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On classical orthogonal polynomials and the Cholesky factorization of a class of Hankel matrices

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Classical moment functionals (Hermite, Laguerre, Jacobi, Bessel) can be characterized as those linear functionals whose moments satisfy a second-order linear recurrence relation. In this work, we use this characterization to link the theory of classical orthogonal polynomials and the study of Hankel matrices whose entries satisfy a second-order linear recurrence relation. Using the recurrent character of the entries of such Hankel matrices, we give several characterizations of the triangular and diagonal matrices involved in their Cholesky factorization and connect them with a corresponding characterization of classical orthogonal polynomials.

Keywords: Orthogonal polynomials; Hankel matrices; Cholesky factorization.

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1. Introduction

Classical orthogonal polynomials (Hermite, Laguerre, Jacobi, and Bessel) have been characterized using different approaches. For instance, they can be characterized in terms of differential equations [2], their derivatives [8, 12], structure relations [1, 5, 14], and a Rodrigues formula [19], among others (see [6, 15] and the references therein). More recently, an approach that uses linear functionals and duality was introduced by Maroni [16]. In all of these approaches, the starting point is to use the basis of monomials to represent polynomials and state the results.

However, an interesting (and most recent) approach is to start from the theory of semi-infinite matrices (see, for instance, [3] and [17]). The bibliography on this subject has grown greatly in the last years, and it has become increasingly difficult to do a comprehensive review of all the references. Hence, we refer the reader to [20, 21] (and the references therein) where the algebra of infinite triangular matrices and the algebra of infinite Hessenberg matrices are used to study some aspects of orthogonal polynomials, and to [4, 13] (and the references therein) where the main tool is the Cholesky factorization of Gram matrices of bilinear forms. We remark that the Cholesky factorization proves to be quite fruitful in the study of nonstandard orthogonality such as multiple, matrix, Sobolev, and multivariate orthogonality as well as orthogonality on the unit circle of the complex plane, and have successfully found its way into applications in random matrices, Toda lattices, integrable systems, Riemann–Hilbert problems, Painlevé equations, and Darboux transformations, among others topics.

Our goal is to contribute to the link between matrix factorization and orthogonal polynomials. In particular, we deal with several characterizations of classical orthogonal polynomials. However, we shift our paradigm from infinite matrices to the *finite* Gram matrix G_n associated with a bilinear form defined on the linear space of polynomials of degree at most $n \geq 0$. For standard orthogonality, G_n is a Hankel matrix (all of its antidiagonals are constant). Taking into account that the moments of a linear functional associated with a family of classical orthogonal polynomials satisfy a second-order linear recurrence relation [14], we can say that this paper deals with Hankel matrices with an additional structure: the entries of G_n satisfy such recurrence relation. In this way, we can extend the bilinear form to the linear space of polynomials of degree at most $n+1$ by constructing a new Gram matrix G_{n+1} by means of bordering G_n with a new row and column whose entries are obtained using the recurrence relation and the entries of G_n. The resulting matrix G_{n+1} will also be a Hankel matrix with the additional structure mentioned above. Consequently, it will be possible to prove by induction that the properties satisfied by G_n are also satisfied by G_{n+1}.

The change from infinite matrices to subsequently bordering finite matrices is motivated by an alternative proof of a classical result about the interlacing of zeros of orthogonal polynomials of consecutive degrees found in [6]. This classical result states that if the zeros of any two polynomials of consecutive degrees interlace, then these polynomials are elements of a sequence of orthogonal polynomials associated with a positive definite moment functional. This result can be proved using the Euclidean division algorithm for polynomials. However, this result can also be deduced using the following theorem about interlacing eigenvalues of Hermitian matrices found in [10, p. 185]:

Theorem 1.1. *Let* n *be a given positive integer, and let* $\{x_{n,k}\}_{k=1}^{n}$ *and* $\{x_{n-1,k}\}_{k=1}^{n-1}$ *be two given sequences of numbers such that*

$$x_{n,1} < x_{n-1,1} < \cdots < x_{n,k} < x_{n-1,k} < x_{n,k+1} < \cdots < x_{n-1,n-1} < x_{n,n}.$$

Let $\Lambda = \operatorname{diag}(x_{n-1,1}, x_{n-1,2}, \cdots, x_{n-1,n-1})$*. Then there exists real number* b *and a real vector* $\bar{y} = (y_1, \ldots, y_{n-1})^\top \in \mathbb{R}^{n-1}$ *such that* $(x_{n,k})_{k=1}^{n}$ *is the set of eigenvalues of the real symmetric matrix*

$$B = \left(\begin{array}{c|c} \Lambda & \bar{y} \\ \hline \bar{y}^\top & b \end{array}\right).$$

From this, it seems reasonable to think that the procedure of bordering matrices encoding information about polynomial sequences is well suited for presenting and deducing results about orthogonal polynomials most likely due to the fact that for all $n \geq 0$, the linear space of polynomials of degree at most n is a subspace of the space of polynomials of degree at most $n+1$. In this way, the Gram matrices of bilinear forms associated with classical orthogonal polynomials possess the adequate structure to start exploring our proposed paradigm.

The paper is organized as follows. Section 2 presents basic background on classical orthogonal polynomials and their associated linear functionals, and we introduce classical sequences of real numbers in Sec. 3. In Sec. 4, we discuss the Cholesky factorization of Hankel matrices obtained from given sequences of real number and its relation to orthogonal polynomials. We present several characterizations of classical sequences of real numbers in Sec. 5.

2. Orthogonal Polynomials and Linear Functionals

For $n \geq 0$, let Π_n be the linear space of polynomials of degree at most n of a real variable and real coefficients, and let $\Pi = \bigcup_{n\geq 0} \Pi_n$.

Let Π^* denote the algebraic dual space of Π. That is, Π^* is the linear space of linear functionals defined on Π,

$$\Pi^* = \{\mathbf{u} : \Pi \to \mathbb{R} : \mathbf{u} \text{ is linear}\}.$$

We denote by $\langle \mathbf{u}, p\rangle$ the image of the polynomials p under the linear functional $\mathbf{u}$.

Any linear functional $\mathbf{u}$ is completely defined by the values

$$\mu_n := \langle \mathbf{u}, x^n\rangle, \quad n \geq 0,$$

and extended by linearity to all polynomials, where μ_n is called the nth moment of $\mathbf{u}$. Therefore, we refer to $\mathbf{u}$ as a moment functional.

