ON SOBOLEV ORTHOGONAL POLYNOMIALS
ON A TRIANGLE

MISAEL E. MARRIAGA

(Communicated by Yuan Xu)

ABSTRACT. We use the invariance of the triangle $\mathbf{T}^2 = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x, y, 1-x-y\}$ under the permutations of $\{x, y, 1-x-y\}$ to construct and study two-variable orthogonal polynomial systems with respect to several distinct Sobolev inner products defined on $\mathbf{T}^2$. These orthogonal polynomials can be constructed from two sequences of univariate orthogonal polynomials. In particular, one of the two univariate sequences of polynomials is orthogonal with respect to a Sobolev inner product and the other is a sequence of classical Jacobi polynomials.

1. INTRODUCTION

The purpose of this paper is to study bivariate orthogonal polynomials associated with several Sobolev inner products, that is, inner products involving partial derivatives of polynomials, defined on the triangle $\mathbf{T}^2 = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x, y, 1-x-y\}$.

Sobolev orthogonal polynomials have been widely studied for the last few decades. We refer the reader to [13] for a detailed survey on the topic. However, the study of Sobolev orthogonal polynomials in several variables is most recent. Some references include studies on the unit ball and the unit sphere [3–5, 11, 12, 15, 16, 18, 19], the simplex [1, 20], product domains [6, 7, 9] and other interesting domains [14]. We remark that some Sobolev orthogonal polynomials are eigenfunctions of second order linear ordinary differential equations (see, for instance, [10] and the references therein) and second order linear partial differential equations [2].

Sobolev orthogonal polynomials in one and several variables have been used in the implementation of spectral methods for boundary value problems for elliptic differential operators. For instance, in the univariate case, generalized Jacobi spectral schemes are proposed in [21] for second- and fourth-order elliptic boundary value problems with Dirichlet or Robin boundary conditions. These schemes consist in the construction of Jacobi–Sobolev orthogonal polynomials which allows the diagonalization of the involved discrete systems. The corresponding error estimates and numerical results illustrate the effectiveness and the spectral accuracy of the method. In several variables, Sobolev orthogonal polynomials on the unit ball have been considered in [18] in the numerical solution of boundary value problems for

elliptic partial differential operators. For details, we refer again to the survey paper [13] and the references therein.

Orthogonal polynomials associated with the classical weight function $W(x,y)$ defined on the triangle have been studied extensively (see for instance, [8]). Contrary to the univariate case, there are several bivariate orthogonal polynomial bases associated with $W(x,y)$. Of special interest for us is the fact that there are orthogonal bases that can be obtained by using the invariance of $\mathbf{T}^2$ under permutations of $\{x, y, 1-x-y\}$ (see Subsection 2.2). We aim to exploit this invariance to extend the study of orthogonal structures on the triangle to the Sobolev realm.

Our work is motivated by the results presented in [14], where the so-called *Koornwinder method* for generating bivariate orthogonal polynomials from univariate orthogonal polynomials was extended to the Sobolev case. In the particular case of the triangle, this extended Koornwinder method can be used to construct an orthogonal polynomial basis associated with a Sobolev inner product:

$$(P,Q) = \int \int_{\mathbf{T}^2} \left[P\,Q + \lambda\,(\nabla P)^\top M\,\nabla Q \right] W(x,y)\,dy dx, \quad \lambda > 0,$$

defined for all polynomials P and Q, where M is a 2×2 positive semidefinite polynomial matrix of a special form. However, the invariance of the triangle under permutations of $\{x, y, 1-x-y\}$ allows for $(\cdot, \cdot)$ to admit more matrices M than those presented in [14] that can be used to study orthogonal structures on the triangle. In the sequel, we explore several Sobolev inner products and the associated orthogonal polynomial bases obtained from such invariance. We must remark that in this paper, we restrict our study to $\lambda > 0$ to ensure that $(\cdot, \cdot)$ is an inner product. However, the study presented here is also valid for values $\lambda \in \mathbb{R}$ such that $(\cdot, \cdot)$ admits an associated orthogonal polynomial basis. Moreover, when $\lambda = 0$ we recover the results about the classical orthogonal polynomials on the triangle.

The paper is organized as follows. Section 2 contains the basic background on bivariate orthogonal polynomials needed to present our results. In Section 3, we present the Sobolev inner products explored in this work, and in Section 4 we construct orthogonal polynomial bases associated with these inner products. Last, we present a comparison of the Sobolev inner products and orthogonal polynomials in Section 5.

2. Bivariate orthogonal polynomials

In this section, we recall the basic tools and results about bivariate orthogonal polynomials needed throughout this work. Our main reference for the basic theory is [8].

2.1. **Basic tools.** We denote by Π^2 the linear space of real bivariate polynomials. For $n \geqslant 0$, let Π_n^2 denote the linear space of real bivariate polynomials of total degree at most n. Evidently,

$$\dim \Pi_n^2 = \binom{n+2}{2} \quad \text{and} \quad \Pi^2 = \bigcup_{n \geqslant 0} \Pi_n^2.$$

We say that a sequence $\mathcal{P} = \{P_{n,m}(x,y) : n \geqslant 0, 0 \leqslant m \leqslant n\}$ of polynomials in Π^2 is a *polynomial system (PS)* if for all $n \geqslant 0$, the set $\mathcal{P}_n = \{P_{n,m}(x,y) : 0 \leqslant m \leqslant n\}$ consists of $n+1$ linearly independent polynomials of total degree n, that is, $\deg P_{n,m} = n, 0 \leqslant m \leqslant n$. In this way, a PS $\mathcal{P}$ is a basis of Π^2.

Let $\langle \cdot, \cdot \rangle : \Pi^2 \times \Pi^2 \to \mathbb{R}$ be a bilinear form defined on polynomials. A polynomial P of degree n is called an *orthogonal polynomial* with respect to the bilinear form if

$$\langle P, Q \rangle \, = \, 0, \quad \forall Q \in \Pi^2_{n-1}.$$

Given a bilinear form $\langle \cdot, \cdot \rangle$ and a PS $\mathcal{P} = \{P_{n,m}(x,y) : n \geqslant 0, \, 0 \leqslant m \leqslant n\}$, we will say that $\mathcal{P}$ is *orthogonal with respect to the bilinear form* if

$$\langle P_{n,m}, Q \rangle \, = \, 0, \quad \forall Q \in \Pi^2_{n-1},$$

for all $n \geqslant 0$ and $0 \leqslant m \leqslant n$. Moreover, if

$$\langle P_{n,m}, P_{i,j} \rangle \, = \, H_{n,m} \, \delta_{n,i} \, \delta_{m,j},$$

where $H_{n,m} \neq 0$ for $n \geqslant 0$, then we say that $\mathcal{P}$ is a *mutually orthogonal polynomial system*. Here, $\delta_{n,k}$ denotes the Kronecker delta.

Let $W(x,y)$ be a weight function defined on a domain $\Omega \subseteq \mathbb{R}^2$. That is, $W(x,y) > 0$ for all $(x,y) \in \Omega$, and Ω has a nonempty interior. If a bilinear form is given by

$$\langle P, Q \rangle \, = \, \int \int_\Omega P(x,y) \, Q(x,y) \, W(x,y) \, dx dy, \quad \forall P, Q \in \Pi^2,$$

we say that the orthogonal polynomials, whenever they exist, are orthogonal with respect to the weight function W.

