}x^*a^2))a^{2r-2} = \\
&= \binom{m-1}{r-1}(-1)^{\frac{m-1}{2}+|x|}a^{2r-1}(x^* + x)a^{2r-1}
\end{aligned}$$

and

$$\begin{aligned}
0 &= \operatorname{ad}_a^{2m-1}(x - x^*)a = \operatorname{ad}_a^{2m-2}\operatorname{ad}_a(x - x^*)a = \operatorname{ad}_{a^2}^{m-1}\operatorname{ad}_a(x - x^*)a = \\
&= \binom{m-1}{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}}a^{m-1}(ax - ax^* - (-1)^{|x|}(xa - x^*a))a^m = \\
&= \binom{m-1}{r-1}(-1)^{r-1}a^{2r-2}(ax - ax^* - (-1)^{|x|}(xa - x^*a))a^{2r-1} = \\
&= \binom{m-1}{r-1}(-1)^{r-1}a^{2r-1}(x - x^*)a^{2r-1}.
\end{aligned}$$

Therefore, since R is free of $\binom{m-1}{r-1}$-torsion, $a^{2r-1}Ra^{2r-1} = 0$, and by semiprimeness of R, $a^{2r-1} = 0$ and a is an ad-nilpotent element of R of index $n = 2m-1$. So $n \equiv_8 1$ and for $s = \frac{n+1}{2}$ we have that $a^s = 0$, $a^{s-1} \neq 0$. By Proposition 3.1.1, a is ad-nilpotent of R of index n, case (2).

- $m \equiv_4 2$ is not possible.

- If $m \equiv_4 3$ and $r = \frac{m+1}{2}$, let us first see that $(a^2)^r = 0$. Suppose otherwise that $(a^2)^r \neq 0$. Then $(a^2)^{r+1} = 0$ and $a^{2r}ka^{2r-2} - a^{2r-2}ka^{2r} = 0$ for every $k \in K$. Let x be any homogeneous element in R and let us consider $ax + (-1)^{|x|}x^*a \in K_{1+|x|}$:

$$
\begin{aligned}
0 = \mathrm{ad}_{a^2}^m(ax + (-1)^{|x|}x^*a)a^3 &= \binom{m}{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}}a^{m+1}(ax + (-1)^{|x|}x^*a)a^{m+2} + \\
&+ \binom{m}{\frac{m+1}{2}}(-1)^{\frac{m+1}{2}}a^{m-1}axa^{m+4} + \binom{m}{\frac{m+1}{2}}(-1)^{\frac{m+1}{2}}a^{m-1}(-1)^{|x|}x^*aa^{m+4} = \\
&= \binom{m}{r-1}(-1)^{r-1}a^{2r}(ax + (-1)^{|x|}x^*a)a^{2r+1} + \binom{m}{\frac{m+1}{2}}(-1)^{\frac{m+1}{2}}a^{2r-2}axa^{2r+3} \\
&+ \binom{m}{\frac{m+1}{2}}(-1)^{\frac{m+1}{2}}a^{2r-2}(-1)^{|x|}x^*aa^{2r+3} = \binom{m}{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}}a^{2r+1}xa^{2r+1}
\end{aligned}
$$

and therefore, since R is free of $\binom{m}{r-1}$-torsion and semiprime, $a^{2r+1} = 0$. Then for every homogeneous $x \in R$

$$
\begin{aligned}
0 = a\mathrm{ad}_{a^2}^m(ax + (-1)^{|x|}x^*a) &= \binom{m}{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}}a^{m+2}(ax + (-1)^{|x|}x^*a)a^{m-1} + \\
&+ \binom{m}{\frac{m+1}{2}}(-1)^{\frac{m+1}{2}}a^{m}axa^{m+1} + \binom{m}{\frac{m+1}{2}}(-1)^{\frac{m+1}{2}}a^{m}(-1)^{|x|}x^*aa^{m+1} = \\
&= \binom{m}{r-1}(-1)^{r-1}a^{2r+1}(ax + (-1)^{|x|}x^*a)a^{2r-2} + \\
&+ \binom{m}{r}(-1)^{r}a^{2r-1}axa^{2r} + \binom{m}{r}(-1)^{r}a^{2r-1}(-1)^{|x|}x^*aa^{2r} = \binom{m}{r}(-1)^{r}a^{2r}xa^{2r}
\end{aligned}
$$

and therefore, since R is free of $\binom{m}{r}$-torsion and semiprime, $a^{2r} = 0$, a contradiction. Thus $(a^2)^r = 0$, $(a^2)^{r-1} \neq 0$ and $\mathrm{ad}_{a^2}^m(R) = 0$.

- If $\mathrm{ad}_a^{2m-1}(K) \neq 0$, then a is ad-nilpotent of K of index $n = 2m$ and there exists $k \in K$ homogeneous such that

$$\begin{aligned}0 \neq \mathrm{ad}_a^{2m-1}(k) &= \mathrm{ad}_a^{2m-2}\mathrm{ad}_a(k) = \mathrm{ad}_{a^2}^{m-1}\mathrm{ad}_a(k) = \\ &= \binom{m-1}{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}}(a^m k a^{m-1} - (-1)^{|k|}a^{m-1}ka^m) = \\ &= \binom{m-1}{r-1}(-1)^{r-1}(a^{2r-1}ka^{2r-2} - (-1)^{|k|}a^{2r-2}ka^{2r-1}).\end{aligned}$$

Therefore, since R is free of $\binom{m-1}{r-1}$-torsion, $a^{2r-1} \neq 0$ so a is nilpotent of index $2r$. So $n \equiv_8 6$ and with $s = \frac{n}{2}$, $a^{s+1} = 0$, $a^s \neq 0$ and by Proposition 3.1.1 a is ad-nilpotent of R of index n, case (5).

- If $\mathrm{ad}_a^{2m-1}(K) = 0$, then a is ad-nilpotent of K of index $n = 2m-1$. Let x be any homogeneous element in R and let us consider $ax + (-1)^{|x|}x^*a \in K_{1+|x|}$:

$$\begin{aligned}0 &= \mathrm{ad}_a^{2m-1}(ax + (-1)^{|x|}x^*a) = \mathrm{ad}_a^{2m-2}\mathrm{ad}_a(ax + (-1)^{|x|}x^*a) = \\ &= \mathrm{ad}_{a^2}^{m-1}\mathrm{ad}_a(ax + (-1)^{|x|}x^*a) = \\ &= \binom{m-1}{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}}a^{m-1}(a^2x + (-1)^{|x|}ax^*a - (-1)^{1+|x|}(axa + (-1)^{|x|}x^*a^2))a^{m-1} = \\ &= \binom{m-1}{r-1}(-1)^{r-1}a^{2r-2}(a^2x + (-1)^{|x|}ax^*a - (-1)^{1+|x|}(axa + (-1)^{|x|}x^*a^2))a^{2r-2} = \\ &= \binom{m-1}{r-1}(-1)^{r-1+|x|}a^{2r-1}(x^* + x)a^{2r-1},\end{aligned}$$

and

$$\begin{aligned}0 &= \mathrm{ad}_a^{2m-1}(x - x^*)a = \mathrm{ad}_a^{2m-2}\mathrm{ad}_a(x - x^*)a = \mathrm{ad}_{a^2}^{m-1}\mathrm{ad}_a(x - x^*)a = \\ &= \binom{m-1}{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}}a^{m-1}(ax - ax^* - (-1)^{|x|}(xa - x^*a))a^m = \\ &= \binom{m-1}{r-1}(-1)^{r-1}a^{2r-2}(ax - ax^* - (-1)^{|x|}(xa - x^*a))a^{2r-1} = \\ &= \binom{m-1}{r-1}(-1)^{\frac{m-1}{2}}a^{2r-1}(x - x^*)a^{2r-1}.\end{aligned}$$

Therefore, since R is free of $\binom{m-1}{r-1}$-torsion, $a^{2r-1}Ra^{2r-1} = 0$, and by semiprimeness of R, $a^{2r-1} = 0$. So in this case $n \equiv_8 5$. For $s = \frac{n+1}{2}$ we have that $a^s = 0$, $a^{s-1} \neq 0$ and, by Proposition 3.1.1, a is an ad-nilpotent element of R of index n, case (4). $\square$

Chapter 4

Examples of ad-nilpotent elements

In this chapter we are going to construct examples of all types of ad-nilpotent elements appearing in Theorems 2.3.6 and 2.2.4 (non-super setting), and all types of ad-nilpotent homogeneous elements appearing in Theorem 3.1.2, and in Theorems 3.2.4 and 3.2.5. The examples of even ad-nilpotent elements of R and of K are based on the examples of ad-nilpotent elements in the non-super setting, see [11]; here we have rewritten those examples to have one example for both non-super and super setting.

4.0.1. Let Φ be a ring of scalars and let r, s be natural numbers. Following the notation of [46], the matrix algebra $\mathcal{M}_{r+s}(\Phi)$ with

$$\mathcal{M}(r|s)_0 := \left\{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} : A \in \mathcal{M}_r(\Phi), D \in \mathcal{M}_s(\Phi) \right\} \text{ and}$$
$$\mathcal{M}(r|s)_1 := \left\{ \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} : B \in \mathcal{M}_{r,s}(\Phi), C \in \mathcal{M}_{s,r}(\Phi) \right\}$$

becomes an $\mathbb{Z}_2$-graded associative algebra. It will be denoted $\mathcal{M}(r|s) = \mathcal{M}(r|s)_0 + \mathcal{M}(r|s)_1$. We will use the notation $\mathcal{M}(r) = \mathcal{M}(r|r)$.

4.0.2. Let r and s be two natural numbers with odd $r > 1$ and even s, let $\mathbb{F}$ be a field with involution (a second-order automorphism) denoted by $\overline{\alpha}$ for any $\alpha \in \mathbb{F}$, and let

R be the superalgebra $\mathcal{M}(r|s)$ over $\mathbb{F}$. Let $\{e_{i,j}\}$ denote the matrix units, and define

$$H = \sum_{i=1}^{r}(-1)^i e_{i,r+1-i} \in \mathcal{M}_r(\mathbb{F}) \quad \text{(notice } H = H^t = H^{-1}\text{)}$$
$$J = \sum_{i=1}^{s}(-1)^i e_{i,s+1-i} \in \mathcal{M}_s(\mathbb{F}) \quad \text{(notice } J^t = -J = J^{-1}\text{).}$$

The map $* : R \to R$ given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} H & 0 \\ 0 & J \end{bmatrix}^{-1} \overline{\begin{bmatrix} A & -B \\ C & D \end{bmatrix}}^t \begin{bmatrix} H & 0 \\ 0 & J \end{bmatrix}$$

defines a superinvolution in R. In particular

$$e^*_{i,j} = (-1)^{j-i} e_{r-j+1,r-i+1} \text{ for every } i, j \in \{1, \dots, r\},$$

$$e^*_{r+i,r+j} = (-1)^{j-i} e_{r+s-j+1,r+s-i+1} \text{ for every } i, j \in \{1, \dots, s\} \text{ and}$$

$$e^*_{i,r+j} = (-1)^{i-j+1} e_{r+s+1-j,r+1-i} \text{ for every } i \in \{1, ..., r\} \text{ and } j \in \{1, ..., s\}.$$

Notice the superinvolution restricted to R_0 is an involution $\star$ such that $K_0 = \mathrm{Skew}(R, *)_0 = \mathrm{Skew}(R_0, \star)$.