A moment functional $\mathbf{u}$ is called positive definite if $\langle \mathbf{u}, p^2\rangle > 0$ for every nonzero polynomial $p \in \Pi$.

Let $\mathbf{u}$ be a moment functional. A sequence of polynomials $\{P_n(x)\}_{n\geq 0}$ is called an orthogonal polynomial sequence (OPS) with respect to $\mathbf{u}$ if

(1) $\deg P_n = n$,
(2) $\langle \mathbf{u}, P_n\, P_m\rangle = h_n\, \delta_{n,m}$, with $h_n \neq 0$.

Here, $\delta_{n,m}$ denotes the Kronecker delta defined as follows:

$$\delta_{n,m} = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

If there exists an OPS associated with $\mathbf{u}$, then $\mathbf{u}$ is called quasi-definite. Positive definite moment functionals are quasi-definite.

Observe that an OPS $\{P_n(x)\}_{n\geq 0}$ constitutes a basis for Π. If for all $n \geq 0$, the leading coefficient of $P_n(x)$ is 1, then $\{P_n(x)\}_{n\geq 0}$ is called a monic orthogonal polynomial sequence (MOPS).

Given a moment functional $\mathbf{u}$ and a polynomial $q(x)$, we define the left multiplication of $\mathbf{u}$ by $q(x)$ as the moment functional $q\,\mathbf{u}$ such that

$$\langle q\,\mathbf{u}, p\rangle = \langle \mathbf{u}, q\,p\rangle, \quad \forall\, p \in \Pi,$$

and we define the distributional derivative $D\mathbf{u}$ by

$$\langle D\mathbf{u}, p\rangle = -\langle \mathbf{u}, p'\rangle, \quad \forall\, p \in \Pi.$$

Moreover, the product rule is satisfied, that is,

$$D(q\,\mathbf{u}) = q'\,\mathbf{u} + q\,D\mathbf{u}.$$

Definition 2.1. Let $\mathbf{u}$ be a quasi-definite moment functional, and let $\{P_n(x)\}_{n\geq 0}$ be an OPS with respect to $\mathbf{u}$. Then $\mathbf{u}$ is classical if there are nonzero polynomials $\phi(x)$ and $\psi(x)$ with $\deg\phi \leq 2$ and $\deg\psi = 1$, such that $\mathbf{u}$ satisfies the distributional Pearson equation

$$D(\phi\,\mathbf{u}) = \psi\,\mathbf{u}. \tag{2.1}$$

The sequence $\{P_n(x)\}_{n\geq 0}$ is called a classical OPS.

The following characterizations of classical moment functionals and OPS will be of central importance in the sequel.

Theorem 2.2. *Let* $\mathbf{u}$ *be a quasi-definite moment functional, and* $\{P_n(x)\}_{n\geq 0}$ *its associated MOPS. The following statements are equivalent*:

(1) $\mathbf{u}$ *is a classical moment functional.*
(2) (*Bochner* [2]) *There are nonzero polynomials* $\phi(x)$ *and* $\psi(x)$ *with* $\deg\phi \leq 2$ *and* $\deg\psi = 1$ *such that, for* $n \geq 0$, $P_n(x)$ *satisfies*

$$\phi(x)\, P_n''(x) + \psi(x)\, P_n'(x) = \lambda_n\, P_n(x), \tag{2.2}$$

where $\lambda_n = n\,(\frac{n-1}{2}\phi'' + \psi')$.

(3) (*Hahn* [8, 9]) *There is a nonzero polynomial* $\phi(x)$ *with* $\deg\phi \leq 2$, *such that* $\{\frac{P'_{n+1}(x)}{n+1}\}_{n\geq 0}$ *is the MOPS associated with the moment functional* $\mathbf{v} = \phi(x)\,\mathbf{u}$.

(4) (*First structure relation*, [1]) *There is a nonzero polynomial with* $\deg\phi \leq 2$, *and real numbers* a_n, b_n, c_n, $n \geq 1$, *with* $c_n \neq 0$, *such that*

$$\phi(x)\,P'_n(x) = a_n\,P_{n+1}(x) + b_n\,P_n(x) + c_n\,P_{n-1}(x), \quad n \geq 1.$$

(5) (*Second structure relation*, [5, 14]) *There are real numbers* α_n *and* β_n, $n \geq 2$, *such that*

$$P_n(x) = \frac{P'_{n+1}(x)}{n+1} + \alpha_n\,\frac{P'_n(x)}{n} + \beta_n\,\frac{P'_{n-1}(x)}{n-1}, \quad n \geq 2. \tag{2.3}$$

(6) (*Rodrigues formula*, [19]) *There is a nonzero polynomial* $\phi(x)$ *with* $\deg\phi \leq 2$, *and a nonzero real number* $k_n \neq 0$ *such that*

$$D^n(\phi^n(x)\,\mathbf{u}) = k_n\,P_n(x)\,\mathbf{u}, \quad n \geq 0.$$

It is well known (see [2] as well as [11]) that, up to affine transformations of the independent variable, the only families of positive definite classical orthogonal polynomials are the Hermite, Laguerre, and Jacobi polynomials. The corresponding moment functionals admit an integral representation of the following form [18]

$$\langle \mathbf{u}, p\rangle = \int_I p(x)\,w(x)\,dx, \quad p \in \Pi,$$

where $I = \mathbb{R}$ and $w(x) = e^{-x^2}$ in the Hermite case, $I = (0, +\infty)$ and $w(x) = x^\alpha e^{-x}$ with $\alpha > -1$ in the Laguerre case, and $I = (-1, 1)$ and $w(x) = (1-x)^\alpha(1+x)^\beta$ with $\alpha, \beta > -1$ in the Jacobi case. We note that in each case, $w(x) > 0$ on I and, thus, we say that $w(x)$ is a weight function.

The definition of classical moment functionals in terms of the distributional Pearson equation not only encompasses positive definite moment functionals associated with weight functions, but includes the nonpositive case as well. Considering the nonpositive definite case gives rise to the Bessel classical moment functional satisfying the distributional Pearson equation (2.1) with $\phi(x) = x^2$ and $\psi(x) = ax + 2$. The Bessel functional is quasi-definite when $a \neq -1, -2, \ldots$. Moreover, it has the following integral representation

$$\langle \mathbf{u}, p\rangle = \int_c p(z)\,w(z)\,dz, \quad p \in \Pi,$$

where $w(z) = (2\,\pi\,i)^{-1} z^{a-2} e^{-2/z}$, and c is the unit circle oriented in the counterclockwise direction.