2.2. **Orthogonal polynomials on a triangle.** Let

$$\mathbf{T}^2 \, = \, \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x, y, \, 1 - x - y\}$$

denote a triangle in $\mathbb{R}^2$. For $\alpha, \beta, \gamma > -1$, define the weight function

$$W_{\alpha,\beta,\gamma}(x,y) \, = \, x^\alpha \, y^\beta \, (1 - x - y)^\gamma, \quad (x,y) \in \mathbf{T}^2,$$

and the bilinear form

$$\langle P, Q \rangle_{\alpha,\beta,\gamma} \, = \, b_{\alpha,\beta,\gamma} \int \int_{\mathbf{T}^2} P(x,y) \, Q(x,y) \, W_{\alpha,\beta,\gamma}(x,y) \, dx dy,$$

where

$$b_{\alpha,\beta,\gamma} \, = \, \left(\int \int_{\mathbf{T}^2} W_{\alpha,\beta,\gamma}(x,y) \, dx dy \right)^{-1} \, = \, \frac{\Gamma(\alpha + \beta + \gamma + 3)}{\Gamma(\alpha + 1) \, \Gamma(\beta + 1) \, \Gamma(\gamma + 1)}.$$

A mutually orthogonal polynomial system on the triangle can be given in terms of the Jacobi polynomials. Hence, let $P_n^{(\alpha,\beta)}(t)$ denote the Jacobi polynomial of degree n, which is orthogonal with respect to the univariate weight function

$$w_{\alpha,\beta}(t) \, = \, (1 - t)^\alpha (1 + t)^\beta, \quad \alpha, \beta > -1, \quad t \in [-1, 1].$$

Proposition 2.1 ([8])**.** *For $n \geqslant 0$, define the polynomials*

$$P_{n,m}^{(\alpha,\beta,\gamma)}(x,y) \, = \, P_{n-m}^{(\beta_m,\alpha)}(2x - 1) \, (1 - x)^m \, P_m^{(\gamma,\beta)} \left(\frac{2y}{1 - x} - 1 \right), \quad 0 \leqslant m \leqslant n,$$

where $\beta_m = \beta + \gamma + 2m + 1$. Then $\{P_{n,m}^{(\alpha,\beta,\gamma)}(x,y) : n \geqslant 0, \, 0 \leqslant m \leqslant n\}$ constitutes a mutually orthogonal polynomial system with respect to $\langle \cdot, \cdot \rangle_{\alpha,\beta,\gamma}$. Moreover,

$$\langle P_{n,m}^{(\alpha,\beta,\gamma)}, P_{k,j}^{(\alpha,\beta,\gamma)} \rangle_{\alpha,\beta,\gamma} \, = \, H_{n,m}^{(\alpha,\beta,\gamma)} \, \delta_{n,k} \, \delta_{m,j},$$

where

$$H_{n,m}^{(\alpha,\beta,\gamma)} = \frac{(\alpha+1)_{n-m}\,(\beta+1)_m\,(\gamma+1)_m\,(\beta+\gamma+2)_{n+m}}{(n-m)!\,k!\,(\beta+\gamma+2)_m\,(\alpha+\beta+\gamma+3)_{n+m}}$$
$$\times \frac{(n+m+\alpha+\beta+\gamma+2)\,(m+\beta+\gamma+1)}{(2n+\alpha+\beta+\gamma+2)\,(2m+\beta+\gamma+1)}.$$

Here, as usual,

$$(\nu)_0 = 1, \quad (\nu)_k = \nu\,(\nu+1)\cdots(\nu+k-1), \quad k = 1,2,\ldots,$$

denotes the Pochhammer symbol.

The bilinear form $\langle\cdot,\cdot\rangle_{\alpha,\beta,\gamma}$ can be expressed in different ways as two interated integrals. In particular, the invariance of the triangle $\mathbf{T}^2$ with respect to permutations of $\{x,y,1-x-y\}$ leads to two other mutually orthogonal polynomial systems.

Proposition 2.2 ([8]). *Define the polynomials*

$$Q_{n,m}^{(\alpha,\beta,\gamma)}(x,y) = P_{n,m}^{(\beta,\alpha,\gamma)}(y,x) \quad and \quad R_{n,m}^{(\alpha,\beta,\gamma)}(x,y) = P_{n,m}^{(\gamma,\beta,\alpha)}(1-x-y,y).$$

Then, $\{Q_{n,m}^{(\alpha,\beta,\gamma)}(x,y):\ n\geqslant 0,\ 0\leqslant m\leqslant n\}$ and $\{R_{n,m}^{(\alpha,\beta,\gamma)}(x,y):\ n\geqslant 0,\ 0\leqslant m\leqslant n\}$ are also mutually orthogonal systems with respect to $\langle\cdot,\cdot\rangle_{\alpha,\beta,\gamma}$. Furthermore,

$$\langle Q_{n,m}^{(\alpha,\beta,\gamma)}, Q_{n,m}^{(\alpha,\beta,\gamma)}\rangle_{\alpha,\beta,\gamma} = H_{n,m}^{(\beta,\alpha,\gamma)} \quad and \quad \langle R_{n,m}^{(\alpha,\beta,\gamma)}, R_{n,m}^{(\alpha,\beta,\gamma)}\rangle_{\alpha,\beta,\gamma} = H_{n,m}^{(\gamma,\beta,\alpha)}.$$

We note that in Proposition 2.2, each permutation of $\{x,y,1-x-y\}$ induces a corresponding permutation of the parameters $\{\alpha,\beta,\gamma\}$.

3. Sobolev inner products on the triangle

The triangle polynomials $Q_{n,m}^{(\alpha,\beta,\gamma)}(x,y)$ and $R_{n,m}^{(\alpha,\beta,\gamma)}(x,y)$ in Proposition 2.2 are obtained from $P_{n,m}^{(\alpha,\beta,\gamma)}(x,y)$ by taking advantage of the invariance of $\mathbf{T}^2$ with respect to the permutations of $\{x,y,1-x-y\}$. In this section, we use this invariance of $\mathbf{T}^2$ to extend the study of orthogonal polynomial bases to the Sobolev realm.

Here, we introduce the Sobolev inner products that we explore throughout this paper. First, we need some notation. Let $x_1 = x$, $x_2 = y$, and $x_3 = 1-x-y$. We denote by S_3 the permutation group on the set $(1,2,3)$ (i.e., the set of bijections defined on $(1,2,3)$).

For each $\sigma \in S_3$, we define the variables (s,t) as follows:

(1) $$(s, t\,(1-s), (1-t)\,(1-s)) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}).$$

Explicitly, using the cyclic notation, we have:

$\sigma_0 := (1):$	$s = x,$	$t\,(1-s) = y,$	$(1-t)\,(1-s) = 1-x-y,$
$\sigma_1 := (2\,3):$	$s = x,$	$t\,(1-s) = 1-x-y,$	$(1-t)\,(1-s) = y,$
$\sigma_2 := (1\,2):$	$s = y,$	$t\,(1-s) = x,$	$(1-t)\,(1-s) = 1-x-y,$
$\sigma_3 := (1\,2\,3):$	$s = y,$	$t\,(1-s) = 1-x-y,$	$(1-t)\,(1-s) = x,$
$\sigma_4 := (1\,3\,2):$	$s = 1-x-y,$	$t\,(1-s) = x,$	$(1-t)\,(1-s) = y,$
$\sigma_5 := (1\,3):$	$s = 1-x-y,$	$t\,(1-s) = y,$	$(1-t)\,(1-s) = x.$

Moreover, denote by $\partial_1 = \partial_x$ and $\partial_2 = \partial_y$ the partial derivatives with respect to x and y, respectively, and define

$$\partial_3 = \partial_y - \partial_x.$$

We organize the expression of partial derivatives ∂_s and ∂_t corresponding to each $\sigma \in S_3$ in Lemma 3.1.