The associative superalgebra R is a simple superalgebra with superinvolution, and its extended centroid $C(R)$, which coincides with $Z(R)$, is isomorphic to $\mathbb{F}$. Moreover, the restriction of the superinvolution $*$ to $Z(R)$ is isomorphic to the involution $-$ of $\mathbb{F}$.

4.1 Examples in the non-super setting and of even ad-nilpotent elements of R^- and of K.

Let k be an even number($k \geq 2$), let $r = 3k + 3$ and $s = 2k$, and let us consider the associative superalgebra $R = \mathcal{M}(r|s)$ over $\mathbb{F}$ with the superinvolution defined in 4.0.2. Let us denote by K the skew-symmetric elements of R with respect to $*$. Consider the following nilpotent matrices:

$$
\begin{aligned}
T &:= \sum_{i=k+2}^{2k+1} e_{i,i+1} \in R_0 \text{ (nilpotent of index } k+1)\\
S &:= \sum_{i=1}^{k-1} (e_{i,i+1} + e_{r-i,r-i+1}) \in R_0 \text{ (nilpotent of index } k)\\
U &:= \sum_{i=1}^{k-1} e_{r+i,r+i+1} + \sum_{i=k+1}^{2k-1} e_{r+i,r+i+1} \in R_0 \text{ (nilpotent of index } k).
\end{aligned}
$$

By Proposition 3.1.1(1), T is ad-nilpotent of R and of R_0 of index $2k+1$, and S and U are ad-nilpotent elements of R and of R_0 of index $2k-1$.

Notice that $T^* = -T$, $S^* = -S$ and $U^* = -U$ so $T, S, U \in K_0$. Let us calculate their indexes of ad-nilpotence in K:

(a) If $\mathrm{Skew}(\mathbb{F}, -) \neq 0$, by Proposition 3.2.3 the index of ad-nilpotence of T in K coincides with its index of ad-nilpotence in R, i.e., $2k+1$.

(b) If $\mathrm{Skew}(\mathbb{F}, -) = 0$, for any $B = \sum_{i,j} \lambda_{i,j} e_{i,j} \in K$ we have that $\lambda_{2k+2,k+2} = 0$ and $\lambda_{2k+1,k+2} = \lambda_{2k+2,k+3}$, so

$$
\begin{aligned}
&\mathrm{ad}_T^{2k-1}(B) = \binom{2k-1}{k}(T^{k-1}BT^k - T^kBT^{k-1}) =\\
&= \binom{2k-1}{k}((e_{k+2,2k+1} + e_{k+3,2k+2})B(e_{k+2,2k+2}))-\\
&- \binom{2k-1}{k}(e_{k+2,2k+2})B(e_{k+2,2k+1} + e_{k+3,2k+2})) =\\
&= \binom{2k-1}{k}(\lambda_{2k+1,k+2}\, e_{k+2,2k+2} + \lambda_{2k+2,k+2}\, e_{k+3,2k+2})-\\
&- \binom{2k-1}{k}(\lambda_{2k+2,k+2}\, e_{k+2,2k+1} + \lambda_{2k+2,k+3}\, e_{k+2,2k+2}) = 0.
\end{aligned}
$$

Furthermore,

$$
\mathrm{ad}_T^{2k-2}(e_{2k+1,k+2} - e^*_{2k+1,k+2}) = \mathrm{ad}_T^{2k-2}(e_{2k+1,k+2} + e_{2k+2,k+3}) \neq 0.
$$

Thus T is ad-nilpotent of K of index $2k-1$.

(c) S is ad-nilpotent of K of index $2k-1$: by its ad-nilpotence in R, we have $\mathrm{ad}_S^{2k-1}(K)=0$. Moreover, $0\neq C=e_{k,1}-e_{k,1}^*=e_{k,1}+e_{r,r-k+1}\in K$ and

$$\begin{aligned}\mathrm{ad}_S^{2k-2}(C)&=-\binom{2k-2}{k-1}S^{k-1}(e_{k,1}+e_{r,r-k+1})S^{k-1}=\\&=-\binom{2k-2}{k-1}(e_{1,k}+e_{r-k+1,r})(e_{k,1}+e_{r,r-k+1})(e_{1,k}+e_{r-k+1,r})=\\&=-\binom{2k-2}{k-1}(e_{1,k}+e_{r-k+1,r})\neq 0,\end{aligned}$$

so S is also ad-nilpotent of K of index $2k-1$.

(d) U is ad-nilpotent of K of index $2k-1$: by its ad-nilpotence in R, we have $\mathrm{ad}_U^{2k-1}(K)=0$. Moreover, $0\neq C=e_{r+k,r+1}-e_{r+k,r+1}^*=e_{r+k,r+1}+e_{r+2k,r+k+1}\in K$ and

$$\begin{aligned}\mathrm{ad}_U^{2k-2}(C)&=\mathrm{ad}_U^{2k-2}(e_{r+k,r+1}+e_{r+2k,r+k+1})=\\&=-\binom{2k-2}{k-1}U^{k-1}(e_{r+k,r+1}+e_{r+2k,r+k+1})U^{k-1}=\\&=-\binom{2k-2}{k-1}(e_{r+1,r+k}+e_{r+k+1,r+2k})\neq 0.\end{aligned}$$

Let us use these matrices T, S and U to get examples of any of models of ad-nilpotent elements in Theorems 2.3.6 and 2.2.4 from non-super setting and of even ad-nilpotent elements in Theorems 3.1.2 and 3.2.4. Here is important to point out in Theorems 3.1.2 and 3.2.4 we gave the index of ad-nilpotency of R_0 and K_0 aswell, therefore if an even element is ad-nilpotent of R or K it will be always ad-nilpotent of the same index of R_0 and K_0 but in the case $n\equiv_4 0$ and ad-nilpotent of K then could be of index $n-1$ of K_0. Thus, we will give examples of even homogeneous elements ad-nilpotent of R and K and will give examples for non-super setting ad-nilpotent of R_0 and K_0.

(i). Suppose $\mathrm{Skew}(\mathbb{F},-)\neq 0$. For any $\lambda\in\mathrm{Skew}(\mathbb{F},-)$, the element $T+\lambda\mathrm{id}$ is ad-nilpotent of R of index $2k+1$, and by Proposition 3.2.3 its index in K is again $n=2k+1$. This is an example that fits case (2) of Theorem 3.2.4 and of Theorem 2.3.6 (a skew element a in K_0 with nilpotent $(a-\lambda)$ of index $k+1$ such that a is ad-nilpotent of index $n \equiv_4 1$ in K, K_0 and the same index in R). It also provides an example of case (1) in Theorem 3.1.2 and of Theorem 2.2.4.

(ii). Suppose $\mathrm{Skew}(\mathbb{F},-) \neq 0$. For any $\lambda \in \mathrm{Skew}(\mathbb{F},-)$, $S + \lambda \mathrm{id}$ is an ad-nilpotent element of R and of K of index $n = 2k-1$. This is an example that fits case (1) of Theorem 3.1.2 and case (4.2) of Theorem 3.2.4 and of Theorem 2.3.6 (a skew element in K_0, which is ad-nilpotent of index $n \equiv_4 3$ in K_0 and in K, and ad-nilpotent of the same index in R and R_0).

(iii). Suppose $\mathrm{Skew}(\mathbb{F},-) = 0$. T is an element of K_0 which is ad-nilpotent of K of index $n = 2k-1$. This is an example that fits case (4.1) of Theorem 3.2.4 (an element in K_0 which is ad-nilpotent of index $n \equiv_4 3$ in K and in K_0, and ad-nilpotent of index $n+2$ in R and R_0).

(iv). Suppose $\mathrm{Skew}(\mathbb{F},-) = 0$. The matrix $A = T + S$, which is an orthogonal sum of T and S, is nilpotent of index $t+1$ and ad-nilpotent of R and of R_0 of index $2k+1$. Let us see that it is ad-nilpotent of K of index $2k$: from the indexes of nilpotence of T and S, their indexes of ad-nilpotence in K and the fact that $TS = 0 = ST$ we get that $\mathrm{ad}_A^{2k}(K) = 0$. Moreover, $C = e_{k,k+2} - e^*_{k,k+2} = e_{k,k+2} - e_{2k+2,2k+4} \in K$ and one can check that $\mathrm{ad}_A^{2k-1}(C) = -\binom{2k-1}{k}(e_{1,2k+2} + e_{k+2,3k+3}) \neq 0$. This is an example that fits case (1) of Theorem 3.2.4 (a skew element in K_0 which is ad-nilpotent of index $n \equiv_4 0$ in K_0 and in K, and ad-nilpotent of index $n+1$ in R and R_0).

(v). Suppose $\mathrm{Skew}(\mathbb{F},-) = 0$. Let us consider $A = T + U$, which is an orthogonal sum of T and U. The nilpotence of $T + U$ implies that the index of ad-nilpotence of A in R (and in R_0) is $2k+1$ (by Proposition 3.1.1(1)). Since both T and U are ad-nilpotent elements of K_0 of indexes $2k-1$, A is ad-nilpotent of K_0 of index $2k-1$. Nevertheless, its index of ad-nilpotence in K is higher: for any $B = \sum \lambda_{i,j} e_{i,j} \in K$ we have that

$$\begin{aligned} \mathrm{ad}_A^{2k}(B) &= \binom{2k}{k} A^k B A^k = \binom{2k}{k} e_{k+2,2k+2}\, B\, e_{k+2,2k+2} = \\ &= \binom{2k}{k} \lambda_{2k+2,k+2} e_{k+2,2k+2} = 0 \end{aligned}$$

because $\lambda_{2k+2,k+2} = 0$. Moreover, if we consider the element $C = e_{2k+2,r+1} -$

$e^*_{2k+2,r+1} = e_{2k+2,r+1} - e_{r+s,k+2} \in K$ one can check that

$$\begin{aligned}\mathrm{ad}_A^{2k-1}(C) = \binom{2k-1}{k}(A^{k-1}CA^k - A^kCA^{k-1}) = \\ = -\binom{2k-1}{k}(e_{r+k+1,2k+2} + e_{k+2,r+k}) \neq 0\end{aligned}$$

because

$$A^{k-1} = T^{k-1} + U^{k-1} = e_{k+2,2k+1} + e_{k+3,2k+2} + e_{r+1,r+k} + e_{r+k+1,r+s}.$$

This means that the index of ad-nilpotence of A in K is $n = 2k$. This gives an example of an element in the conditions of Theorem 3.2.4 (1) and a case, again, (4.1) of Theorem 2.3.6 (a skew element in K_0, which ad-nilpotent of K of index $n \equiv_4 0$, ad-nilpotent of K_0 of index $n-1$, and ad-nilpotent of R index $n+1$).