Observe that from Theorem 2.2, if $\mathbf{u}$ is a classical moment functional satisfying (2.1), then $\mathbf{v} = \phi(x)\,\mathbf{u}$ is a classical moment functional satisfying the Pearson equation

$$D(\phi\,\mathbf{v}) = (\psi + \phi')\,\mathbf{v}.$$

Iterating this idea, we get that the high-order derivatives of classical orthogonal polynomials are again classical orthogonal polynomials of the same type.

Theorem 2.3 ([8, 11, 12]). *Let* $\mathbf{u}$ *be a classical moment functional satisfying* (2.1), *and* $\{P_n(x)\}_{n\geq 0}$ *its corresponding MOPS. For* $k \geq 0$, *let* $\mathbf{v}_k = \phi^k(x)\,\mathbf{u}$ *and* $\{Q_{n,k}(x)\}_{n\geq 0}$ *be the sequence of polynomials given by*

$$Q_{n,k}(x) := \frac{1}{(n+1)_k} P^{(k)}_{n+k}(x), \quad n \geq 0, \tag{2.4}$$

where $p^{(k)}$ *is the* k*th derivative of* p, *and* $(\nu)_k = \nu\,(\nu+1)\cdots(\nu+k-1)$, $(\nu)_0 = 1$, *denotes the Pochhammer symbol. Then, for each* $k \geq 0$, $\{Q_{n,k}(x)\}_{n\geq 0}$ *is a MOPS associated with the moment functional* $\mathbf{v}_k$ *satisfying*

$$D(\phi\,\mathbf{v}_k) = \psi_k\,\mathbf{v}_k,$$

where $\psi_k(x) = \psi(x) + k\,\phi'(x)$. *Hence,* $\mathbf{v}_k$ *is a classical moment functional.*

3. Classical Sequences of Numbers

This section is devoted to presenting the definition of classical moment functionals from a different approach. We start by introducing sequences of real numbers that satisfy a second-order recurrence relation and use them to construct linear functionals defined on Π.

Definition 3.1. Let $\{\mu_n\}_{n\geq 0}$ be a sequence of real numbers with $\mu_0 \neq 0$. Then $\{\mu_n\}_{n\geq 0}$ is a pre-classical sequence if there are real numbers a, b, c, d, e satisfying

$$|a| + |b| + |c| > 0, \quad n\,a + d \neq 0 \quad n \geq 0,$$

such that the following holds

$$(n\,a + d)\mu_{n+1} + (n\,b + e)\,\mu_n + n\,c\,\mu_{n-1} = 0, \quad n \geq 0. \tag{3.1}$$

By convention, $\mu_n = 0$ whenever $n < 0$.

Let $\{\mu_n\}_{n\geq 0}$ be a pre-classical sequence of real numbers. Then it is possible to define a functional $\mathbf{u}$ as follows:

$$\mu_n := \langle \mathbf{u}, x^n \rangle, \quad n \geq 0,$$

and extend it by linearity to all polynomials, where μ_n is called the nth moment of $\mathbf{u}$. Therefore, we refer to $\mathbf{u}$ as a pre-classical moment functional. Observe that the condition $n\,a + d \neq 0$, $n \geq 0$, guarantees that $\mathbf{u}$ is completely defined since each moment

$$\mu_{n+1} = -\frac{1}{n\,a + d}[(n\,b + e)\mu_n + n\,c\,\mu_{n-1}], \quad n \geq 0,$$

is well defined.

The recurrence relation (3.1) can be passed down to the pre-classical moment functional associated with $\{\mu_n\}_{n\geq 0}$.

Theorem 3.2. *A sequence* $\{\mu_n\}_{n\geq 0}$ *is pre-classical if and only if there are nonzero polynomials* $\phi(x)$ *and* $\psi(x)$ *with* $\deg\phi \leq 2$, $\deg\psi = 1$, *and* $\frac{n}{2}\phi'' + \psi' \neq 0$ *for*

$n \geq 0$, *such that the moment functional* $\mathbf{u}$ *defined by* $\mu_n = \langle \mathbf{u}, x^n \rangle$ *satisfies the distributional Pearson equation*

$$D(\phi\,\mathbf{u}) = \psi\,\mathbf{u}.$$

Proof. Suppose that $\mathbf{u}$ satisfies (2.1) with nonzero $\phi(x) = a\,x^2 + b\,x + c$ and $\psi(x) = d\,x + e$ with $\deg \psi = 1$ such that $0 \neq \frac{n}{2}\phi'' + \psi' = n\,a + d$ for $n \geq 0$. Then

$$\langle \mathbf{u}, \phi\,p' + \psi\,p \rangle = 0, \quad \forall\, p \in \Pi.$$

In particular,

$$0 = \langle \mathbf{u}, n\,\phi\,x^{n-1} + \psi\,x^n \rangle = (n\,a + d)\mu_{n+1} + (n\,b + e)\mu_n + n\,c\,\mu_{n-1}, \quad n \geq 0.$$

Therefore $\{\mu_n\}_{n\geq 0}$ is a pre-classical sequence of real numbers and, thus, $\mathbf{u}$ is a pre-classical moment functional. It is easy to verify that the implications in the opposite direction hold by inverting each of the previous steps. □

For any sequence $\{\mu_n\}_{n\geq 0}$, we can define the sequence of matrices $\{G_n\}_{n\geq 0}$ where G_n is an $(n+1)\times(n+1)$ matrix given by $G_0 = \mu_0$ and

$$G_n = \left[\begin{array}{ccc|c} & & & \mu_n \\ & G_{n-1} & & \vdots \\ & & & \mu_{2n-1} \\ \hline \mu_n & \cdots & \mu_{2n-1} & \mu_{2n} \end{array}\right] = \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix}, \quad n \geq 0. \tag{3.2}$$

In particular, consider the sequence $\{\mu_n\}_{n\geq 0}$ with $\mu_0 = 1$ and $\mu_n = 0$ for $n \geq 1$. Observe that this sequence corresponds to the linear functional $\boldsymbol{\delta} \in \Pi^*$, known as the Dirac delta, defined as follows:

$$\langle \boldsymbol{\delta}, p \rangle = p(0), \quad \forall\, p \in \Pi.$$

In this case, we have $G_0 = 1$,

$$G_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix},$$

and $\det G_n = 0$ for $n \geq 1$. In the sequel, we will need to exclude this and other similar cases and, therefore, we impose that $\det G_n \neq 0$ for $n \geq 0$. Hence, we have the following definition.