Lemma 3.1. *We have:*

$$
\begin{array}{llll}
\sigma_0: & \partial_s = \partial_x - t\,\partial_y = \partial_1 - t\,\partial_2, & \partial_t = (1-s)\partial_y = (1-s)\,\partial_2, \\
\sigma_1: & \partial_s = \partial_x - (1-t)\,\partial_y = -\partial_3 + t\,\partial_2, & \partial_t = -(1-s)\,\partial_y = -(1-s)\,\partial_2, \\
\sigma_2: & \partial_s = -t\,\partial_x + \partial_y = \partial_2 - t\,\partial_1, & \partial_t = (1-s)\,\partial_x = (1-s)\,\partial_1, \\
\sigma_3: & \partial_s = -(1-t)\,\partial_x + \partial_y = \partial_3 + t\,\partial_1, & \partial_t = -(1-s)\,\partial_x = -(1-s)\,\partial_1, \\
\sigma_4: & \partial_s = -t\,\partial_x - (1-t)\,\partial_y = -\partial_2 + t\,\partial_3, & \partial_t = (1-s)\,\partial_x - (1-s)\,\partial_y = -(1-s)\,\partial_3, \\
\sigma_5: & \partial_s = -(1-t)\,\partial_x - t\,\partial_y = -\partial_1 + t\,\partial_3, & \partial_t = -(1-s)\,\partial_x + (1-s)\,\partial_y = -(1-s)\,\partial_3.
\end{array}
$$

Motivated by Lemma 3.1, we define the following differential operators.

Definition 3.2. We define the gradient-type differential operators

$$
\nabla_{\sigma_0} = (\partial_1 \ \ \partial_2)^\top, \qquad \nabla_{\sigma_1} = (-\partial_3 \ \ -\partial_2)^\top, \qquad \nabla_{\sigma_2} = (\partial_2 \ \ \partial_1)^\top,
$$

$$
\nabla_{\sigma_3} = (\partial_3 \ \ -\partial_1)^\top, \qquad \nabla_{\sigma_4} = (-\partial_2 \ \ -\partial_3)^\top, \qquad \nabla_{\sigma_5} = (-\partial_1 \ \ -\partial_3)^\top,
$$

where $\top$ denotes matrix transpose.

We are ready to introduce the Sobolev inner products that we will study in this paper.

Definition 3.3. For $\alpha, \beta, \gamma > -1$, let $W_{\alpha,\beta,\gamma}(x,y)$ be the weight function on $\mathbf{T}^2$ and $b_{\alpha,\beta,\gamma}$ be the corresponding normalization constant as defined in Subsection 2.2. Let $\lambda > 0$ be a real number. For each $\sigma \in S_3$ and all $P, Q \in \Pi^2$, we define two types of Sobolev inner products on the triangle:

Type I:

$$
(2) \qquad \langle P, Q \rangle_\sigma^I = b_{\alpha,\beta,\gamma} \iint_{\mathbf{T}^2} \left[P\,Q + \lambda (\nabla_\sigma P)^\top M_\sigma \nabla_\sigma Q \right] W_{\alpha,\beta,\gamma}(x,y)\,dydx,
$$

where

$$
M_\sigma = \begin{pmatrix} (1 - x_{\sigma(1)})^2 & -x_{\sigma(2)}(1 - x_{\sigma(1)}) \\ -x_{\sigma(2)}(1 - x_{\sigma(1)}) & x_{\sigma(2)}^2 \end{pmatrix}.
$$

Type II:

$$
(3) \qquad \langle P, Q \rangle_\sigma^{II} = b_{\alpha,\beta,\gamma} \iint_{\mathbf{T}^2} \left[P\,Q + \lambda (\nabla_\sigma P)^\top N_\sigma \nabla_\sigma Q \right] W_{\alpha,\beta,\gamma}(x,y)\,dydx,
$$

where

$$
N_\sigma = \begin{pmatrix} 0 & 0 \\ 0 & (1 - x_{\sigma(1)})^2 \end{pmatrix}.
$$

4. Sobolev orthogonal polynomials on the triangle

Now we focus on studying orthogonal polynomial bases associated with (2) and (3). The strategy that we will use for constructing these orthogonal bases is motivated by the construction in Proposition 2.2. More concretely, for each $\sigma \in S_3$, we will consider two sequences of univariate orthogonal polynomials associated with some bilinear forms, and use them to construct mutually orthogonal polynomial systems, analogous to the polynomials in Proposition 2.2, associated with (2) and (3). Here, we will determine the involved univariate orthogonal polynomials.

To this end, we will consider the Type I and Type II Sobolev inner products separately.

4.1. **Type I Sobolev orthogonal polynomials.** Let $\alpha, \beta, \gamma > -1$, and let $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = \gamma$. For $\sigma \in S_3$ and $m \geqslant 0$, let us define the univariate Sobolev inner product

(4)
$$(f, g)_m^\sigma = \int_0^1 \left[f\,g + \lambda\,(f\ f') \begin{pmatrix} m^2 & -m\,(1-s) \\ -m\,(1-s) & (1-s)^2 \end{pmatrix} \begin{pmatrix} g \\ g' \end{pmatrix} \right] w_m^\sigma(s)\,ds, \quad \lambda > 0,$$

where

$$w_m^\sigma(s) = (1-s)^{\alpha_{\sigma(2)}+\alpha_{\sigma(3)}+2m+1}\,s^{\alpha_{\sigma(1)}},$$

or, equivalently,

$$(f, g)_m^\sigma = \int_0^1 f(s)\,g(s)\,w_m^\sigma(s)\,ds$$
$$+ \lambda \int_0^1 ((1-s)^m\,f)'\,((1-s)^m\,g)'\,(1-s)^{\alpha_{\sigma(2)}+\alpha_{\sigma(3)}+3}\,s^{\alpha_{\sigma(1)}}\,ds.$$

For $m \geqslant 0$, let $\{p_{n,\sigma}^{(m)}(s)\}_{n\geqslant 0}$ be a univariate Sobolev orthogonal polynomial sequence associated with the inner product $(\cdot, \cdot)_m^\sigma$. In addition, we define

(5)
$$(p_{n,\sigma}^{(m)}, p_{n,\sigma}^{(m)})_m^\sigma = h_{n,\sigma}^{(m)},$$

with $h_{n,\sigma}^{(m)} = h_{n,\sigma}^{(m)}(\lambda) > 0$, $n, m \geqslant 0$.

Since $\{p_{n,\sigma}^{(m)}(s)\}_{n\geqslant 0}$ is unique up to a constant factor, we choose it such that, for $n \geqslant 0$, $p_{n,\sigma}^{(m)}(s)$ has the same leading coefficient as the Jacobi polynomial $P_n^{(\beta_m^\sigma, \alpha_{\sigma(1)})}(2s - 1)$ with $\beta_m^\sigma = \alpha_{\sigma(2)} + \alpha_{\sigma(3)} + 2m + 1$, and therefore, $p_{0,\sigma}^{(m)}(s) = P_0^{(\beta_m^\sigma, \alpha_{\sigma(1)})}(2s - 1)$.