4.2 Examples of odd ad-nilpotent elements of R^- and of K.

Let $\mathbb{F}$ be a field with identity involution, let $r > 1$ be an odd number, let $s = r-1$, and consider the superalgebra $R = \mathcal{M}(r|s)$ with the superinvolution given in 4.0.2. Again, let us denote by K the skew-symmetric elements of R with respect to $*$.

Let us consider $T := \sum_{i=1}^{r-1} e_{i,r+i} \in R_1$. Then

$$A = T - T^* = \sum_{i=1}^{r-1} e_{i,r+i} + \sum_{i=2}^{r} e_{r+i-1,i} \in K_1 \text{ (nilpotent of index } 2r-1).$$

We have that

$$A^2 = \sum_{i=1}^{r-1} e_{i,i+1} + \sum_{i=2}^{r-1} e_{r+i-1,r+i},$$

$$A^{2r-7} = e_{1,2r-3} + e_{2,2r-2} + e_{3,2r-1} + e_{r+1,r-2} + e_{r+2,r-1} + e_{r+3,r},$$

$$A^{2r-6} = e_{1,r-2} + e_{2,r-1} + e_{3,r} + e_{r+1,2r-2} + e_{r+2,2r-1},$$

$$A^{2r-3} = e_{1,2r-1} + e_{r+1,r},$$

$$A^{2r-2} = e_{1,r} \text{ and}$$

$$A^{2r-1} = 0.$$

By Proposition 3.1.1(2b) A is ad-nilpotent in R of index $m = 4r-3$. For every $B = \sum_{i,j} \lambda_{i,j} e_{i,j} \in K_0 \cup K_1$,

$$\begin{aligned}
&\mathrm{ad}_A^{4r-5}(B) = \mathrm{ad}_{A^2}^{2r-3}\mathrm{ad}_A(B) = \\
&= \binom{2r-3}{r-1}((A^2)^{r-2}\mathrm{ad}_A(B)(A^2)^{r-1} - (A^2)^{r-1}\mathrm{ad}_A(B)(A^2)^{r-2}) = \\
&= \binom{2r-3}{r-1}(A^{2r-3}BA^{2r-2} + (-1)^{|B|}A^{2r-2}BA^{2r-3}) = \\
&= \binom{2r-3}{r-1}((e_{1,2r-1} + e_{r+1,r})Be_{1,r} + (-1)^{|B|}e_{1,r}B(e_{1,2r-1} + e_{r+1,r})) = \\
&= \binom{2r-3}{r-1}(\lambda_{2r-1,1}e_{1,r} + \lambda_{r,1}e_{r+1,r} + (-1)^{|B|}\lambda_{r,1}e_{1,2r-1} + (-1)^{|B|}\lambda_{r,r+1}e_{1,r}) = 0
\end{aligned}$$

because when $B \in K_0$ we always have that $\lambda_{2r-1,1} = \lambda_{r,r+1} = 0$ (by grading) and $\lambda_{r,1} = 0$, and when $B \in K_1$, $\lambda_{r,1} = 0$ (by grading) and $\lambda_{2r-1,1} = \lambda_{r,r+1}$. Moreover, by Theorem 3.2.5, the index of ad-nilpotence of A in K can be m, $m-1$ or $m-2$, so it is $m-2 = 4r-5$.

(i). The element $A \in K_1$ is an example of an element in the conditions of Theorem 3.2.5(6) (a nilpotent element of index $2r-1$, which is ad-nilpotent of index $n = 4r-5 \equiv_8 7$ in K and ad-nilpotent of index $n+2$ in R, and such that $A^{2r-3}BA^{2r-2} + (-1)^{|B|}A^{2r-2}BA^{2r-3} = 0$ for every $B \in K_0 \cup K_1$).

To produce examples for the rest of the cases of Theorem 3.2.5, let us consider $A^5 \in K_1$ for some particular cases of odd $r > 1$.

(ii). Fix $r = 10t+1$ for some $t \in \mathbb{N}$. Then

$$(A^5)^{4t+1} = A^{2r+3} = 0,$$

$$(A^5)^{4t} = A^{2r-2} \in H_0,$$

$$(A^5)^{4t-1} = A^{2r-7}.$$

In particular, A^5 is nilpotent of index $4t+1$ and ad-nilpotent of R of index $8t+1$. Notice that for every $B = \sum_{i,j} \lambda_{i,j} e_{i,j} \in K$

$$(A^5)^{4t} B (A^5)^{4t} = e_{1,r} B e_{1,r} = \lambda_{r,1} e_{1,r} = 0$$

because every $B \in K$ has $\lambda_{r,1} = 0$. Therefore, for every $B \in K$ we have

$$\mathrm{ad}_{A^5}^{8t}(B) = \mathrm{ad}_{A^{10}}^{4t}(B) = \binom{4t}{2t} (A^{10})^{2t} B (A^{10})^{2t} = 0.$$

Furthermore, considering $C = e_{r,r+1} - e^*_{r,r+1} = e_{r,r+1} + e_{2r-1,1} \in K_1$

$$\begin{aligned}
&\mathrm{ad}_{A^5}^{8t-1}(C) = \mathrm{ad}_{A^5}^{8t-2}(\mathrm{ad}_{A^5}(e_{r,r+1} + e_{2r-1,1})) = \\
&= \mathrm{ad}_{A^{10}}^{4t-1}(\mathrm{ad}_{A^5}(e_{r,r+1} + e_{2r-1,1})) = \\
&= \binom{4t-1}{2t} (A^{10})^{2t-1} (\mathrm{ad}_{A^5}(e_{r,r+1} + e_{2r-1,1})) (A^{10})^{2t} - \\
&- \binom{4t-1}{2t} (A^{10})^{2t} (\mathrm{ad}_{A^5}(e_{r,r+1} + e_{2r-1,1})) (A^{10})^{2t-1} = \\
&= \binom{4t-1}{2t} \left(A^{20t-5} (e_{r,r+1} + e_{2r-1,1}) A^{20t}) - (A^{20t} (e_{r,r+1} + e_{2r-1,1}) A^{20t-5} \right) = \\
&= \binom{4t-1}{2t} (e_{3,r} - e_{1,r-2}) \neq 0.
\end{aligned}$$

The element A^5 gives an example of an element in the conditions of Theorem 3.2.5(1) (a nilpotent element of index $4t+1$, ad-nilpotent element in K_1 of index $n = 8t \equiv_8 0$, ad-nilpotent in R of index $n+1 = 8t+1$ and such that $(A^5)^{4t} K (A^5)^{4t} = 0$).

(iii). Fix $r = 10t + 3$ for some $t \in \mathbb{N}$. Then

$$(A^5)^{4t+1} = A^{2r-1} = 0$$

$$(A^5)^{4t} = A^{2r-6}.$$

In particular, A^5 is nilpotent of index $4t+1$ and ad-nilpotent of R of index $8t+1$ (see Proposition 3.1.1(2b)). In this case the index of ad-nilpotence of A^5 in K is the same as in R because for $C = e_{r,r+1} - e^*_{r,r+1} = e_{r,r+1} + e_{2r-1,1} \in K_1$ we have

$$\begin{aligned}\mathrm{ad}_{A^5}{}^{8t}(C) &= \mathrm{ad}^{4t}_{A^{10}}(e_{r,r+1} + e_{2r-1,1}) = \\ &= \binom{4t}{2t}(A^{10})^{2t}(e_{r,r+1} + e_{2r-1,1})(A^{10})^{2t} = \\ &= \binom{4t}{2t}(e_{3,2r-2} + e_{r+2,r-2}) \neq 0.\end{aligned}$$

The element A^5 gives an example of an element in the conditions of Theorem 3.2.5(2) (a nilpotent element in K_1 of index $4t+1$, ad-nilpotent of K and of R of the same index $n = 8t+1 \equiv_8 1$).

(iv). Fix $r = 10t + 5$ for some $t \in \mathbb{N}$. Then A^5 is nilpotent of index $4t + 2$. Since the index of nilpotence of A^5 is even, we know by Proposition 3.1.1(2a) that A^5 is ad-nilpotent of R of index $2(4t + 2) - 2 = 8t + 2$. Moreover, from the fact that A^5 is ad-nilpotent of R of index $8t + 2 \equiv_8 2$ we get from Theorem 3.2.5 that its index of ad-nilpotence in K is the same as in R. The element A^5 gives an example of an element in the conditions of Theorem 3.2.5(3) with $\lambda = 0$ (a nilpotent element of K_1 of index $4t+2$ which is ad-nilpotent of K and of R of the same index $n = 8t+2 \equiv_8 2$.)

(v). Fix $r = 10t + 7$ for some $t \in \mathbb{N}$. Then A^5 is nilpotent of index $4t + 3$. Since the index of nilpotence of A^5 is odd, we know by Proposition 3.1.1(2a) that A^5 is ad-nilpotent of R of index $2(4t + 3) - 1 = 8t + 5$. Moreover, from the fact that A^5 is ad-nilpotent of R of index $8t + 5 \equiv_8 5$ we get from Theorem 3.2.5 that its index of ad-nilpotence in K is the same as in R. The element A^5 gives an example of an element in the conditions of Theorem 3.2.5(4) (a nilpotent element of K_1 of index $4t + 3$ which is ad-nilpotent of K and of R of the same index $n = 8t + 5 \equiv_8 5$).

(vi). Fix $r = 10t+9$ for some $t \in \mathbb{N}$. Then A^5 is nilpotent of $4t+4$. Since the index of nilpotence of A^5 is even, we know by Proposition 3.1.1(2a) that A^5 is ad-nilpotent of R of index $2(4t+4)-2 = 8t+6$. Moreover, from the fact that A^5 is ad-nilpotent of R of index $8t + 6 \equiv_8 6$ we get from Theorem 3.2.5 that its index of ad-nilpotence in K is the same as in R. The element A^5 gives an example of an element in the conditions of Theorem 3.2.5(5) with $\lambda = 0$ (a nilpotent element of K_1 of index $4t+4$

which is ad-nilpotent of K and of R of the same index $n = 8t + 6 \equiv_8 6$).