Definition 3.3. A pre-classical sequence $\{\mu_n\}_{n\geq 0}$ is classical if the sequence of matrices $\{G_n\}_{n\geq 0}$ defined as in (3.2) satisfy $\det G_n \neq 0$ for $n \geq 0$. The moment functional defined by $\mu_n = \langle \mathbf{u}, x^n \rangle$ is called a classical moment functional.

In the sequel, some orthogonal bases for $\mathbb{R}^{n+1}$ associated with a classical sequence of numbers will play an important role in our study of such sequences. Therefore, in the following section we discuss the orthogonal structure of $\mathbb{R}^{n+1}$ induced by general sequences of numbers.

4. Sequences of Numbers and Orthogonality

Given an integer $n \geq 0$, a sequence of numbers induces a bilinear form in $\mathbb{R}^{n+1}$ whose Gram matrix is G_n defined in (3.2). In this section, we explore orthogonal bases of $\mathbb{R}^{n+1}$ associated with a sequence of numbers and use it to construct bases of polynomials in Π_n orthogonal with respect to its corresponding moment functional.

For $n \geq 0$, we will denote by $\bar{e}_0, \ldots, \bar{e}_n$ the columns of the identity matrix I_{n+1}, that is, $I_{n+1} = [\bar{e}_0\ \bar{e}_1 \cdots \bar{e}_n]$. The set of column vectors $\mathcal{E}_n = \{\bar{e}_0, \ldots, \bar{e}_n\}$ is called the canonical basis for $\mathbb{R}^{n+1}$.

Definition 4.1. Let $\{\mu_n\}_{n\geq 0}$ be a sequence of numbers. For $n \geq 0$, $\mathfrak{B}_n(\cdot,\cdot)$ denotes the bilinear form defined by

$$\mathfrak{B}_n(\bar{u},\bar{v}) = \bar{u}^\top G_n\, \bar{v}, \quad \forall\, \bar{u},\bar{v} \in \mathbb{R}^{n+1},$$

and is called the bilinear form associated with $\{\mu_n\}_{n\geq 0}$ (relative to the canonical basis of $\mathbb{R}^{n+1}$).

For $n \geq 0$, if $\det G_n > 0$, then $\mathfrak{B}_n(\cdot,\cdot)$ is an inner product on $\mathbb{R}^{n+1}$. In this case, we can define the norm

$$\|\bar{u}\|_n = \sqrt{\mathfrak{B}_n(\bar{u},\bar{u})}, \quad \forall\, \bar{u} \in \mathbb{R}^{n+1}.$$

Of course, there are many orthogonal bases for $\mathbb{R}^{n+1}$ associated with $\mathfrak{B}_n$. However, we are interested in orthogonal bases obtained from the Cholesky factorization of G_n. Recall that if $\det G_n \neq 0$, there is an $(n+1)\times(n+1)$ unit lower triangular matrix (that is, with 1s in its main diagonal) S_n^{-1} and an $(n+1)\times(n+1)$ nonsingular diagonal matrix H_n such that

$$G_n = S_n^{-1}\, H_n\, S_n^{-\top},$$

where $S_n^{-\top} = (S_n^{-1})^\top$ (see Theorem 4.1.3 in [7]). Moreover, this matrix factorization is unique. We can immediately observe that if we write the above identity as

$$S_n\, G_n\, S_n^\top = H_n,$$

then we have an orthogonality relation for the columns of $S_n^\top$ as we show in the following theorem.

Theorem 4.2. *Let $G_n = S_n^{-1}\, H_n\, S_n^{-\top}$ be the Cholesky decomposition of G_n. Then the columns of $S_n^\top$ form an orthogonal basis for $\mathbb{R}^{n+1}$ with respect to the bilinear form $\mathfrak{B}_n$.*

Proof. If $\bar{s}_0, \bar{s}_1, \ldots, \bar{s}_n$ are the columns of $S_n^\top$, then

$$H_n = S_n\, G_n\, S_n^\top = \begin{bmatrix} \bar{s}_0^\top \\ \bar{s}_1^\top \\ \vdots \\ \bar{s}_n^\top \end{bmatrix} G_n\ \begin{bmatrix} \bar{s}_0\ \bar{s}_1\ \cdots\ \bar{s}_n \end{bmatrix}.$$

Since H_n is a diagonal matrix and G_n is the representation of the bilinear form $\mathfrak{B}_n$, it follows that

$$\mathfrak{B}_n(\bar{s}_i, \bar{s}_j) = 0 \quad \text{for } i \neq j$$

and

$$\mathfrak{B}_n(\bar{s}_i, \bar{s}_i) = h_i, \quad i = 0, 1, \ldots, n,$$

where h_i is the ith nonzero entry of H_n, that is, $H_n = \operatorname{diag}(h_0, h_1, \ldots, h_n)$. □

The following theorem shows that the Cholesky factorization of G_n and G_{n+1} are related. In fact, the Cholesky factorization of G_{n+1} is obtained by bordering S_n and H_n with a new row and a new column. The proof relies heavily on the expressions for G_n^{-1} and $\det G_n$ in terms of cofactors. If $(G_n)_{i,j}$ is the (i,j)-cofactor of G_n with $i, j = 0, 1, \ldots, n$, then

$$\det G_n = \sum_{k=0}^{n} \mu_{i+k}\,(G_n)_{i,k}, \quad G_n^{-1} = \frac{1}{\det G_n}\begin{bmatrix} (G_n)_{0,0} & (G_n)_{1,0} & \cdots & (G_n)_{n,0} \\ (G_n)_{0,1} & (G_n)_{1,1} & \cdots & (G_n)_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ (G_n)_{0,n} & (G_n)_{1,n} & \cdots & (G_n)_{n,n} \end{bmatrix}.$$

The result also involves a *formal determinant* of size $(n+2) \times (n+2)$ with vectors appearing in the last row. By this, we mean the symbolic Laplace expansion of a determinant across the last row, which results in a linear combination of the involved vectors.