Now, we present the first type of Sobolev orthogonal polynomials.

Theorem 4.1. *For $\sigma \in S_3$, the two-variable polynomials defined as*

$$S_{n,m}^\sigma(x, y) = p_{n-m,\sigma}^{(m)}(x_{\sigma(1)})\,(1 - x_{\sigma(1)})^m\,P_m^{(\alpha_{\sigma(3)}, \alpha_{\sigma(2)})}\left(\frac{2\,x_{\sigma(2)}}{1 - x_{\sigma(1)}} - 1 \right), \quad 0 \leqslant m \leqslant n,$$

constitute a mutually orthogonal polynomial system associated with the Sobolev inner product $\langle \cdot, \cdot \rangle_\sigma^I$ defined in (2).

Moreover,

$$H_{n,m}^\sigma = \langle S_{n,m}^\sigma, S_{n,m}^\sigma \rangle_\sigma^I = b_{\alpha,\beta,\gamma}\,h_{n-m,\sigma}^{(m)}\,h_m,$$

where $h_{n-m,\sigma}^{(m)}$ is defined in (5) and

$$h_m = \int_0^1 \left(P_m^{(\alpha_{\sigma(3)}, \alpha_{\sigma(2)})}(2t - 1) \right)^2 (1 - t)^{\alpha_{\sigma(3)}}\,t^{\alpha_{\sigma(2)}}\,dt.$$

Proof. By (1), we have

$$W_{\alpha,\beta,\gamma}(x, y)\,dydx = (1-s)^{\alpha_{\sigma(2)}+\alpha_{\sigma(3)}+1}\,s^{\alpha_{\sigma(1)}}\,(1-t)^{\alpha_{\sigma(3)}}\,t^{\alpha_{\sigma(2)}}\,dsdt.$$

Using Lemma 3.1 and Definition 3.2, we get

$$\langle S_{n,m}^{\sigma}, S_{i,j}^{\sigma}\rangle_{\sigma}^{I}$$

$$= b_{\alpha,\beta,\gamma} \int\!\!\int_{\mathbf{T}^2} \left[S_{n,m}^{\sigma}\, S_{i,j}^{\sigma} + \lambda (\nabla_{\sigma} S_{n,m}^{\sigma})^{\top} M_{\sigma} \nabla_{\sigma} S_{i,j}^{\sigma}\right] W_{\alpha,\beta,\gamma}(x,y)\, dy dx$$

$$= b_{\alpha,\beta,\gamma} \left[\int_0^1 p_{n-m,\sigma}^{(m)}(s)\, p_{i-j,\sigma}^{(j)}(s)\,(1-s)^{\alpha_{\sigma(2)}+\alpha_{\sigma(3)}+m+j+1} s^{\alpha_{\sigma(1)}} ds \right.$$

$$\left. + \lambda \int_0^1 \left((1-s)^m\, p_{n-m,\sigma}^{(m)}(s)\right)' \left((1-s)^j\, p_{i-j,\sigma}^{(j)}(s)\right)' (1-s)^{\alpha_{\sigma(2)}+\alpha_{\sigma(3)}+3}\, s^{\alpha_{\sigma(1)}}\, ds \right]$$

$$\times \int_0^1 P_m^{(\alpha_{\sigma(3)},\alpha_{\sigma(2)})}(2t-1)\, P_j^{(\alpha_{\sigma(3)},\alpha_{\sigma(2)})}(2t-1)\,(1-t)^{\alpha_{\sigma(3)}}\, t^{\alpha_{\sigma(2)}}\, dt$$

$$= b_{\alpha,\beta,\gamma}\, (p_{n-m,\sigma}^{(m)}, p_{i-m,\sigma}^{(m)})_m\, h_m\, \delta_{m,j}.$$

Since $\{p_{n,\sigma}^{(m)}\}_{n\geqslant 0}$ is orthogonal with respect to $(\cdot,\cdot)_m^{\sigma}$, the theorem is proved. $\square$

The univariate Sobolev inner product (4) is a particular case of a univariate bilinear form studied in [14], where relations between the univariate Sobolev and classical orthogonal polynomials involving up to three consecutive terms of each family were obtained. In our case, from [14, Corollary 4.3], we get the following result.

Proposition 4.2. *For $\sigma \in S_3$ and $n \geqslant 1$, there are constants $a_{n,1}^{(m)} = a_{n,1}^{(m)}(\sigma), a_{n,2}^{(m)} = a_{n,2}^{(m)}(\sigma), b_{n,1}^{(m)} = b_{n,1}^{(m)}(\sigma), b_{n,2}^{(m)} = b_{n,2}^{(m)}(\sigma)$, depending on σ, such that the following relation holds:*

$$(6) \quad \begin{aligned} P_n^{(\beta_m^{\sigma},\alpha_{\sigma(1)})}(2s-1) &+ a_{n,1}^{(m)} P_{n-1}^{(\beta_m^{\sigma},\alpha_{\sigma(1)})}(2s-1) + a_{n,2}^{(m)} P_{n-2}^{(\beta_m^{\sigma},\alpha_{\sigma(1)})}(2s-1) \\ &= p_{n,\sigma}^{(m)}(s) + b_{n,1}^{(m)} p_{n-1,\sigma}^{(m)}(s) + b_{n,2}^{(m)} p_{n-2,\sigma}^{(m)}(s), \end{aligned}$$

where $P_{-1}^{(\beta_m^{\sigma},\alpha_{\sigma(1)})}(2s-1) = p_{-1,\sigma}^{(m)}(s) = 0$ and $p_{0,\sigma}^{(m)}(s) = P_0^{(\beta_m^{\sigma},\alpha_{\sigma(1)})}(2s-1)$.

Observe that although $P_0^{(\beta_m^{\sigma},\alpha_{\sigma(1)})}(s)$ is a nonzero constant, the value of this constant depends on the standardization of the Jacobi polynomials $P_n^{(\beta_m^{\sigma},\alpha_{\sigma(1)})}(s)$.

Using the expression for $S_{n,m}^{\sigma}(x,y)$, relation (6) can be extended to the bivariate case by multiplying both sides by $(1-x_{\sigma(1)})^m\, P_m^{(\alpha_{\sigma(3)},\alpha_{\sigma(2)})}\left(\frac{2\,x_{\sigma(2)}}{1-x_{\sigma(1)}}-1\right)$.

Proposition 4.3. *For $n \geqslant 0$ and $0 \leqslant m \leqslant n$, define the bivariate Jacobi polynomials*

$$P_{n,m}^{\sigma}(x,y) = P_{n-m}^{(\beta_m^{\sigma},\alpha_{\sigma(1)})}(2x_{\sigma(1)}-1)\,(1-x_{\sigma(1)})^m\, P_m^{(\alpha_{\sigma(3)},\alpha_{\sigma(2)})}\left(\frac{2\,x_{\sigma(2)}}{1-x_{\sigma(1)}}-1\right).$$

The following relation holds:

$$\begin{aligned} P_{n,m}^{\sigma}(x,y) &+ a_{n-m-1,1}^{(m)} P_{n-1,m}^{\sigma}(x,y) + a_{n-m-1,2}^{(m)} P_{n-2,m}^{\sigma}(x,y) \\ &= S_{n,m}^{\sigma}(x,y) + b_{n-m-1,1}^{(m)} S_{n-1,m}^{\sigma}(x,y) + b_{n-m-1,2}^{(m)} S_{n-2,m}^{\sigma}(x,y), \end{aligned}$$

and $S_{0,0}^{\sigma}(x,y) = P_{0,0}^{\sigma}(x,y)$.