The matrices given in **(i)**, **(ii)**, **(iii)** and **(v)** provide examples of (2.a) in Theorem 3.1.2. Moreover, the matrices of **(iv)** and **(vi)** fit in case (2.b) of Theorem 3.1.2 with $\lambda = 0$.

4.2.1. Some other examples of odd ad-nilpotent elements of K and of R.

The examples **(iv)** and **(vi)** in the previous section are ad-nilpotent elements of K of indexes $n \equiv_8 2$ and $n \equiv_8 6$, and fit in Theorem 3.2.5(3) and (5) with $\lambda = 0$. To get examples of such types of elements with nonzero λ's, we will work with matrices over a field with nontrivial involution.

Let r be a natural number, let $\mathbb{C}$ be the field of complex numbers with involution given by conjugation, and let us consider the simple superalgebra $R = \mathcal{M}(r)$ over $\mathbb{C}$. The map trp given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{trp} = \begin{bmatrix} D^t & -B^t \\ C^t & A^t \end{bmatrix},$$

where $A, B, C, D \in \mathcal{M}_r(\mathbb{C})$ and $(\ )^t$ denotes the usual matrix transposition, defines a superinvolution in R known as the transpose superperinvolution (see [36, Example 2.2]).

Let us denote by K the set of skew-symmetric elements of $\mathcal{M}(r)$ with respect trp. Note that any element of K_1 has the form $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ where B is a symmetric matrix and C is a skew-symmetric matrix in $\mathcal{M}_r(\mathbb{C})$ with respect to the usual transposition.

Let us consider a symmetric matrix $B \in \mathcal{M}_r(\mathbb{C})$ with $B^r = 0$ and $B^{r-1} \neq 0$ (it is shown in [51, Corollary 5] that for every r there exist symmetric nilpotent matrices in $\mathcal{M}_r(\mathbb{C})$ of rank $r - 1$). Let $0 \neq \lambda \in \mathbb{R}$ and let i denote the square root of -1. Then

$$a = \begin{bmatrix} 0 & B + \mathrm{id} \\ (\lambda i)\mathrm{id} & 0 \end{bmatrix} \in K_1 \text{ and } a^2 = \begin{bmatrix} (\lambda i)B + (\lambda i)\mathrm{id} & 0 \\ 0 & (\lambda i)B + (\lambda i)\mathrm{id} \end{bmatrix}$$

i.e., $(a^2 - \lambda i)$ is nilpotent of index r.

When r is odd a is an example for Theorem 3.2.5 (3), and when r is even a is an example for Theorem 3.2.5 (5). Both cases are examples of elements of the form

(2.b) of Theorem 3.1.2.

Chapter 5

Local superalgebra of Lie superalgebras at ad-nilpotent elements

This chapter is part of an article that has been published in the journal *Communications in Algebra* and can be found in [30].

In this chapter we extend the ideas of local algebras of Jordan algebras to the super setting, and Jordan superstructures are attached to Lie superalgebras at ad-nilpotent homogeneous elements.

We also generalize in the section 6.3 the notion of subquotient to the Lie superalgebra. It comes attached to an abelian Lie inner ideal of a Lie superalgebra, and it is indeed a Jordan superpair. Moreover, in the particular case of an abelian inner ideal of the form $[a, [a, L]]$, the subquotient agrees with the Jordan superobject obtained in the section 6.2.

The chapter is organized as follows. When a is even, we easily obtain a Jordan superalgebra by using the Grassmann envelope. But when we deal with an odd ad-nilpotent element a of index less than or equal to 4 we first define a triple product in $[a, [a, L]]$, and then we double this triple and change a sign in one of the associated triple products to get a Jordan superpair. We introduce subquotients associated to abelian inner ideals of Lie superalgebras and show that they are Jordan superpairs.

Finally, we show that the Jordan superalgebras/superpairs obtained in the previous section agree with the subquotients associated to abelian inner ideals of the form $[a, [a, L]]$.

In addition, we will assume that $\frac{1}{3} \in \Phi$.

5.0.1. Let $M = M_0 \oplus M_1$ be a supermodule over Φ. Then the associative algebra $\mathrm{End}(M)$ is provided with the induced $\mathbb{Z}_2$-grading $\mathrm{End}(M) = \mathrm{End}(M)_0 \oplus \mathrm{End}(M)_1$, in which

$$\mathrm{End}(M)_i = \{f \in \mathrm{End}(M) \mid f(M_j) \subseteq M_{i+j}\}.$$

Let $L = L_0 \oplus L_1$ be a Lie superalgebra over Φ then $\mathrm{End}(L)$ becomes an associative superalgebra and $(\mathrm{End}(L))^-$ with product $[f, g] = fg - (-1)^{|f||g|} gf$ for homogeneous elements $f, g \in \mathrm{End}(L)$ becomes a Lie superalgebra. The set $\mathrm{ad}\, L$ of adjoint maps is a Lie superideal of $(\mathrm{End}(L))^-$, so if we denote by capital letters the adjoint maps associated to elements, i.e., $A = \mathrm{ad}_a$, $B = \mathrm{ad}_b$, etc., we have $[A, B] = AB - (-1)^{|a||b|} BA$ for homogeneous elements $a, b \in L_0 \cup L_1$. This notation will be useful because it allows us to think in an associative way when we are doing calculations.

5.1 A Jordan superalgebra at an even homogeneous ad-nilpotent element

5.1.1. Let $L = L_0 + L_1$ be a Lie superalgebra, and let $a \in L_0$ such that $\mathrm{ad}_a^3 L = 0$. Such an element will be called Jordan element of L. In the Φ-module $[a, [a, L]]$ we can define a new product

$$[a, [a, x]] \cdot [a, [a, y]] = \frac{1}{2}[a, [a, [x, [a, y]]]].$$

The (nonassociative) algebra $([a, [a, L]], \cdot)$ is $\mathbb{Z}_2$-graded with homogeneous parts $[a, [a, L]]_0 = [a, [a, L_0]]$ and $[a, [a, L]]_1 = [a, [a, L_1]]$. The parity of an homogeneous element $\bar{x}$ coincides with the parity of x as an element in the Lie superalgebra L, i.e., $|\bar{x}| = |x|$ for every homogeneous element $x \in L_0 \cup L_1$. In the next proposition we prove that this superalgebra is in fact a Jordan superalgebra.

Proposition 5.1.2. *Let $L = L_0 + L_1$ be a Lie superalgebra and $a \in L_0$ be a Jordan element. Then $([a,[a,L]],\cdot)$ is a Jordan superalgebra.*

Proof. Let us check that the Grassmann envelope of $[a,[a,L]]$ is a Jordan algebra with the induced product. Let us consider $\tilde{a} = a \otimes 1 \in G(L)$, which is a Jordan element of the Lie algebra $G(L)$. By Theorem [24, 2.4(ii)] and Remark [24, 2.44] we can consider the Jordan algebra $[\tilde{a},[\tilde{a},G(L)]]$ of $G(L)$ at $\tilde{a}$ with product

$$[\tilde{a},[\tilde{a},\tilde{x}]] \cdot [\tilde{a},[\tilde{a},\tilde{y}]] = \frac{1}{2}[\tilde{a},[\tilde{a},[\tilde{x},[\tilde{a},\tilde{y}]]]]$$

for any $\tilde{x},\tilde{y} \in G(L)$.

The map $\varphi : [\tilde{a},[\tilde{a},G(L)]] \to G([a,[a,L]])$ given by $\varphi([\tilde{a},[\tilde{a},x\otimes \xi_{i_1}\xi_{i_2}\dots\xi_{i_k}]]) = [a,[a,x]] \otimes \xi_{i_1}\xi_{i_2}\dots\xi_{i_k}$ is an isomorphism, so $G([a,[a,L]])$ is a Jordan algebra, giving that $[a,[a,L]]$ is a Jordan superalgebra. □

Remark 5.1.3. The induced triple product on $[a,[a,L]]$ is given by

$$\{\bar{x},\bar{y},\bar{z}\} = (-1)^{|y||z|}\frac{1}{4}A^2XZA^2(y)$$

for homogeneous $\bar{x},\bar{y},\bar{z} \in [a,[a,L]]$. Indeed,

$$\begin{aligned}
4\,\{\bar{x},\bar{y},\bar{z}\} &= 4\left(\bar{x}\cdot(\bar{y}\cdot\bar{z}) + (-1)^{|y||z|+|x||z|}\bar{z}\cdot(\bar{x}\cdot\bar{y}) - (-1)^{|x||y|}\bar{y}\cdot(\bar{x}\cdot\bar{z})\right) = \\
&= A^2[x,[a,[y,[a,z]]]] + (-1)^{|y||z|+|x||z|}A^2[z,[a,[x,[a,y]]]] - \\
&- (-1)^{|x||y|}A^2[y,[a,[x,[a,z]]]] = \\
&= \overline{[[x,a],[[y,a],z]]} + (-1)^{|y||z|+|x||z|}\overline{[[z,a],[[x,a],y]]} - (-1)^{|x||y|}\overline{[[y,a],[[x,a],z]]} = \\
&= A^2[[[x,a],[y,a]],z] + (-1)^{|x||y|+|y||z|+|x||z|}A^2[z,[a,[[y,a],x]]] = \\
&= A^2[[x,[a,[y,a]]],z] = (-1)^{|z|(|x|+|y|)}A^2[z,[x,[a,[a,y]]]] = (-1)^{|y||z|}A^2XZA^2(y).
\end{aligned}$$

Remark 5.1.4. An equivalent construction of the Jordan superalgebra $([a,[a,L]],\cdot)$

is the following: in L define a new product by $x \bullet y = \frac{1}{2}[x,[a,y]]$ for any $x, y \in L$, and denote $L^{(a)}$ the (nonassociative) $\mathbb{Z}_2$-graded algebra $(L, \bullet)$, with $L_0^{(a)} = L_0$ and $L_1^{(a)} = L_1$. If we define $\mathrm{Ker}_L(a) := \{x \in L | [a,[a,x]] = 0\}$, then $\mathrm{Ker}_L(a)$ is the kernel of the $\mathbb{Z}_2$-graded algebra homomorphism $\varphi : L^{(a)} \to [a,[a,L]]$ given by $\varphi(x) = [a,[a,x]]$, so $L^{(a)}/\mathrm{Ker}_L(a)$ and $[a,[a,L]]$ are isomorphic as Jordan superalgebras.

5.2 Jordan superalgebras at odd homogeneous ad-nilpotent elements

Now we turn to odd ad-nilpotent elements. Notice that for every homogeneous element $a \in L_1$ we have $\mathrm{ad}_{[a,a]} = 2\mathrm{ad}_a^2$. When dealing with ad-nilpotent elements of L_1 we will require $\mathrm{ad}_a^4 L = 0$. In this case the element $b = [a,a] \in L_0$ verifies $\mathrm{ad}_b^2 = \mathrm{ad}_{[a,a]}^2 = 4\mathrm{ad}_a^4 = 0$.