Hereon, $\mathsf{0}$ will denote the zero matrix of appropriate size.

Theorem 4.3. *Let $n \geq 0$, and let $S_n^{-1} H_n S_n^{-\top}$ be the Cholesky factorization of G_n. Then, $G_{n+1} = S_{n+1}^{-1} H_{n+1} S_{n+1}^{-\top}$ with*

$$S_{n+1}^{\top} = \left[\begin{array}{c|c} S_n^{\top} & \hat{s}_{n+1} \\ \hline \mathsf{0} & 1 \end{array}\right], \quad H_{n+1} = \left[\begin{array}{c|c} H_n & \mathsf{0} \\ \hline \mathsf{0} & h_{n+1} \end{array}\right],$$

where $[\hat{s}_{n+1}^{\top}\ 1]^{\top}$ is a vector given by the formal determinant

$$\begin{aligned} \begin{bmatrix} \hat{s}_{n+1} \\ 1 \end{bmatrix} &= \frac{1}{\det G_n} \det \left[\begin{array}{ccc|c} & & & \mu_{n+1} \\ & G_n & & \vdots \\ & & & \mu_{2n+1} \\ \hline \bar{e}_0 & \cdots & \bar{e}_n & \bar{e}_{n+1} \end{array}\right] \\ &= \bar{e}_{n+1} + \frac{1}{\det G_n}\left((G_{n+1})_{n+1,0}\,\bar{e}_0 + \cdots + (G_{n+1})_{n+1,n}\,\bar{e}_n\right) \end{aligned} \tag{4.1}$$

and

$$h_{n+1} = \frac{\det G_{n+1}}{\det G_n}. \tag{4.2}$$

Proof. Observe that the columns of $[S_n \ \ 0]^\top$ are linear combinations of the vectors $\bar{e}_0, \ldots, \bar{e}_n \in \mathbb{R}^{n+2}$ (that is, the first $n+1$ columns of I_{n+2}). Then $H_{n+1} = S_{n+1}\, G_{n+1}\, S_{n+1}^\top$ with $\hat{s}_{n+1}$ and H_{n+1} as in (4.1) and (4.2) if and only if

$$\mathfrak{B}_{n+1}([\hat{s}_{n+1}^\top \ \ 1]^\top, \bar{e}_i) = 0, \quad i = 0, 1, \ldots, n \tag{4.3}$$

and

$$\mathfrak{B}_{n+1}([\hat{s}_{n+1}^\top \ \ 1]^\top, [\hat{s}_{n+1}^\top \ \ 1]^\top) = \frac{\det G_{n+1}}{\det G_n}.$$

The condition (4.3) can be written as a system of linear equations:

$$G_n\, \hat{s}_{n+1} = -\begin{bmatrix} \mu_{n+1} \\ \vdots \\ \mu_{2n+1} \end{bmatrix}. \tag{4.4}$$

By Cramer's rule,

$$\hat{s}_{n+1} = \frac{1}{\det G_n}((G_{n+1})_{n+1,0}\, \bar{e}_0 + \cdots + (G_{n+1})_{n+1,n}\, \bar{e}_n), \tag{4.5}$$

which is the formal expansion of the determinant (4.1) across the last row.

Moreover, since $\det G_n \neq 0$, from (4.4) we also have

$$\hat{s}_{n+1} = -G_n^{-1} \begin{bmatrix} \mu_{n+1} \\ \vdots \\ \mu_{2n+1} \end{bmatrix} \tag{4.6}$$

and

$$S_{n+1}\, G_{n+1}\, S_{n+1}^\top = \left[\begin{array}{c|c} S_n & 0 \\ \hline \hat{s}_{n+1}^\top & 1 \end{array}\right] \left[\begin{array}{ccc|c} & & & \mu_{n+1} \\ & G_n & & \vdots \\ & & & \mu_{2n+1} \\ \hline \mu_{n+1} & \cdots & \mu_{2n+1} & \mu_{2n+2} \end{array}\right] \left[\begin{array}{c|c} S_n^\top & \hat{s}_{n+1} \\ \hline 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} H_n & S_n\, B_n \\ \hline C_n\, S_n^\top & h_{n+1} \end{array}\right],$$

where

$$B_n = G_n\, \hat{s}_{n+1} + \begin{bmatrix} \mu_{n+1} \\ \vdots \\ \mu_{2n+1} \end{bmatrix}, \quad C_n = \hat{s}_{n+1}^\top\, G_n + \begin{bmatrix} \mu_{n+1} \\ \vdots \\ \mu_{2n+1} \end{bmatrix}^\top,$$

$$h_{n+1} = [\hat{s}_{n+1}^\top \,|\, 1] \left[\begin{array}{c} \mu_{n+1} \\ \vdots \\ \mu_{2n+1} \\ \hline \mu_{2n+2} \end{array}\right].$$

It follows from (4.6) that $B_n = \mathtt{0}$ and $C_n = \mathtt{0}$. Finally, using (4.5) and the fact that $(G_{n+1})_{n+1,n+1} = \det G_n$, we obtain

$$h_{n+1} = \frac{1}{\det G_n} \sum_{j=0}^{n+1} \mu_{n+1+j} (G_{n+1})_{n+1,j} = \frac{\det G_{n+1}}{\det G_n},$$

which proves (4.2). □

We can reformulate the above discussion in the context of Π_n as follows. Given a sequence of numbers $\{\mu_n\}_{n\geq 0}$, denote by $\mathbf{u}$ the moment functional defined by $\mu_n = \langle \mathbf{u}, x^n \rangle$. If $\det G_n \neq 0$, then

$$\langle \mathbf{u}, p\, q \rangle, \quad p, q \in \Pi_n,$$

is a (nondegenerate) bilinear form defined on Π_n. It is easy to see that its Gram matrix relative to the basis of monomials is G_n. Let $G_n = S_n^{-1} H_n S_n^{-\top}$ be the Cholesky decomposition of G_n ($\det G_n \neq 0$) and let