We note that the case when $\sigma = \sigma_0$ was studied in [14, Section 6.4].

4.2. **Type II Sobolev orthogonal polynomials.** For $\sigma \in S_3$, let us define the univariate Sobolev inner product

$$(7) \qquad (f,g)^\sigma = \int_0^1 [f(t)\,g(t) + \lambda\,f'(t)\,g'(t)]\,(1-t)^{\alpha_{\sigma(3)}}\,t^{\alpha_{\sigma(2)}}\,dt, \quad \lambda > 0.$$

Let $\{q_{n,\sigma}(t)\}_{n\geqslant 0}$ be the corresponding sequence of univariate orthogonal polynomials, standardized in such a way that the leading coefficient of $q_{n,\sigma}(t)$ is the same as the leading coefficient of the Jacobi polynomial $P_n^{(\alpha_{\sigma(3)},\alpha_{\sigma(2)})}(2t-1)$. Therefore, $q_{0,\sigma}(t) = P_0^{(\alpha_{\sigma(3)},\alpha_{\sigma(2)})}(2t-1)$. In addition, let

$$(8) \qquad\qquad h_{n,\sigma}^{(q)} = h_{n,\sigma}^{(q)}(\lambda) = (q_{n,\sigma}, q_{n,\sigma})^\sigma > 0, \quad n \geqslant 0.$$

We can construct Sobolev orthogonal polynomials as follows:

Theorem 4.4. *For $\sigma \in S_3$, the two-variable polynomials*

$$\widetilde{S}_{n,m}^\sigma(x,y) = P_{n-m}^{(\beta_m^\sigma,\alpha_{\sigma(1)})}(2\,x_{\sigma(1)}-1)\,(1-x_{\sigma(1)})^m\,q_{m,\sigma}\left(\frac{x_{\sigma(2)}}{1-x_{\sigma(1)}}\right), \quad 0 \leqslant m \leqslant n,$$

with $\beta_m^\sigma = \alpha_{\sigma(2)} + \alpha_{\sigma(3)} + 2m + 1$, constitute a mutually orthogonal polynomial system with respect to the Sobolev inner product $\langle\cdot,\cdot\rangle_\sigma^{II}$ defined in (3).

Moreover,

$$\widetilde{H}_{n,m}^\sigma = \langle\widetilde{S}_{n,m}^\sigma, \widetilde{S}_{n,m}^\sigma\rangle_\sigma^{II} = b_{\alpha,\beta,\gamma}\,\widetilde{h}_{n-m}^{(m)}\,h_{m,\sigma}^{(q)},$$

where $h_{m,\sigma}^{(q)}$ is defined in (8), and

$$\widetilde{h}_{n-m}^{(m)} = \int_0^1 \left(P_{n-m}^{(\beta_m^\sigma,\alpha_{\sigma(1)})}(2s-1)\right)^2 (1-s)^{\beta_m^\sigma}\,s^{\alpha_{\sigma(1)}}ds.$$

Proof. We compute

$$\langle\widetilde{S}_{n,m}^\sigma, \widetilde{S}_{i,j}^\sigma\rangle_\sigma^{II}$$
$$= b_{\alpha,\beta,\gamma}\iint_{\mathbf{T}^2}\left[\widetilde{S}_{n,m}^\sigma\,\widetilde{S}_{i,j}^\sigma + \lambda(\nabla_\sigma\widetilde{S}_{n,m}^\sigma)^\top N_\sigma\nabla_\sigma\widetilde{S}_{i,j}^\sigma\right] W_{\alpha,\beta,\gamma}(x,y)\,dydx.$$

Using (1), Lemma 3.1, and Definition 3.2, we get

$$\iint_{\mathbf{T}^2}\widetilde{S}_{n,m}^\sigma\,\widetilde{S}_{i,j}^\sigma\,W_{\alpha,\beta,\gamma}(x,y)\,dydx$$
$$= \int_0^1 P_{n-m}^{(\beta_m^\sigma,\alpha_{\sigma(1)})}(2s-1)\,P_{i-j}^{(\beta_j^\sigma,\alpha_{\sigma(1)})}(2s-1)\,(1-s)^{\alpha_{\sigma(2)}+\alpha_{\sigma(3)}+m+j+1}\,s^{\alpha_{\sigma(1)}}ds$$
$$\times \int_0^1 q_{m,\sigma}(t)\,q_{j,\sigma}(t)\,(1-t)^{\alpha_{\sigma(3)}}\,t^{\alpha_{\sigma(2)}}\,dt,$$

and

$$\iint_{\mathbf{T}^2}(\nabla_\sigma\widetilde{S}_{n,m}^\sigma)^\top N_\sigma\nabla_\sigma\widetilde{S}_{i,j}^\sigma\,W_{\alpha,\beta,\gamma}(x,y)\,dydx$$
$$= \int_0^1 P_{n-m}^{(\beta_m^\sigma,\alpha_{\sigma(1)})}(2s-1)\,P_{i-j}^{(\beta_j^\sigma,\alpha_{\sigma(1)})}(2s-1)\,(1-s)^{\alpha_{\sigma(2)}+\alpha_{\sigma(3)}+m+j+1}\,s^{\alpha_{\sigma(1)}}ds$$
$$\times \int_0^1 q'_{m,\sigma}(t)\,q'_{j,\sigma}(t)\,(1-t)^{\alpha_{\sigma(3)}}\,t^{\alpha_{\sigma(2)}}\,dt.$$

Putting both terms together, we get

$$\int\int_{\mathbf{T}^2} \left[\widetilde{S}^\sigma_{n,m}\, \widetilde{S}^\sigma_{i,j} + \lambda (\nabla_\sigma \widetilde{S}^\sigma_{n,m})^\top N_\sigma \nabla_\sigma \widetilde{S}^\sigma_{i,j} \right] W_{\alpha,\beta,\gamma}(x,y)\, dy dx$$

$$= \int_0^1 P^{(\beta^\sigma_m,\alpha_{\sigma(1)})}_{n-m}(2s-1)\, P^{(\beta^\sigma_j,\alpha_{\sigma(1)})}_{i-j}(2s-1)\, (1-s)^{\alpha_{\sigma(2)}+\alpha_{\sigma(3)}+m+j+1}\, s^{\alpha_{\sigma(1)}} ds$$

$$\times \int_0^1 \left[q_{m,\sigma}(t)\, q_{j,\sigma}(t) + \lambda\, q'_{m,\sigma}(t)\, q'_{j,\sigma}(t) \right] (1-t)^{\alpha_{\sigma(3)}}\, t^{\alpha_{\sigma(2)}}\, dt.$$

Therefore, $\langle \widetilde{S}^\sigma_{n,m}, \widetilde{S}^\sigma_{i,j} \rangle^{II}_\sigma = b_{\alpha,\beta,\gamma}\, \widetilde{h}^{(m)}_{n-m}\, h^{(q)}_{m,\sigma}\, \delta_{n,i}\, \delta_{m,j}.$ $\qquad\square$

The Sobolev inner product (7) is also a particular case of a univariate bilinear form studied in [14]. From [14, Corollary 5.3], we deduce the following result.