Remark 5.2.1. Given such an element $a \in L_1$ with $\mathrm{ad}_a^4 = 0$, if we consider the Φ-module $[a,[a,L]]$ and we define the bilinear product as in 5.1.1 ($[a,[a,x]] \cdot [a,[a,y]] = \frac{1}{2}[a,[a,[x,[a,y]]]]$ for every $x, y \in L$) then $[a,[a,L]]$ is $\mathbb{Z}_2$-graded with $[a,[a,L]]_0 = [a,[a,L_1]]$ and $[a,[a,L]]_1 = [a,[a,L_0]]$. The parity of the homogeneous elements of $[a,[a,L]]$ changes and $|[a,[a,x]]| = |x| + 1$ for any homogeneous element $x \in L_0 \cup L_1$. Moreover,

$$\bar{x} \cdot \bar{y} = -(-1)^{|\bar{x}||\bar{y}|} \bar{y} \cdot \bar{x}$$

for homogeneous $\bar{x} = [a,[a,x]], \bar{y} = [a,[a,y]] \in [a,[a,L]]$, i.e., $([a,[a,L]], \cdot)$ is super-anticommutative. To avoid this situation and get a Jordan superstructure, we define a trilinear product on $[a,[a,L]$.

5.2.2. For an element $a \in L_1$ with $\mathrm{ad}_a^4 = 0$, we consider the trilinear map $\{\ ,\ ,\ \}$ on $[a,[a,L]]$ defined by

$$\{\bar{x}, \bar{y}, \bar{z}\} := \frac{1}{4}[[a,[a,x]],[y,[a,[a,z]]]] = \frac{1}{4} A^2 X Y A^2(z) \tag{5.2.1}$$

for every homogeneous $\bar{x} = [a,[a,x]]$, $\bar{y} = [a,[a,y]]$ and $\bar{z} = [a,[a,z]] \in [a,[a,L]]$ (notice that $[[a,[a,x]],[y,[a,[a,z]]]] = \frac{1}{4}[[[a,a],x],[y,[[a,a],z]]] = A^2XYA^2(z)$ because $\mathrm{ad}_{[a,a]}\mathrm{ad}_y\mathrm{ad}_{[a,a]} = 0$). The Φ-module $[a,[a,L]]$ is $\mathbb{Z}_2$-graded with respect to this trilinear product and $[a,[a,L]]_i = [a,[a,L_i]]$, $i \in \{0,1\}$.

We have that

$$[[a,[a,x]],[y,[a,[a,z]]]] = [[[a,[a,x]],y],[a,[a,z]]]$$

for every $x,y,z \in L_0 \cup L_1$ because $[[a,[a,x]],[a,[a,z]]] = \frac{1}{4}[[[a,a],x],[[a,a],z]] = 0$ since $[a,a]$ is an absolute zero divisor. This implies that the triple product is supersymmetric in the outer variables:

$$\{\bar{x},\bar{y},\bar{z}\} = (-1)^{|x||y|+|x||z|+|y||z|}\{\bar{z},\bar{y},\bar{x}\}. \tag{5.2.2}$$

Moreover,

$$\{\bar{x},\bar{y},\bar{z}\} = (-1)^{|y||z|}\{\bar{x},\bar{z},\bar{y}\} \tag{5.2.3}$$

because $4\{\bar{x},\bar{y},\bar{z}\} = [[a,[a,x]],[y,[a,[a,z]]]] = (-1)^{|y||z|+1}[[a,[a,x]],[[a,[a,z]],y]] =$
$= \frac{1}{4}(-1)^{|y||z|+1}[[[a,a],x],[[[a,a],z],y]] = \frac{1}{4}(-1)^{|y||z|}[[a,a],[x,[z,[[a,a],y]]]] =$
$= (-1)^{|y||z|}A^2XZA^2(y) = 4(-1)^{|y||z|}\{\bar{x},\bar{z},\bar{y}\}$. From equations (5.2.2) and (5.2.3) we get that the triple product defined in (5.2.1) is supercommutative on its three variables.

Lemma 5.2.3. *For a homogeneous element $a \in L_1$ with $\mathrm{ad}_a^4 = 0$, the trilinear map given in (5.2.1) satisfies*

$$\begin{aligned}\{\bar{x},\bar{y},\{\bar{z},\bar{u},\bar{v}\}\} = \{\{\bar{x},\bar{y},\bar{z}\},\bar{u},\bar{v}\} &+ (-1)^{|x||y|+|x||z|+|y||z|}\{\bar{z},\{\bar{y},\bar{x},\bar{u}\},\bar{v}\} \\ &+ (-1)^{|x||z|+|x||u|+|y||z|+|y||u|}\{\bar{z},\bar{u},\{\bar{x},\bar{y},\bar{v}\}\} \qquad (*)\end{aligned}$$

for every $x,y,z,u,v \in L_0 \cup L_1$.

Proof. For every $x, y, z, u, v \in L_0 \cup L_1$,

$$
\begin{aligned}
&8\,\{\bar{x},\bar{y},\{\bar{z},\bar{u},\bar{v}\}\} = [[[a,[a,x]],y],[[[a,[a,z]],u],[a,[a,v]]]] = \\
&= [[[[a,[a,x]],y],[[a,[a,z]],u]],[a,[a,v]]] + \\
&+ (-1)^{(|x|+|y|)(|z|+|u|)}[[[a,[a,z]],u],[[[a,[a,x]],y],[a,[a,v]]]] = \\
&= [[[[[a,[a,x]],y],[a,[a,z]]],u],[a,[a,v]]] + \\
&+ (-1)^{(|x|+|y|)|z|}[[[a,[a,z]],[[[a,[a,x]],y],u]],[a,[a,v]]] + \\
&+ (-1)^{(|x|+|y|)(|z|+|u|)}[[[a,[a,z]],u],[[[a,[a,x]],y],[a,[a,v]]]] =
\end{aligned}
$$

$$
\begin{aligned}
&= 8\,\{\{\bar{x},\bar{y},\bar{z}\},\bar{u},\bar{v}\} + (-1)^{(|x|+|y|)|z|}[[[a,[a,z]],[[[a,[a,x]],y],u]],[a,[a,v]]] + \\
&+ (-1)^{|x||z|+|x||u|+|y||z|+|y||u|}8\,\{\bar{z},\bar{u},\{\bar{x},\bar{y},\bar{v}\}\}
\end{aligned}
$$

Let us see that $(-1)^{(|x|+|y|)|z|}[[[a,[a,z]],[[[a,[a,x]],y],u]],[a,[a,v]]]$ coincides with the second term on the right side of equality $(*)$: from the definition of the triple product,

$$
[[[a,[a,z]],[[[a,[a,x]],y],u]],[a,[a,v]]] = 4\,\{\bar{z},\overline{[[[a,[a,x]],y],u]},\bar{v}\}
$$

and

$$
\begin{aligned}
&\overline{[[[a,[a,x]],y],u]} = [a,[a,[[[a,[a,x]],y],u]]] = (-1)^{1+|y|}[[a,[a,x]],[a,y]],[a,u]] + \\
&+ (-1)^{|y|}[[a,[a,x],[a,y]],[a,u]] + [[[a,[a,x]],y],[a,[a,u]]] = \\
&= 4\,\{\bar{x},\bar{y},\bar{u}\} = (-1)^{|x||y|}4\,\{\bar{y},\bar{x},\bar{u}\}
\end{aligned}
$$

hence

$$
\begin{aligned}
(-1)^{(|x|+|y|)|z|}&[[[a,[a,z]],[[[a,[a,x]],y],u]],[a,[a,v]]] = \\
&= (-1)^{|x||y|+|x||z|+|y||z|}8\,\{\bar{z},\{\bar{y},\bar{x},\bar{u}\},\bar{v}\}
\end{aligned}
$$

and we have shown $(*)$. $\square$

5.2.4. A pair of $\mathbb{Z}_2$-graded Φ-modules $V = (V^+, V^-)$ is a (linear) Jordan superpair if there exist two trilinear maps $\{\ ,\ ,\}^\sigma : V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma$, $\sigma = \pm$, both supersymmetric in the outer variables, and that satisfy (JSP15):

$$\{a,b,\{c,d,e\}^\sigma\}^\sigma = \{\{a,b,c\}^\sigma,d,e\}^\sigma - (-1)^{|a||b|+|a||c|+|b||c|}\{c,\{b,a,d\}^{-\sigma},e\}^\sigma \\ + (-1)^{|a||c|+|a||d|+|b||c|+|b||d|}\{c,d,\{a,b,e\}^\sigma\}^\sigma, \quad \sigma = \pm$$

for homogeneous $a, c, e \in V^\sigma$ and homogeneous $b, d \in V^{-\sigma}$, $\sigma = \pm$.

We have just shown that when $a \in L_1$ has $\mathrm{ad}_a^4 = 0$, $[a,[a,L]]$ with the trilinear map $\{,,\}$ given in (5.2.1) is a (1,1)-Jordan supertriple in the sense of [47, §3], which are a particular case (ϵ,δ)-Freudenthal-Kantor supertriple systems, $\epsilon = \pm 1$, $\delta = \pm 1$ [47, §3]. We say that a $\mathbb{Z}_2$-graded Φ-module $M = M_0 + M_1$ with a graded triple product $\{\ ,\ ,\ \} : M \times M \times M \to M$ is a $(1,1)$-Jordan supertriple if

- $\{a,b,c\} = (-1)^{|a||b|+|a||c|+|b||c|}\{c,b,a\}$ and
- $\{a,b,\{c,d,e\}\} = \{\{a,b,c\},d,e\} + (-1)^{|a||b|+|a||c|+|b||c|}\{c,\{b,a,d\},e\}$
  $+ (-1)^{|a||c|+|a||d|+|b||c|+|b||d|}\{c,d,\{a,b,e\}\}$

for homogeneous elements $a, b, c, d, e \in M$. The second identity resembles (JSP15) but there is a change of sign in the second summand of its right side. Notice that every $(1,1)$-Jordan supertriple M with triple product $\{\ ,\ ,\ \}$ gives rise to a Jordan superpair $V = (V^+, V^-) = (M, M)$ with products $\{a,b,c\}^+ := \{a,b,c\}$ and $\{b,c,d\}^- := -\{b,c,d\}$ for every $a, c \in V^+$ and $b, d \in V^-$. In our case we have shown that if we double $[a,[a,L]]$ and twist one of the triple products we have that $([a,[a,L]],[a,[a,L]])$ is a Jordan superpair.