$$S_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ s_{1,0} & 1 & 0 & \cdots & 0 \\ s_{2,0} & s_{2,1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n,0} & s_{n,1} & s_{n,2} & \cdots & 1 \end{bmatrix}$$

be the explicit expression of the matrix S_n. It follows from Theorem 4.2 that the set of polynomials $\{P_0(x), P_1(x), \ldots, P_n(x)\}$ where

$$P_k(x) = s_{k,0} + s_{k,1} x + \cdots + s_{k,k-1} x^{k-1} + x^k, \quad k = 0, 1, 2, \ldots, n,$$

form an orthogonal basis for Π_n with respect to $\mathbf{u}$; that is, $\deg P_k = k$ and

$$\langle \mathbf{u}, P_j\, P_i \rangle = h_j\, \delta_{i,j}, \quad 0 \leq i, j \leq n,$$

with $h_j \neq 0$. Moreover, by Theorem 4.3 the polynomial

$$P_{n+1}(x) = \frac{1}{\det G_n} \det \left[\begin{array}{ccc|c} & & & \mu_{n+1} \\ & G_n & & \vdots \\ & & & \mu_{2n+1} \\ \hline 1 & \cdots & x^n & x^{n+1} \end{array} \right],$$

has degree exactly $n+1$, is orthogonal to every polynomial in Π_n and

$$h_{n+1} = \langle \mathbf{u}, P_{n+1}^2 \rangle = \frac{\det G_{n+1}}{\det G_n}.$$

5. Characterizations of Classical Sequences

Our goal for this section is to recast Theorem 2.2 in terms of classical sequences by shifting our point of view from the moment functional $\mathbf{u}$ to the Gram matrix G_n

associated with a bilinear form defined on Π_n. Recall that G_n is a Hankel matrix (all of its antidiagonals are constant). Hence, we can say that this section deals with Hankel matrices with an additional structure: the entries of G_n satisfy the recurrence relation (3.1). In this way, we can extend the bilinear form to Π_{n+1} by constructing a new Gram matrix G_{n+1} by means of bordering G_n with a new row and column whose entries are obtained with (3.1) from the entries of G_n. The resulting matrix G_{n+1} will also be a Hankel matrix with the additional structure mentioned above. Consequently, it will be possible to prove by induction that the properties satisfied by G_n are also satisfied by G_{n+1}.

The following matrices will play an important role in the sequel.

Definition 5.1. Let a, b, c, d, and e be real numbers such that $|a|+|b|+|c|>0$ and $n\,a+d\neq 0$ for $n\geq 0$. For $n\geq 1$, we define the $n\times(n+1)$ matrix R_n recursively as follows:

$$R_n = \left[\begin{array}{c|c} R_{n-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0\cdots 0\ (n-1)\,c\ (n-1)\,b+e & (n-1)\,a+d \end{array}\right], \quad R_1 = \begin{bmatrix} e & d \end{bmatrix}.$$

Consider the differential operator $\mathcal{R}:\Pi\to\Pi$ defined as follows:

$$\mathcal{R}[p] = \phi(x)\,p' + \psi(x)\,p, \quad \forall\, p\in\Pi,$$

where $\phi(x) = a\,x^2 + b\,x + c$ and $\psi(x) = d\,x + e$. Observe that

$$\mathcal{R}[x^n] = (n\,a+d)\,x^{n+1} + (n\,b+e)\,x^n + n\,c\,x^{n-1}, \quad n\geq 0.$$

In this way, the matrix $R_{n+1}^{\top}$ is the matrix representation relative to the basis of monomials of $\mathcal{R}$ restricted to Π_n.

Let $\{\mu_n\}_{n\geq 0}$ be a sequence of real numbers. Define the vector of moments

$$\mathtt{M}_n = [\mu_0\ \mu_1 \cdots \mu_n]^{\top}, \quad n\geq 0.$$

If $\{\mu_n\}_{n\geq 0}$ is pre-classical, then Eq. (3.1) for $\{\mu_0,\ldots,\mu_n\}$ can be written as follows:

$$R_n\,\mathtt{M}_n = \mathtt{0}.$$

This implies that if $\mathbf{u}$ is the moment functional defined as $\mu_n := \langle \mathbf{u}, x^n\rangle$, then by (3.1), the following holds

$$\langle \mathbf{u}, \mathcal{R}[x^n]\rangle = 0, \quad n\geq 0.$$

Now, consider the vector whose entries are the monomials in Π_n:

$$\mathtt{X}_n = [1\ x\cdots x^n]^{\top}.$$

If we define the $n\times(n+1)$ matrix

$$N_n = \left[\begin{array}{c|c} N_{n-1} & \mathtt{0} \\ \hline \mathtt{0} & n \end{array}\right], \quad N_1 = [0\ 1], \quad N_0 = 0,$$

then

$$\frac{d}{dx}\mathtt{X}_n = N_n^\top \,\mathtt{X}_{n-1}.$$

5.1. *Bochner-type characterization*

For $n \geq 0$, let us introduce the $(n+1)\times(n+1)$ matrix $\mathtt{D}_n$ defined as follows:

$$\mathtt{D}_0 = 0, \quad \mathtt{D}_n := R_n^\top N_n, \quad n \geq 1.$$

It is possible to express $\mathtt{D}_n$ recursively as follows:

$$\mathtt{D}_n = \left[\begin{array}{ccc|c} & & & 0 \\ & & & \vdots \\ & \mathtt{D}_{n-1} & & 0 \\ & & & n\,(n-1)\,c \\ & & & n\,((n-1)\,b+e) \\ \hline 0 & \cdots & 0 & n\,((n-1)\,a+d) \end{array}\right], \quad n \geq 1.$$

Observe that $\mathtt{D}_n$ is the matrix representation relative to the basis of monomials of the operator

$$\mathcal{D}[p] := \mathcal{R}\left[\frac{d}{dx}p\right] = \phi(x)\,p'' + \psi(x)\,p', \quad \forall\, p \in \Pi, \tag{5.1}$$

restricted to Π_n.

In the following theorem, we show that $\mathtt{D}_n$ is a self-adjoint matrix with respect to the bilinear form $\mathfrak{B}_n$ given in Definition 4.1.