Proposition 4.5. *For $\sigma \in S_3$ and $n \geqslant 1$, there are constants $c^\sigma_{n,1}, c^\sigma_{n,2}, d^\sigma_{n,1}, d^\sigma_{n,2}$, such that the following relation holds:*

(9)
$$P^{(\alpha_{\sigma(3)},\alpha_{\sigma(2)})}_n(2t-1) + c^\sigma_{n,1}\, P^{(\alpha_{\sigma(3)},\alpha_{\sigma(2)})}_{n-1}(2t-1) + c^\sigma_{n,2}\, P^{(\alpha_{\sigma(3)},\alpha_{\sigma(2)})}_{n-2}(2t-1)$$
$$= q_{n,\sigma}(t) + d^\sigma_{n,1}\, q_{n-1,\sigma}(t) + d^\sigma_{n,2}\, q_{n-2,\sigma}(t),$$

with $P^{(\alpha_{\sigma(3)},\alpha_{\sigma(2)})}_{-1}(2t-1) = q_{-1,\sigma}(t) = 0$ and $q_{0,\sigma}(t) = P^{(\alpha_{\sigma(3)},\alpha_{\sigma(2)})}_0(2t-1).$

We can extend relation (9) to the bivariate case by following the proof of [14, Theorem 5.5].

Proposition 4.6. *For $n \geqslant 1$ and $0 \leqslant m \leqslant n$, there are real numbers depending on σ, such that the following relation holds:*

$$\sum_{i=0}^4 \left[\eta^{(m)}_{n-i}\, P^\sigma_{n+2-i,m} + c^\sigma_{m,1}\, \theta^{(m)}_{n-i}\, P^\sigma_{n+2-i,m-1} + c^\sigma_{m,2}\, \vartheta^{(m)}_{n-i}\, P^\sigma_{n+2-i,m-2} \right]$$

$$= \sum_{i=0}^4 \left[\eta^{(m)}_{n-i}\, \widetilde{S}^\sigma_{n+2-i,m} + d^\sigma_{m,1}\, \theta^{(m)}_{n-i}\, \widetilde{S}^\sigma_{n+2-i,m-1} + d^\sigma_{m,2}\, \vartheta^{(m)}_{n-i}\, \widetilde{S}^\sigma_{n+2-i,m-2} \right],$$

with $\widetilde{S}^\sigma_{0,0}(x,y) = P^\sigma_{0,0}(x,y)$ and $\widetilde{S}^\sigma_{i,j}(x,y) = 0$ for $i < j$.

Proof. The result follows from the proof of [14, Theorem 5.5] by taking $\mathbf{u}_m$ and $\mathbf{v}$ as the moment functionals associated with the Jacobi weight functions $w_1(s) = (1-s)^{\beta^\sigma_m}\, s^{\alpha_{\sigma(1)}}$ and $w_2(t) = (1-t)^{\alpha_{\sigma(2)}}\, t^{\alpha_{\sigma(3)}}$, respectively, and $\rho(s) = (1-s) = (1-x_{\sigma(1)}).$ $\qquad\square$

5. Comparison of Sobolev inner products

In Definition 3.3, we have introduced a type I and a type II Sobolev inner product for each $\sigma \in S_3$. Hence, there are six inner products of each type. Moreover, in Theorem 4.1 (respectively, Theorem 4.4), we constructed a polynomial system orthogonal with respect to each type I (resp., type II) Sobolev inner product corresponding to σ.

In this section, we compare the Sobolev inner products of type I (resp., type II). We do this by writing all the inner products in terms of the usual gradient operator $\nabla = (\partial_x\ \partial_y)^\top$. Recasting the inner products in this way shows that there are only three distinct Sobolev inner products of type I (resp., type II).

We organize these results in the following propositions.

Proposition 5.1. *For all $P, Q \in \Pi^2$,*

$$\langle P, Q \rangle_{\sigma_0}^I = \langle P, Q \rangle_{\sigma_1}^I$$

$$= b_{\alpha,\beta,\gamma} \int\int_{\mathbf{T}^2} \left[P\,Q + \lambda (\nabla P)^\top \begin{pmatrix} (1-x)^2 & -y\,(1-x) \\ -y\,(1-x) & y^2 \end{pmatrix} \nabla Q \right] W_{\alpha,\beta,\gamma}(x,y)\,dy dx.$$

Moreover, let $S_{n,m}^{\sigma_0}(x,y)$ and $S_{n,m}^{\sigma_1}(x,y)$ be the polynomials defined in Theorem 4.1. Then

$$S_{n,m}^{\sigma_1}(x,y) = (-1)^m S_{n,m}^{\sigma_0}(x,y), \quad n \geqslant 0, \quad 0 \leqslant m \leqslant n.$$

Proof. Using Lemma 3.1 and Definition 3.2, it is easy to check that $\langle P, Q \rangle_{\sigma_0}^I = \langle P, Q \rangle_{\sigma_1}^I$ for all $P, Q \in \Pi^2$.

Now, recall that

$$S_{n,m}^{\sigma_0}(x,y) = p_{n-m,\sigma_0}^{(m)}(x)\,(1-x)^m\,P_m^{(\gamma,\beta)}\left(\frac{2\,y}{1-x} - 1\right), \quad 0 \leqslant m \leqslant n,$$

and

$$S_{n,m}^{\sigma_1}(x,y) = p_{n-m,\sigma_1}^{(m)}(x)\,(1-x)^m\,P_m^{(\beta,\gamma)}\left(\frac{2\,(1-x-y)}{1-x} - 1\right), \quad 0 \leqslant m \leqslant n.$$

On one hand, for $w_m^\sigma(s)$ defined in (4), we have $w_m^{\sigma_0}(s) = w_m^{\sigma_1}(s)$ for $m \geqslant 0$. Since we have standardized $p_{n,\sigma}^{(m)}(s)$ as having the same leading coefficient as the Jacobi polynomial $P_n^{(\beta_m^\sigma, \alpha_{\sigma(1)})}(2s-1)$ with $\beta_m^\sigma = \alpha_{\sigma(2)} + \alpha_{\sigma(3)} + 2m + 1$, we have that for $n, m \geqslant 0$, $p_{n,\sigma_0}^{(m)}(s) = p_{n,\sigma_1}^{(m)}(s)$.

On the other hand, the Jacobi polynomials satisfy ([17, p. 59])

$$P_n^{(a,b)}(x) = (-1)^n P_n^{(b,a)}(-x).$$

Therefore,

$$P_m^{(\beta,\gamma)}\left(\frac{2\,(1-x-y)}{1-x} - 1\right) = P_m^{(\beta,\gamma)}\left(1 - \frac{2\,y}{1-x}\right) = (-1)^m P_m^{(\gamma,\beta)}\left(\frac{2\,y}{1-x} - 1\right).$$

It follows that $S_{n,m}^{\sigma_1}(x,y) = (-1)^m S_{n,m}^{\sigma_0}(x,y)$. $\square$

The following auxiliary lemma is used to prove the analogous result for type II Sobolev inner product with σ_0 and σ_1.