5.2.5. Another Jordan structure can be defined from an ad-nilpotent element $a \in L_1$: suppose that $a \in L_1$ has $\mathrm{ad}_a^6 = 0$. Then $b = [a,a] \in L_0$ is a Jordan element ($\mathrm{ad}_b^3 = (2\mathrm{ad}_a^2)^3 = 0$), and we can define a Jordan superalgebra on the Φ-module

$[b,[b,L]] = \mathrm{ad}_a^4 L$ as in 5.1.2. The product is now given by

$$[b,[b,x]] \cdot [b,[b,y]] = \frac{1}{2}[b,[b,[x,[b,y]]]]$$

or, equivalently,

$$\mathrm{ad}_a^4 x \cdot \mathrm{ad}_a^4 y = 2\mathrm{ad}_a^4[x, \mathrm{ad}_a^2 y].$$

5.3 Subquotients associated to abelian inner ideals of Lie superalgebras

5.3.1. Let $L = L_0 + L_1$ be a Lie superalgebra. We say that $B = B_0 + B_1 \subset L$ is an inner ideal of L if $[B,[B,L]] \subset B$, and B is abelian if $[B,B] = 0$. Inner ideals can be easily produced from homogeneous ad-nilpotent elements.

Example 5.3.2. Let $L = L_0 + L_1$ a Lie superalgebra and let $a \in L_0$ with $\mathrm{ad}_a^3 = 0$ or $a \in L_1$ with $\mathrm{ad}_a^4 = 0$. Then

$$[a] := [a,[a,L]] \qquad (a) := \Phi a + [a,[a,L]]$$

are inner ideals of L. Moreover, $[a]$ is an abelian inner ideal.

Conversely, given an abelian inner ideal $B = B_0 + B_1$, any homogeneous $b \in B_0$ is a Jordan element and gives rise to the inner ideals $[b]$ and (b) contained in B. If $b \in B_1$ then $0 = [b,b]$ implies $0 = \mathrm{ad}_{[b,b]} = 2\mathrm{ad}_b^2$ so $[b] = 0$ and $(b) = \Phi b$.

Proposition 5.3.3. *Let L be a Lie superalgebra and B an abelian inner ideal of L. Let us consider* $\mathrm{Ker}B := \{x \in L \mid [B,[B,x]] = 0\}$. *Then $(B, L/\mathrm{Ker}B)$ is a Jordan superpair with products:*

$$\{a,\overline{x},b\} := [a,[x,b]] = [[a,x],b]$$
$$\{\overline{x},a,\overline{y}\} := \overline{[x,[a,y]]} = \overline{[[x,a],y]}$$

for $a,b \in B$ and $x,y \in L$ (here $\bar{x}$, $\bar{y}$, $\overline{[x,[a,y]]}$ and $\overline{[[x,a],y]}$ denote equivalence *classes in the quotient $L/\mathrm{Ker}B$). This Jordan superpair is called the* subquotient of L associated to B.

Proof. First notice that $[a,[x,b]] = [[a,x],b]$ and $\overline{[x,[a,y]]} = \overline{[[x,a],y]}$ for every $a, b \in B$ and every $x, y \in L$ because B is abelian and the definition of $\mathrm{Ker}B$.

The products are well defined: clearly $\{a, 0, b\} = 0$, and if we take homogeneous $\overline{x}, \overline{y} \in L/\mathrm{Ker}B$ with $\overline{x} = \overline{0}$ or $\overline{y} = \overline{0}$ then for homogeneous $a, b, c \in L$ we have that

$$
\begin{aligned}
&[b,[c,[x,[a,y]]]] = [b,[[c,x],[a,y]]] + (-1)^{|c||x|}[b,[x,[c,[a,y]]]] = \\
&= (-1)^{|c||x|}[[b,x],[c,[a,y]]] + (-1)^{|c||x|+|b||x|}[x,[b,[c,[a,y]]]] = 0
\end{aligned}
$$

Let us see that the triple products are supersymmetric in the outer variables:

$$
\begin{aligned}
\{a,\overline{x},b\} &= [a,[x,b]] = (-1)^{1+|x||b|}[a,[b,x]] = (-1)^{|b||x|+|a|(|b|+|x|)}[[b,x],a]] = \\
&= (-1)^{|b||x|+|a||b|+|a||x|}[b,[x,a]] = (-1)^{|b||x|+|a||b|+|a||x|}\{b,\overline{x},a\} \\
\{\overline{x},a,\overline{y}\} &= \overline{[x,[a,y]]} = \overline{[[x,a],y]} + (-1)^{|x||a|}\overline{[a,[x,y]]} = \\
&= (-1)^{|x||y|+|x||a|+|y||a|}\overline{[y,[a,x]]} = (-1)^{|x||y|+|x||a|+|y||a|}\{y,a,\overline{x}\}
\end{aligned}
$$

Let us prove (JSP15). For homogeneous $a, b, c \in B$ and homogeneous $x, y, z \in L$,

- $\{a,\overline{x},\{b,\overline{y},c\}\} = [[a,x],[[b,y],c]] =$
  $= [[[[a,x],b],y],c] + (-1)^{(|a|+|x|)b}[[b,[[a,x],y]],c] + (-1)^{(|b|+|y|)(|a|+|x|)}[[b,y],[[a,x],c]]$
  $= \{\{a,\overline{x},b\},\overline{y},c\} - (-1)^{|a||b|+|x||b|+|a||x|}\{b,\{\overline{x},a,\overline{y}\},c\}+$
  $+ (-1)^{(|b|+|y|)(|a|+|x|)}\{b,\overline{y},\{a,\overline{x},c\}\}.$
- $\{\overline{x},a,\{\overline{y},b,\overline{z}\}\} = \overline{[[x,a],[[y,b],z]]} =$
  $= \overline{[[[[x,a],y],b],z]} + (-1)^{|y|(|x|+|a|)}\overline{[[y,[[x,a],b]],z]} + (-1)^{(|y|+|b|)(x+a)}\overline{[[y,b],[[x,a],z]]}$
  $= \{\{\overline{x},a,\overline{y}\},b,\overline{z}\} - (-1)^{|y||x|+|y||a|+|a||x|}\{\overline{y},\{a,\overline{x},b\},\overline{z}\}+$
  $+ (-1)^{(|y|+|b|)(|x|+|a|)}\{\overline{y},b,\{\overline{x},a,\overline{z}\}\}$

Therefore, $(B, L/\mathrm{Ker}B)$ is a Jordan superpair. $\square$

Remark 5.3.4. Let $a \in L_0$ be a Jordan element or $a \in L_1$ with $\mathrm{ad}_a^4 = 0$. Then $B = [a] = [a, [a, L]]$ is an abelian inner ideal and we can build the subquotient $([a], L/\mathrm{Ker}[a])$. In this particular case, for homogeneous $x, y, z \in L$ the triple product

$$\begin{aligned}\{[a,[a,x]], y + \mathrm{Ker}[a],\ [a,[a,z]]\} &= [[a,[a,x]],[y,[a,[a,z]]]] = \\ &= (-1)^{|y||z|+1+|a|} A^2 XZA^2(y)\end{aligned}$$

coincides, up to a scalar, with the triple product we have already defined in $[x]$, see Remark 5.1.3 when $[a]$ is even and 5.2.2 when a is odd. In the following result we are going to prove that the Jordan superpair structures defined in this section and in the previous ones coincide.

Corollary 5.3.5. *Let L be a Lie superalgebra, take $a \in L_0$ with $\mathrm{ad}_a^3 = 0$ or $a \in L_1$ with $\mathrm{ad}_a^4 = 0$, and let us consider the subquotient associated to the abelian inner ideal $[a]$.*

(a) When $a \in L_0$, if we consider the Jordan superpair structure induced on $([a,[a,L]],[a,[a,L]])$ by Remark 5.1.3, then the pair of maps

$$(\Psi_1, \Psi_2) : ([a,[a,L]],[a,[a,L]]) \to ([a], L/\mathrm{Ker}[a])$$

given by

$$\Psi_1 = -\frac{1}{2}\mathrm{id} \quad \text{and} \quad \Psi_2([a,[a,x]]) = \frac{1}{2}x + \mathrm{Ker}[a]$$

is an isomorphism of Jordan superpairs.

(b) *When $a \in L_1$, if we consider the Jordan superpair structure defined on* $([a,[a,L]],[a,[a,L]])$ *by5.2.3, then the pair of maps*

$$(\Psi_1, \Psi_2) : ([a,[a,L]],[a,[a,L]]) \to ([a], L/\mathrm{Ker}[a])$$

given by

$$\Psi_1 = \frac{1}{2}\mathrm{id} \quad \textit{and} \quad \Psi_2([a,[a,x]]) = \frac{1}{2}x + \mathrm{Ker}[a]$$

is an isomorphism of Jordan superpairs.

Proof. In both cases, the pair of maps given by

$$\Psi_1([a,[a,x]]) = (-1)^{|a|+1}\frac{1}{2}[a,[a,x]] \in [a], \text{ and}$$
$$\Psi_2([a,[a,x]]) = \frac{1}{2}x + \mathrm{Ker}[a] \in L/\mathrm{Ker}[a],$$

for every $x \in L$, are well defined (if $[a,[a,x]] = [a,[a,y]]$, then $[a,[a,x-y]] = 0$ implies $x - y \in \mathrm{Ker}[a]$). They are clearly bijective. Let us see that they are Jordan superpair homomorphisms.

(a) Suppose that $a \in L_0$ and take homogeneous $x, y, z \in L$.

- $\Psi_1(\{[a,[a,x]],[a,[a,y]],[a,[a,z]]\}) = \Psi_1((-1)^{|y||z|}\frac{1}{4}A^2XZA^2(y)) =$
$= (-1)^{|y||z|+1}\frac{1}{8}A^2XZA^2(y) = \{-\frac{1}{2}[a,[a,x]], \frac{1}{2}y + \mathrm{Ker}[a], -\frac{1}{2}[a,[a,z]]\} =$
$= \{\Psi_1([a,[a,x]]), \Psi_2([a,[a,y]]), \Psi_1([a,[a,z]])\}.$
- $\Psi_2(\{[a,[a,x]],[a,[a,y]],[a,[a,z]]\}) = \Psi_2((-1)^{|y||z|}\frac{1}{4}A^2XZA^2(y)) =$
$= (-1)^{|y||z|}\frac{1}{8}XZA^2(y) + \mathrm{Ker}[a] = -\frac{1}{8}[x,[[a,[a,y]],z]] + \mathrm{Ker}[a] =$
$= \{\frac{1}{2}x + \mathrm{Ker}[a], -\frac{1}{2}[a,[a,y]], \frac{1}{2}z + \mathrm{Ker}[a]\} =$
$= \{\Psi_2([a,[a,x]]), \Psi_1([a,[a,y]]), \Psi_2([a,[a,z]])\}.$

(b) Suppose that $a \in L_1$ and take homogeneous $x, y, z \in L$.