Theorem 5.2. *Let $\{\mu_n\}_{n\geq 0}$ be a classical sequence satisfying* (3.1), *and let $\mathfrak{B}_n$ be the operator defined in Definition* 4.1. *Then, for $n \geq 0$, the matrix $\mathtt{D}_n$ satisfies*

$$\mathfrak{B}_n(\mathtt{D}_n\bar{u},\bar{v}) = \mathfrak{B}_n(\bar{u},\mathtt{D}_n\bar{v}), \quad \bar{u},\bar{v} \in \mathbb{R}^{n+1}. \tag{5.2}$$

Proof. Observe that proving (5.2) is equivalent to proving

$$\mathtt{D}_n^\top\, G_n = G_n\,\mathtt{D}_n.$$

We prove this for $n \geq 0$ by induction.

It is obvious that $\mathtt{D}_0^\top\, G_0 = G_0\,\mathtt{D}_0$. We also prove the case $n=1$ for the sake of clarity since it is the first nontrivial case. We compute

$$\mathtt{D}_1^\top\, G_1 = \left[\begin{array}{c|c} 0 & 0 \\ \hline e & d \end{array}\right] \left[\begin{array}{c|c} \mu_0 & \mu_1 \\ \hline \mu_1 & \mu_2 \end{array}\right] = \left[\begin{array}{c|c} \mathtt{D}_0^\top & 0 \\ \hline e & d \end{array}\right] \left[\begin{array}{c|c} G_0 & \mu_1 \\ \hline \mu_1 & \mu_2 \end{array}\right].$$

Again, multiplying by blocks, we get

$$\mathtt{D}_1^\top\, G_1 = \left[\begin{array}{c|c} \mathtt{D}_0^\top\, G_0 & \mathtt{D}_0^\top\,\mu_1 \\ \hline d\,\mu_1 + e\,\mu_0 & e\,\mu_1 + d\,\mu_2 \end{array}\right] = \left[\begin{array}{c|c} G_0\,\mathtt{D}_0^\top & 0 \\ \hline d\,\mu_1 + e\,\mu_0 & e\,\mu_1 + d\,\mu_2 \end{array}\right].$$

Since $\{\mu_n\}_{n\geq 0}$ is a classical sequence, by condition (3.1) with $n=0$, we have

$$\mu_1\,\mathtt{D}_0 = 0 = d\,\mu_1 + e\,\mu_0.$$

Therefore, we can write

$$\mathtt{D}_1^\top G_1 = \left[\begin{array}{c|c} G_0\,\mathtt{D}_0^\top & e\,\mu_0 + d\,\mu_1 \\ \hline \mu_1\,\mathtt{D}_0 & e\,\mu_1 + d\,\mu_2 \end{array}\right] = \left[\begin{array}{c|c} G_0 & \mu_1 \\ \hline \mu_1 & \mu_2 \end{array}\right] \left[\begin{array}{c|c} \mathtt{D}_0 & e \\ \hline 0 & d \end{array}\right] = G_1\,\mathtt{D}_1.$$

This proves $\mathtt{D}_1^\top G_1 = G_1\,\mathtt{D}_1$.

Now, suppose that $\mathtt{D}_k^\top G_k = G_k\,\mathtt{D}_k$ holds for some $k\geq 0$. We compute

$$\mathtt{D}_{k+1}^\top G_{k+1} = \left[\begin{array}{c|c} \mathtt{D}_k^\top & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0\ (k+1)\,k\,c\ (k+1)\,(k\,b+e) & (k+1)\,(k\,a+d) \end{array}\right] \times \left[\begin{array}{c|c} G_k & \begin{matrix} \mu_{k+1} \\ \vdots \\ \mu_{2k+1} \end{matrix} \\ \hline \mu_{k+1}\ \cdots\ \mu_{2k+1} & \mu_{2k+2} \end{array}\right].$$

Multiplying by blocks, we get

$$\mathtt{D}_{k+1}^\top G_{k+1} = \left[\begin{array}{c|c} \mathtt{D}_k^\top G_k & \bar{x}_k \\ \hline \bar{y}_k^\top & \bar{z}_k \end{array}\right],$$

where

$$\bar{x}_k = \mathtt{D}_k^\top \begin{bmatrix} \mu_{k+1} \\ \vdots \\ \mu_{2k+1} \end{bmatrix},$$

$$\bar{y}_k^\top = [(k+1)\,k\,c\ (k+1)(k\,b+e)(k+1)\,(k\,a+d)] \left[\begin{array}{ccc} \mu_{k-1} & \cdots & \mu_{2k-1} \\ \mu_k & \cdots & \mu_{2k} \\ \hline \mu_{k+1} & \cdots & \mu_{2k+1} \end{array}\right],$$

and

$$\bar{z}_k = (k+1)\,k\,c\,\mu_{2k} + (k+1)\,(k\,b+e)\,\mu_{2k+1} + (k+1)\,(k\,a+d)\,\mu_{2k+2} = \bar{z}_k^\top.$$

Since by our induction hypothesis we have

$$\mathtt{D}_{k+1}^\top G_{k+1} = \left[\begin{array}{c|c} G_k\,\mathtt{D}_k & \bar{x}_k \\ \hline \bar{y}_k^\top & \bar{z}_k^\top \end{array}\right],$$

it is clear that our main efforts should focus on showing that $\bar{x}_k = \bar{y}_k$. Observe that

$$\bar{x}_k = \left[\begin{array}{c|c} \mathtt{D}_{k-1}^{\top} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 \; k\,(k-1)\,c \; k\,((k-1)\,b+e) & k\,((k-1)\,a+d) \end{array}\right] \begin{bmatrix} \mu_{k+1} \\ \vdots \\ \mu_{2k+1} \end{bmatrix}.$$

Then, the kth entry of $\bar{x}_k$ is

$$k\,(k-1)\,c\,\mu_{2k-1} + k\,((k-1)\,b+e)\,\mu_{2k} + k\,((k-1)\,a+d)\,\mu_{2k+1},$$

while the kth entry of $\bar{y}_k$ is

$$(k+1)\,k\,c\,\mu_{2k-1} + (k+1)\,(k\,b+e)\,\mu_{2k} + (k+1)\,(k\,a+d)\,\mu_{2k+1},$$

which means that the kth entry of $\bar{y}_k - \bar{x}_k$ is

$$(2k\,a+d)\mu_{2k+1} + (2k\,b+e)\,\mu_{2k} + 2k\,c\,\mu_{2k-1} = 0,$$

where we have used condition (3.1).