Lemma 5.2. *Let $\{q_{n,\sigma_0}(t)\}_{n\geqslant 0}$ and $\{q_{n,\sigma_1}(t)\}_{n\geqslant 0}$ be the sequences of orthogonal polynomials associated with the bilinear form defined in (7) with σ_0 and σ_1. Then,*

$$q_{n,\sigma_1}(t) = (-1)^n q_{n,\sigma_0}(1-t), \quad n \geqslant 0.$$

Proof. Recall that (7) with σ_0 and σ_1 reads

$$(f, g)^{\sigma_0} = \int_0^1 [f(t)\,g(t) + \lambda\,f'(t)\,g'(t)]\,(1-t)^\gamma t^\beta dt,$$

and

$$(f, g)^{\sigma_1} = \int_0^1 [f(t)\,g(t) + \lambda\,f'(t)\,g'(t)]\,(1-t)^\beta t^\gamma dt.$$

We can write

$$q_{n,\sigma_0}(1-t) = \sum_{k=0}^n c_{n,k}\,q_{k,\sigma_1}(t), \quad n \geqslant 0,$$

where
$$c_{n,k} = \frac{(q_{n,\sigma_0}(1-t), q_{k,\sigma_1}(t))^{\sigma_1}}{(q_{k,\sigma_1}(t), q_{k,\sigma_1}(t))^{\sigma_1}}, \quad 0 \leqslant k \leqslant n.$$

We compute

$$(q_{n,\sigma_0}(1-t), q_{k,\sigma_1}(t))^{\sigma_1}$$
$$= \int_0^1 \left[q_{n,\sigma_0}(1-t)\, q_{k,\sigma_1}(t) + \lambda\, q'_{n,\sigma_0}(1-t)\, q'_{k,\sigma_1}(t) \right] (1-t)^{\beta} t^{\gamma}\, dt.$$

Setting $y = 1 - t$, we obtain

$$(q_{n,\sigma_0}(1-t), q_{k,\sigma_1}(t))^{\sigma_1}$$
$$= \int_0^1 \left[q_{n,\sigma_0}(y)\, q_{k,\sigma_1}(1-y) + \lambda\, q'_{n,\sigma_0}(y)\, q'_{k,\sigma_1}(1-y) \right] y^{\beta}(1-y)^{\gamma}\, dy$$
$$= (q_{n,\sigma_0}(y), q_{k,\sigma_1}(1-y))^{\sigma_0}.$$

Since $q_{n,\sigma_0}(y)$ is orthogonal with respect to $(\cdot, \cdot)^{\sigma_0}$, we get

$$c_{n,k} = 0, \quad 0 \leqslant k \leqslant n-1.$$

Since the leading coefficients of $q_{n,\sigma_0}(1-t)$ and $q_{n,\sigma_1}(t)$ are the same as the leading coefficients of $P_n^{(\gamma,\beta)}(1-2t)$ and $P_n^{(\beta,\gamma)}(2t-1)$, respectively, we have $c_{n,n} = (-1)^n$, and the result follows. $\qquad \square$

We are ready to state the analogous result of Proposition 5.1 for the type II Sobolev inner product with σ_0 and σ_1.

Proposition 5.3. *For all* $P, Q \in \Pi^2$,

$$\langle P, Q \rangle_{\sigma_0}^{II} = \langle P, Q \rangle_{\sigma_1}^{II} = b_{\alpha,\beta,\gamma} \iint_{\mathbf{T}^2} \left[PQ + \lambda (1-x)^2\, \partial_y P\, \partial_y Q \right] W_{\alpha,\beta,\gamma}(x,y)\, dy dx.$$

Moreover, let $\widetilde{S}_{n,m}^{\sigma_0}(x,y)$ *and* $\widetilde{S}_{n,m}^{\sigma_1}(x,y)$ *be the polynomials defined in Theorem 4.4. Then*

$$\widetilde{S}_{n,m}^{\sigma_1}(x,y) = (-1)^m \widetilde{S}_{n,m}^{\sigma_0}(x,y), \quad n \geqslant 0, \quad 0 \leqslant m \leqslant n.$$

Proof. As in Proposition 5.1, it is easy to check that $\langle P, Q \rangle_{\sigma_0}^{II} = \langle P, Q \rangle_{\sigma_1}^{II}$ for all $P, Q \in \Pi^2$.

Recall that

$$\widetilde{S}_{n,m}^{\sigma_0}(x,y) = P_{n-m}^{(\beta+\gamma+2m+1,\alpha)}(2x-1)(1-x)^m\, q_{m,\sigma_0}\left(\frac{y}{1-x}\right), \quad 0 \leqslant m \leqslant n,$$

and

$$\widetilde{S}_{n,m}^{\sigma_1}(x,y) = P_{n-m}^{(\beta+\gamma+2m+1,\alpha)}(2x-1)(1-x)^m\, q_{m,\sigma_1}\left(\frac{1-x-y}{1-x}\right), \quad 0 \leqslant m \leqslant n.$$

By Lemma 5.2, we have that

$$q_{m,\sigma_1}\left(\frac{1-x-y}{1-x}\right) = (-1)^m\, q_{m,\sigma_0}\left(\frac{y}{1-x}\right).$$

Hence, $\widetilde{S}_{n,m}^{\sigma_1}(x,y) = (-1)^m \widetilde{S}_{n,m}^{\sigma_0}(x,y)$. $\qquad \square$

Propositions 5.4 and 5.5 state the analogous results for the remaining cases, and they can be proved similarly to Proposition 5.1 and Proposition 5.3.

Proposition 5.4. *For all $P, Q \in \Pi^2$,*

$$\langle P, Q \rangle_{\sigma_2}^I = \langle P, Q \rangle_{\sigma_3}^I$$

$$= b_{\alpha,\beta,\gamma} \int\int_{\mathbf{T}^2} \left[P\,Q + \lambda (\nabla P)^\top \begin{pmatrix} (1-y)^2 & -x\,(1-y) \\ -x\,(1-y) & x^2 \end{pmatrix} \nabla Q \right] W_{\alpha,\beta,\gamma}(x, y)\, dy dx.$$

Moreover, let $S_{n,m}^{\sigma_2}(x, y)$ and $S_{n,m}^{\sigma_3}(x, y)$ be the polynomials defined in Theorem 4.1. Then

$$S_{n,m}^{\sigma_2}(x, y) = (-1)^m S_{n,m}^{\sigma_3}(x, y), \quad n \geqslant 0, \quad 0 \leqslant m \leqslant n.$$

Similarly,

$$\langle P, Q \rangle_{\sigma_2}^{II} = \langle P, Q \rangle_{\sigma_3}^{II} = b_{\alpha,\beta,\gamma} \int\int_{\mathbf{T}^2} \left[P\,Q + \lambda\,(1-y)^2\, \partial_x P\, \partial_x Q \right] W_{\alpha,\beta,\gamma}(x, y)\, dy dx.$$

Moreover, let $\widetilde{S}_{n,m}^{\sigma_2}(x, y)$ and $\widetilde{S}_{n,m}^{\sigma_3}(x, y)$ be the polynomials defined in Theorem 4.4. Then

$$\widetilde{S}_{n,m}^{\sigma_2}(x, y) = (-1)^m \widetilde{S}_{n,m}^{\sigma_3}(x, y), \quad n \geqslant 0, \quad 0 \leqslant m \leqslant n.$$

Proposition 5.5. *For all $P, Q \in \Pi^2$,*

$$\langle P, Q \rangle_{\sigma_4}^I = \langle P, Q \rangle_{\sigma_5}^I = b_{\alpha,\beta,\gamma} \int\int_{\mathbf{T}^2} \left[P\,Q + \lambda (\nabla P)^\top \begin{pmatrix} x^2 & x\,y \\ x\,y & y^2 \end{pmatrix} \nabla Q \right] W_{\alpha,\beta,\gamma}(x, y)\, dy dx.$$