- $\Psi_1(\{[a,[a,x]],[a,[a,y]],[a,[a,z]]\}) = \Psi_1((-1)^{|y||z|}\frac{1}{4}A^2XZA^2(y)) =$
  $= (-1)^{|y||z|}\frac{1}{8}A^2XZA^2(y) = \{\frac{1}{2}[a,[a,x]], \frac{1}{2}y + \mathrm{Ker}[a], \frac{1}{2}[a,[a,z]]\} =$
  $= \{\Psi_1([a,[a,x]]), \Psi_2([a,[a,y]]), \Psi_1([a,[a,z]])\}.$
- $\Psi_2(\{[a,[a,x]],[a,[a,y]],[a,[a,z]]\}) = \Psi_2(-\frac{1}{4}[[[a,[a,x]],y],[a,[a,z]]]) =$
  $= -\frac{1}{4}\Psi_2((-1)^{|y||z|}A^2XZA^2(y)) = \frac{1}{8}(-1)^{1+|y||z|}XZA^2(y) + \mathrm{Ker}[a] =$
  $= \frac{1}{8}[x,[[a,[a,y]],z]] + \mathrm{Ker}[a] = \{\frac{1}{2}x + \mathrm{Ker}[a], \frac{1}{2}[a,[a,y]], \frac{1}{2}z + \mathrm{Ker}[a]\} =$
  $= \{\Psi_2([a,[a,x]]), \Psi_1([a,[a,y]]), \Psi_2([a,[a,z]])\}.$

$\square$

Results and discussion

In this thesis we have studied ad-nilpotent elements belonging to semiprime associative algebras with involution or prime associative superalgebras with superinvolution, and ad-nilpotent elements in Lie superalgebras. First, we have dealt with semiprime associative algebras with involution. In these algebras we have defined the notion of pure ad-nilpotent element; this notion will be a relevant definition throughout Chapter 2 because it will allow us to give a more precise description of such elements, and to weaken the torsion conditions required to the whole algebra.

We have described the pure ad-nilpotent elements belonging to a semiprime associative algebra R with involution $*$ and belonging to $K := \mathrm{Skew}(R, *)$. Indeed, if a is a pure ad-nilpotent element in R of index n, with R free of $\binom{n}{s}$ and s-torsion, with $s = [\frac{n+1}{2}]$, then there exists λ in the extended centroid of R such that $a-\lambda$ is nilpotent of index s. On the other hand, if a is a pure ad-nilpotent element in K of index n, the description of a depends on the equivalence class of n modulo 4, and there are three posibilities: If $n \equiv_4 0$ then the index of ad-nilpotence of a in R is greater than n and there exists a corner of R that satisfies a PI. If $n \equiv_4 0$ then the index of ad-nilpotence of a in R is n and we can conclude that there exists λ in the extended centroid such that $a - \lambda$ is nilpotent. If $n \equiv_4 3$ then a can be descomposed as an orthogonal sum of an ad-nilpotent element of R of index n and another ad-nilpotent element of R of index greater than n. It is important to note that in semiprime associative algebras the extended centroid is not a field, but it is a von Neumann regular ring.

In the next chapter, we have studied homogeneous ad-nilpotent elements in prime associative algebras $R = R_0 + R_1$ with superinvolution $*$. We have started by studying the homogeneous ad-nilpotent elements a of index n in R. If a is even, since R_0 is an algebra, we can use the above description of ad-nilpotent elements in associative algebras. Although it is an almost direct implication of the previous chapter, we have to deepen into the structure of the extended centroid $C(R)$ to ensure that $a-\lambda$ is nilpotent with λ an even element in the extended centroid. On the other hand, if $a \in R_1$, we have focused on a^2. Thus, unlike the descriptions of even elements, two different cases appear: If n is even then $a^2 \in R_0$ is ad-nilpotent of R of index $\frac{n}{2}$ which implies that there exists $\lambda \in C(R)_0$ such that $a^2-\lambda$ is nilpotent of index $\frac{n+2}{4}$. If n is odd then a is ad-nilpotent of R of index $\frac{n+1}{2}$ and hence a is nilpotent of index $\frac{n+1}{2}$.

Continuing in the super setting, we have described the homogeneous ad-nilpotent elements $a \in K := \mathrm{Skew}(R,*)$ of index n of K. Once we have shown that any homogeneous ad-nilpotent element of K is either an ad-nilpotent element of R of the same index or nilpotent, we can describe these elements in depth. This description depends on the parity of the element: In the even case, the proof and the description is strongly supported by the non-super case. While, in the odd case, we will focus on a^2 and thus use the description of even ad-nilpotent elements. More precisely, if $a \in K_1$ is an ad-nilpotent element of K of index n and R has characteristic $p > n$, there are seven possibilities depending on the equivalence class of n modulo 8:

(1) If $n \equiv_8 0$ then a is nilpotent of index $\frac{n}{2}+1$, ad-nilpotent of R of index $n+1$ and $a^{\frac{n}{2}}Ka^{\frac{n}{2}}=0$ (so $a^{\frac{n}{2}}Ra^{\frac{n}{2}}$ is a commutative trivial local superalgebra).

(2) If $n \equiv_8 1$ then $a^{\frac{n-1}{2}} \in H_0$, and a is nilpotent of index $\frac{n+1}{2}$ and ad-nilpotent of R of index n.

(3) If $n \equiv_8 2$ then there exists λ, a skew-symmetric element in the extended centroid, such that $a^2-\lambda$ is nilpotent of index $\frac{n+2}{4}$ and a is ad-nilpotent of R of index n.

(4) If $n \equiv_8 5$ then $a^{\frac{n-1}{2}} \in K_0$, and a is nilpotent of index $\frac{n+1}{2}$ and ad-nilpotent of R of index n.

(5) If $n \equiv_8 6$ then there exists λ, a skew-symmetric element in the extended centroid, such that $a^2-\lambda$ is nilpotent of index $\frac{n+2}{4}$ and a is ad-nilpotent of R of

index n.

(6) If $n \equiv_8 7$ then a is nilpotent of index $\frac{n+1}{2}+1$, ad-nilpotent of R of index $n+2$ and $a^{\frac{n+1}{2}}ka^{\frac{n-1}{2}} + (-1)^{|k|}a^{\frac{n-1}{2}}ka^{\frac{n+1}{2}} = 0$ for every homogeneous $k \in K$ (so $a^{\frac{n+1}{2}}Ra^{\frac{n+1}{2}}$ is a commutative trivial local superalgebra).

(7) The cases $n \equiv_8 3$ and $n \equiv_8 4$ do not occur.

Afterwards, we have given examples of elements fitting these descriptions. Our examples are matrices considered in the superalgebra $\mathcal{M}(r|s)$ over a field with a nontrivial superinvolution. Although these examples are considered in superalgebras, restricting to the even part yields examples in the non-super setting. These examples allow us to ensure that all the cases appearing in our descriptions hold.

Finally, in Chapter 5, given any Lie superalgebra over Φ with $\frac{1}{6} \in \Phi$, we have studied the even ad-nilpotent elements of index 3 and the odd ad-nilpotent elements of index 4. For the even elements it is possible to associate a Jordan superalgebra to the initial Lie superalgebra by transferring to super setting the existing result in Lie and Jordan algebras due to A. Fernández, E. García and M. Gómez Lozano in [24]. However, for odd ad-nilpotent elements of index 4 we have obtained a Jordan superpair. We have also introduced the notion of subquotient of a Lie superalgebra associated to an abelian inner ideal. Furthermore, the subquotient of a Lie superalgebra associated to an abelian inner ideal is a Jordan superpair, generalizing the structure defined above by the homogeneous ad-nilpotent elements.

Future work

We have studied homogeneous ad-nilpotent elements in prime associative superalgebras but we can also study these descriptions in semiprime associative superalgebras. We note that the main difficulty of working on semiprime superalgebras is that the extended centroid drops the property of its elements being invertible with all that this entails. Another possibility could be to study these descriptions for non-homogeneous ad-nilpotent elements.

On the other hand, the subquotients of a Lie superalgebra associated to an abelian inner ideal could be a starting point to study the concept of socle and a Wedderburn-Artin theory for Lie superalgebras following the ideas of C. Draper, A. Fernández, E. García, and M. Gómez Lozano [22]. Some other future research could be to study the relationship between Jordan superstructures and Leibniz superalgebras, as R. Velásquez and R. Felipe have done in the algebra settings [65].

General conclusions

The main conclusions of this thesis can be summarized as follows:

- We have defined the notion of pure ad-nilpotent element in Chapter 2. The extended centroid plays an important role in this definition. It is a technical condition, since every ad-nilpotent element can be expressed as an orthogonal sum of pure ad-nilpotent elements of decreasing indices. Furthermore, this definition allows us to give a more precise description of such elements, and to weaken the torsion conditions required to the whole algebra. For more details we refer the reader to Section 2.1.

- We have proved that for any pure ad-nilpotent element a in a semiprime associative algebra R of index n with R free of $\binom{n}{s}$ and s-torsion, where $s = [\frac{n+1}{2}]$, there exists λ in the extended centroid such that $a - \lambda$ is nilpotent of index s. This fact is proved in the Theorem 2.2.4. We have weakened the conditions of Theorem [54, Theorem 1.3].

- Considering a semiprime associative algebra R with involution $*$ we have described any skew-symmetric pure ad-nilpotent element a of index n depending on n modulo 4: If $n \equiv_4 0$ then the indexes of ad-nilpotence of a in R and K do not coincide and there exists a corner of R satisfying a PI. If $n \equiv_4 1$ then the indexes of ad-nilpotence of a in R and K coincide and there exists λ in the extended centroid such that $a-\lambda$ is nilpotent. If $n \equiv_4 3$ then we can decompose a in an orthogonal sum $a = a_1 + a_2$ such that, if $a_1 \neq 0$, a_1 is ad-nilpotent of R of index n (so there exists λ in the extended centroid such that $a_1 - \lambda$ is nilpotent) and, if $a_2 \neq 0$, a_2 is ad-nilpotent of R of index $n+2$ (therefore there exists a corner of R that satisfies a PI). The case $n \equiv_4 2$ cannot occur. This description has been proved in Theorem 2.3.6.

- In the same spirit as in the non-super setting, we have given descriptions of ad-nilpotent elements in prime associative superalgebras with superinvolution. These descriptions relate the index of ad-nilpotence of a homogeneous element with its nilpotence index. An important remark related with these descriptions is that every ad-nilpotent element has a minimal polinomial in the central closure with one root in the extended centroid. We refer readers to Theorems 3.1.2, 3.2.4, 3.2.5 for more details.

- To conclude our study about ad-nilpotent elements in associative algebras and superalgebras, in Chapter 4, we have given examples of elements appearing in these descriptions. The examples are matrices considered in the associative superalgebra $\mathcal{M}(r|s)$ over a field with a nontrivial superinvolution. Although we have considered a superalgebra, we also provide examples for descriptions of ad-nilpotent elements in associative algebras when we restrict the examples to the even part of the matrices $\mathcal{M}(r|s)$.

- For a Lie superalgebra L with an even ad-nilpotent element a of index 3 we have shown that $([a,[a,L]],\cdot)$ with a new product $\cdot$ defined by $[a,[a,x]]\cdot[a,[a,y]] := \frac{1}{2}[a,[a,[x,[a,y]]$ is a Jordan superalgebra isomorphic to $L_a = L^{(a)}/\mathrm{Ker}_L(a)$ where $L^{(a)} = (L,\bullet)$ with $x \bullet y := [x,[a,y]]$ and $\mathrm{Ker}_L(a) := \{x \in L \mid [a,[a,x]] = 0\}$. This result has been proved in Proposition 5.1.2.

- However, for a Lie superalgebra with an odd ad-nilpotent element the same construction gives a super anticommutative superalgebra, hence it cannot be a Jordan superalgebra. Instead, we have proved that it is possible to construct a Jordan superpair, see 5.2.3.

- Finally, we have defined the subquotient of a Lie superalgebra associated to an abelian inner ideal and we have proved that it is a Jordan superpair (Proposition 5.3.3). Moreover, we have shown that the subquotient corresponds to the construction made before (Corollary 5.3.5).

Resumen de la tesis en castellano

Esta tesis se enmarca en el estudio de los elementos ad-nilpotentes en álgebras y superálgebras asociativas con involución y superinvolución y elementos ad-nilpotentes en superálgebras de Lie. La primera parte encaja con la rama de teoría de Herstein que estudia las derivaciones internas nilpotentes en álgebras. Son muchos los estudios sobre este área, destacando para nuestro trabajo los artículos de W. S. Martindale y C. R. Miers [55], [56] y de T. K. Lee [54]. Posteriormente, en la segunda parte, estudiamos cómo asociar estructuras Jordan a una superalgebra de Lie, siguiendo la idea del artículo de A. Fernández, E. García y M. Gómez Lozano [24].

Objetivos

Se han desarrollado tres objetivos a lo largo de esta tesis, todos con la misma premisa, trabajar con elementos ad-nilpotentes. En primera instancia buscamos describir detalladamente los elementos ad-nilpotentes en álgebras asociativas semiprimas con involución. En el segundo objetivo, trasladamos el estudio que hemos realizado previamente sobre álgebras asociativas semiprimas a las superálgebras asociativas primas, es decir, se pretende dar una descripción con detalle análoga para los elementos ad-nilpotentes homogéneos. Y por último, asociamos a una superálgebra de Lie con un elemento ad-nilpotente de cierto índice una super estructura de Jordan.

Metodología

Para desarrollar los dos primeros objetivos hemos trabajado en el marco de las álgebras semiprimas con involución y en las superálgebras asociativas primas con superinvolución. Además, el centroide extendido tendrá una importancia esencial en esta tesis. Para el último de los objetivos, hemos trabajado con super estructuras no asociativas como las superálgebras de Jordan y Lie, que se definen mediante la envolvente de Grassmann, y super pares de Jordan. Podemos destacar el alto contenido combinatorio a lo largo de toda la tesis.

Resultados

Hemos abarcado con éxito los tres objetivos iniciales. En primer lugar, hemos des crito con detalle los elementos ad-nilpotentes pertenecientes a un álgebra asociativa semiprima. Además, se ha conseguido reducir la torsión en la clasificación de los elementos ad-nilpotentes en álgebras asociativas semiprimas con involución gracias al nuevo concepto de elemento ad-nilpotente puro, introducido en esta tesis. Se ha pasado de pedir libre de $n!$ torsión para un elemento ad-nilpotente de índice n a pedir libre de $\binom{n}{s}$ y s torsión con $s = [\frac{n+1}{2}]$. Por otra parte, para los elementos ad-nilpotentes antisimétricos de una álgebra asociativa semiprima, R, con involución, $*$, hemos dado una descripción que depende de su índice de ad-nilpotencia módulo 4. En esta descripción podemos destacar lo siguiente: Si un elemento antisimétrico, a, es ad-nilpotente tal que su índice de ad-nilpotencia sobre $K := \mathrm{Skew}(R, *)$ y R no coinciden, es decir, $\mathrm{ad}_a^n K = 0$ pero $\mathrm{ad}_a^n R \neq 0$, (sólo puede ocurrir para los índices de ad-nilpotencia sobre K congruentes con 0 ó 3 módulo 4) entonces un cierto corner del álgebra verifica una PI y por tanto el álgebra inicial satisface una GPI. Estos resultados se han desarrollado a lo largo del capítulo 2 y han originado un artículo que ya está publicado en la revista *Bulletin of the Malaysian Mathematical Sciences Society* ([12]). El segundo objetivo, describir en superálgebras asociativas primas con superinvolución las derivaciones internas nilpotentes, también se ha resuelto positivamente en el capítulo 3. Esta descripción depende a su vez de la paridad del elemento homogéneo: Si el elemento es par se rescata en gran medida lo desarrollado en el capítulo anterior sobre álgebras asociativas ([12]). Sin embargo, si el elemento es impar se trabajará sobre el cuadrado del elemento, que es un elemento ad-nilpotente par, y se le aplicará la descripción de los elementos ad-nilpotentes pares. Este capítulo ha dado lugar a un artículo que está publicado on-line en la revista *Linear and Multilinear Algebra* ([28]). En el capítulo 4, se han dado ejemplos para cada uno de los casos que aparecen en las descripciones de los elementos, tanto en álgebras como en superálgebras, demostrando así que estas descripciones no son triviales. Por último, en el capítulo 5, hemos asociado una superestructura Jordan a una superálgebra de Lie con un elemento ad-nilpotente homogéneo, a, de índice 3 ó 4, según su paridad. Además, el super par de Jordan que construimos siguiendo la filosofía del artículo de A. Fernández, E. García y M. Gómez Lozano [24] coincide con el subcociente de la superalgebra de Lie asociado a un ideal interno abeliano $[a, [a, L]]$. Este último capítulo ha sido publicado y puede consultarse en la revista *Communications in Algebra* ([30]).

Conclusiones

Las principales conclusiones de esta tesis se pueden resumir de la siguiente manera:

- Hemos definido la noción de elemento ad-nilpotente puro en el capítulo 2. El centroide extendido juega un papel muy importante en esta definición. Es una condición técnica, ya que todo elemento ad-nilpotente puede ser expresado como una suma ortogonal de elementos ad-nilpotentes puros de índices decrecientes. Además, esta definición nos permite dar una descripción más precisa de dichos elementos, y debilitar las condiciones de torsión del álgebra. Para más detalles consultar la sección 2.1.

- Hemos probado que para cualquier elemento ad-nilpotente puro a en un álgebra asociativa semiprima R de índice n con R libre de $\binom{n}{s}$ y s-torsión, donde $s = [\frac{n+1}{2}]$, existe λ en el centroide extendido tal que $a - \lambda$ es nilpotente de índice s. Este hecho se demuestra en el Teorema 2.2.4. Hemos debilitado las condiciones del Teorema [54, Theorem 1.3].

- Considerando un álgebra asociativa semiprima R con involución $*$, hemos descrito cualquier elemento antisimétrico ad-nilpotente puro a de índice n dependiendo de n en módulo 4. Si $n \equiv_4 0$ entonces los índices de ad-nilpotencia de a

en R y en K no coinciden y existe un corner de R que satisface una PI. Si $n \equiv_4 1$ entonces los índices de ad-nilpotencia de a en R y K coinciden y existe λ en el centroide extendido tal que $a - \lambda$ es nilpotente. Si $n \equiv_4 3$ entonces podemos descomponer a en una suma ortogonal $a = a_1 + a_2$ tal que, si $a_1 \neq 0$, a_1 es ad-nilpotente de R de índice n (y por tanto existe λ en el centroide extendido tal que $a_1 - \lambda$ es nilpotente) y, si $a_2 \neq 0$, a_2 es ad-nilpotente de R de índice $n + 2$ (y entonces existe un corner de R que satisface una PI). El caso $n \equiv_4 2$ no puede ocurrir. Esta descripción se ha demostrado en el Teorema 2.3.6.

- Siguiendo la misma idea que en el ambiente no super, hemos dado una descripción de elementos ad-nilpotentes en superalgebras asociativas primas con superinvolución. Estas descripciones relacionan el índice de ad-nilpotencia de un elemento homogéneo con su índice de nilpotencia. Una observación importante sobre estas descripciones es que todo elemento ad-nilpotente tiene un polinomio minimal en la clausura central con una única raíz perteneciente al centroide extendido. Para más detalles ver los teoremas 3.1.2, 3.2.4 y 3.2.5.

- Para terminar nuestro estudio sobre elementos ad-nilpotentes en álgebras y superálgebras asociativas, en el capítulo 4, hemos dado ejemplos de cada uno de los casos que aparecen en estas descripciones. Los ejemplos son matrices consideradas en la superálgebra asociativa $\mathcal{M}(r|s)$ sobre un cuerpo con una superinvolución no trivial. Aunque hemos considerado una superálgebra, también se construyen ejemplos para las descripciones de los elementos ad-nilpotentes en álgebras asociativas cuando restringimos los ejemploes a la parte par de las matrices $\mathcal{M}(r|s)$.

- Para una superálgebra de Lie L con un elemento ad-nilpotente par a de índice 3 hemos demostrado que $([a, [a, L]], \cdot)$ con un nuevo producto $\cdot$ definido por $[a, [a, x]] \cdot [a, [a, y]] := \frac{1}{2}[a, [a, [x, [a, y]]$ es una superálgebra de Jordan isomorfa a $L_a = L^{(a)}/\mathrm{Ker}_L(a)$ donde $L^{(a)} = (L, \bullet)$ con $x \bullet y := [x, [a, y]]$ y $\mathrm{Ker}_L(a) := \{x \in L \mid [a, [a, x]] = 0\}$. Este resultado se ha demostrado en la Proposición 5.1.2.

- Sin embargo, para una superálgebra de Lie con un elemento ad-nilpotente impar, la misma construcción genera una superálgebra anticonmutativa, y por tanto no puede ser una superálgebra de Jordan. Se demuestra que esta construcción es un superpar de Jordan, ver 5.2.3.

- Finalmente, hemos definido el subcociente de una superálgebra de Lie asociada a un ideal interno abeliano y hemos probado que es un superpar de Jordan (Proposición 5.3.3). Además, hemos demostrado que el subcociente coincide con la construcción realizada anteriormente (Corolario 5.3.5).

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