Moreover, let $S_{n,m}^{\sigma_4}(x, y)$ and $S_{n,m}^{\sigma_5}(x, y)$ be the polynomials defined in Theorem 4.1. Then

$$S_{n,m}^{\sigma_4}(x, y) = (-1)^m S_{n,m}^{\sigma_5}(x, y), \quad n \geqslant 0, \quad 0 \leqslant m \leqslant n.$$

Similarly,

$$\langle P, Q \rangle_{\sigma_2}^{II} = \langle P, Q \rangle_{\sigma_3}^{II}$$

$$= b_{\alpha,\beta,\gamma} \int\int_{\mathbf{T}^2} \left[P\,Q + \lambda\,(x+y)^2\, (\nabla P)^\top \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \nabla Q \right] W_{\alpha,\beta,\gamma}(x, y)\, dy dx.$$

Moreover, let $\widetilde{S}_{n,m}^{\sigma_4}(x, y)$ and $\widetilde{S}_{n,m}^{\sigma_5}(x, y)$ be the polynomials defined in Theorem 4.4. Then

$$\widetilde{S}_{n,m}^{\sigma_4}(x, y) = (-1)^m \widetilde{S}_{n,m}^{\sigma_5}(x, y), \quad n \geqslant 0, \quad 0 \leqslant m \leqslant n.$$

References

[1] Rabia Aktaş and Yuan Xu, *Sobolev orthogonal polynomials on a simplex*, Int. Math. Res. Not. IMRN **13** (2013), 3087–3131, DOI 10.1093/imrn/rns141. MR3073001

[2] Cleonice F. Bracciali, Antonia M. Delgado, Lidia Fernández, Teresa E. Pérez, and Miguel A. Piñar, *New steps on Sobolev orthogonality in two variables*, J. Comput. Appl. Math. **235** (2010), no. 4, 916–926, DOI 10.1016/j.cam.2010.07.006. MR2727629

[3] Feng Dai and Yuan Xu, *Polynomial approximation in Sobolev spaces on the unit sphere and the unit ball*, J. Approx. Theory **163** (2011), no. 10, 1400–1418, DOI 10.1016/j.jat.2011.05.001. MR2832732

[4] Antonia M. Delgado, Lidia Fernández, Doron S. Lubinsky, Teresa E. Pérez, and Miguel A. Piñar, *Sobolev orthogonal polynomials on the unit ball via outward normal derivatives*, J. Math. Anal. Appl. **440** (2016), no. 2, 716–740, DOI 10.1016/j.jmaa.2016.03.041. MR3484991

[5] Antonia M. Delgado, Teresa E. Pérez, and Miguel A. Piñar, *Sobolev-type orthogonal polynomials on the unit ball*, J. Approx. Theory **170** (2013), 94–106, DOI 10.1016/j.jat.2012.05.013. MR3044047

[6] Herbert Dueñas Ruiz, Omar Salazar-Morales, and Miguel Piñar, *Sobolev orthogonal polynomials of several variables on product domains*, Mediterr. J. Math. **18** (2021), no. 5, Paper No. 227, 21, DOI 10.1007/s00009-021-01852-z. MR4313603

[7] Herbert Dueñas Ruiz, Natalia Pinzón-Cortés, and Omar Salazar-Morales, *Sobolev orthogonal polynomials of high order in two variables defined on product domains*, Integral Transforms Spec. Funct. **28** (2017), no. 12, 988–1008, DOI 10.1080/10652469.2017.1389924. MR3737464

[8] Charles F. Dunkl and Yuan Xu, *Orthogonal polynomials of several variables*, 2nd ed., Encyclopedia of Mathematics and its Applications, vol. 155, Cambridge University Press, Cambridge, 2014, DOI 10.1017/CBO9781107786134. MR3289583

[9] Lidia Fernández, Francisco Marcellán, Teresa E. Pérez, Miguel A. Piñar, and Yuan Xu, *Sobolev orthogonal polynomials on product domains*, J. Comput. Appl. Math. **284** (2015), 202–215, DOI 10.1016/j.cam.2014.09.015. MR3319504

[10] J. C. García-Ardila and M. E. Marriaga, *On Sobolev bilinear forms and polynomial solutions of second-order differential equations*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **115** (2021), no. 4, Paper No. 191, 31, DOI 10.1007/s13398-021-01137-w. MR4314232

[11] Huiyuan Li and Yuan Xu, *Spectral approximation on the unit ball*, SIAM J. Numer. Anal. **52** (2014), no. 6, 2647–2675, DOI 10.1137/130940591. MR3276427

[12] Fátima Lizarte, Teresa E. Pérez, and Miguel A. Piñar, *The radial part of a class of Sobolev polynomials on the unit ball*, Numer. Algorithms **87** (2021), no. 4, 1369–1389, DOI 10.1007/s11075-020-01011-7. MR4287895

[13] Francisco Marcellán and Yuan Xu, *On Sobolev orthogonal polynomials*, Expo. Math. **33** (2015), no. 3, 308–352, DOI 10.1016/j.exmath.2014.10.002. MR3360352

[14] Misael E. Marriaga, Teresa E. Pérez, and Miguel A. Piñar, *Bivariate Koornwinder-Sobolev orthogonal polynomials*, Mediterr. J. Math. **18** (2021), no. 6, Paper No. 234, 36, DOI 10.1007/s00009-021-01875-6. MR4320531

[15] Teresa E. Pérez, Miguel A. Piñar, and Yuan Xu, *Weighted Sobolev orthogonal polynomials on the unit ball*, J. Approx. Theory **171** (2013), 84–104, DOI 10.1016/j.jat.2013.03.004. MR3053718

[16] Miguel Piñar and Yuan Xu, *Orthogonal polynomials and partial differential equations on the unit ball*, Proc. Amer. Math. Soc. **137** (2009), no. 9, 2979–2987, DOI 10.1090/S0002-9939-09-09932-8. MR2506456

[17] Gabor Szegö, *Orthogonal polynomials*, American Mathematical Society Colloquium Publications, Vol. 23, American Mathematical Society, Providence, R.I., 1959. Revised ed. MR0106295

[18] Yuan Xu, *A family of Sobolev orthogonal polynomials on the unit ball*, J. Approx. Theory **138** (2006), no. 2, 232–241, DOI 10.1016/j.jat.2005.11.009. MR2201161

[19] Yuan Xu, *Sobolev orthogonal polynomials defined via gradient on the unit ball*, J. Approx. Theory **152** (2008), no. 1, 52–65, DOI 10.1016/j.jat.2007.11.001. MR2419297

[20] Yuan Xu, *Approximation and orthogonality in Sobolev spaces on a triangle*, Constr. Approx. **46** (2017), no. 2, 349–434, DOI 10.1007/s00365-017-9377-3. MR3691233

[21] Xuhong Yu, Zhongqing Wang, and Huiyuan Li, *Jacobi-Sobolev orthogonal polynomials and spectral methods for elliptic boundary value problems*, Commun. Appl. Math. Comput. **1** (2019), no. 2, 283–308, DOI 10.1007/s42967-019-00016-x. MR4022334

DEPARTAMENTO DE MATEMÁTICA APLICADA, CIENCIA E INGENIERÍA DE MATERIALES Y TECNOLOGÍA ELECTRÓNICA, UNIVERSIDAD REY JUAN CARLOS, SPAIN

Email address: `misael.marriaga@urjc.es`