Now, noticing that the last row of $\mathtt{D}_{k-1}^{\top}$ is

$$\big[0 \cdots 0 \; (k-1)\,(k-2)\,c \; (k-1)\,((k-2)\,b+2) \; (k-1)\,((k-2)\,a+d)\big],$$

we have that the $(k-1)$th entry of $\bar{x}_k$ is

$$(k-1)\,(k-2)\,c\,\mu_{2k-2} + (k-1)\,((k-2)\,b+2)\,\mu_{2k-1} + (k-1)\,((k-2)\,a+d)\,\mu_{2k},$$

while the $(k-1)$th entry of $\bar{y}_k$ is

$$(k+1)k\,c\,\mu_{2k-2} + (k+1)(k\,b+e)\mu_{2k-1} + (k+1)(k\,a+d)\mu_{2k}.$$

Then the $(k-1)$th entry of $\bar{y}_k - \bar{x}_k$ is

$$2[((2k-1)a+d)\mu_{2k} + ((2k-1)b+e)\mu_{2k-1} + (2k-1)c\,\mu_{2k-2}] = 0,$$

where, again, we have used (3.1) with $n = k-1$.

If we continue in this way, then for $i = 0, 1, \ldots, k$, the $(i+1)$th entry of $\bar{y}_k - \bar{x}_k$ is

$$(k+1-i)[((k+i)a+d)\mu_{k+1+i} + ((k+i)b+e)\mu_{k+i} + (k+i)c\,\mu_{k-1+i}] = 0.$$

This means that $\bar{x}_k = \bar{y}_k$ and, consequently,

$$\mathtt{D}_{k+1}^{\top}\,G_{k+1} = \left[\begin{array}{c|c} G_k\,\mathtt{D}_k & \bar{y}_k \\ \hline \bar{x}_k^{\top} & \bar{z}_k^{\top} \end{array}\right] = G_{k+1}\,\mathtt{D}_{k+1},$$

which proves that $\mathtt{D}_n^{\top}\,G_n = G_n\,\mathtt{D}_n$ holds for $n \geq 0$. □

Let us now consider the eigenvectors of $\mathtt{D}_n$. Suppose that $\bar{u}, \bar{v} \in \mathbb{R}^{n+1}$ are eigenvectors corresponding to distinct eigenvalues λ and $\overline{\lambda}$, respectively. Then,

$$\lambda\,\mathfrak{B}_n(\bar{u},\bar{v}) = \mathfrak{B}_n(\lambda\,\bar{u},\bar{v}) = \mathfrak{B}_n(\mathtt{D}_n\bar{u},\bar{v}) = \mathfrak{B}_n(\bar{u},\mathtt{D}_n\bar{v}) = \mathfrak{B}_n(\bar{u},\overline{\lambda}\,\bar{v}) = \overline{\lambda}\,\mathfrak{B}_n(\bar{u},\bar{v}),$$

which implies

$$(\lambda - \overline{\lambda})\, \mathfrak{B}_n(\bar{u}, \bar{v}) = 0.$$

Since $\lambda \neq \overline{\lambda}$, we must have that $\mathfrak{B}_n(\bar{u}, \bar{v}) = 0$ and, therefore, $\bar{u}$ and $\bar{v}$ are orthogonal with respect to $\mathfrak{B}_n$. We have already encountered the eigenvectors of $\mathtt{D}_n$, as the following theorem shows. This theorem is, in fact, a characterization of classical sequences in terms of $\mathtt{D}_n$.

Theorem 5.3. *For $n \geq 0$, let $S_n^{-1} H_n S_n^{-\top}$ be the Cholesky factorization of G_n and let $\bar{s}_{n,0}, \bar{s}_{n,1}, \ldots, \bar{s}_{n,n}$ denote the columns of $S_n^{\top}$. Then $\{\mu_n\}_{n\geq 0}$ is a classical sequence if and only if*

$$\mathtt{D}_n\, \bar{s}_{n,j} = \lambda_j\, \bar{s}_{n,j}, \quad n \geq 0, \quad 0 \leq j \leq n,$$

where $\lambda_j = j\,[(j-1)\,a + d]$.

Proof. Suppose that $\{\mu_n\}_{n\geq 0}$ is a classical sequence. For $n = 0$, then $\mathtt{D}_0 = 0$ and $\bar{s}_{0,0} = 1$. Thus it is obvious that $\mathtt{D}_0\, \bar{s}_{0,0} = \lambda_0\, \bar{s}_{0,0}$.

For $n = 1$,

$$\mathtt{D}_1 = \left[\begin{array}{c|c} \mathtt{D}_0 & e \\ \hline 0 & d \end{array}\right], \quad \bar{s}_{1,0} = \left[\begin{array}{c} \bar{s}_{0,0} \\ \hline 0 \end{array}\right].$$

Then, multiplying by blocks, we have

$$\mathtt{D}_1\, \bar{s}_{1,0} = \left[\begin{array}{c} \mathtt{D}_0\, \bar{s}_{0,0} \\ \hline 0 \end{array}\right] = \lambda_0 \left[\begin{array}{c} \bar{s}_{0,0} \\ \hline 0 \end{array}\right] = \lambda_0\, \bar{s}_{1,0}.$$

Since $S_1^{\top}$ is upper triangular with 1s on its diagonal, then $\{\bar{s}_{1,0}, \bar{s}_{1,1}\}$ constitutes a basis for $\mathbb{R}^2$. Then we can write

$$\mathtt{D}_1\, \bar{s}_{1,1} = a_{1,1}\, \bar{s}_{1,1} + a_{1,0}\, \bar{s}_{1,0},$$

for some constants $a_{1,1}$ and $a_{1,0}$. Using the orthogonality of the columns of $S_1^{\top}$ with respect to $\mathfrak{B}_1$ and Theorem 5.2, we obtain

$$a_{1,j} = \frac{\mathfrak{B}_1(\mathtt{D}_1\, \bar{s}_{1,1}, \bar{s}_{1,j})}{h_j} = \frac{\mathfrak{B}_1(\bar{s}_{1,1}, \mathtt{D}_1\, \bar{s}_{1,j})}{h_j}, \quad j = 0, 1.$$

It follows that $a_{1,0} = 0$ and, thus,

$$\mathtt{D}_1\, \bar{s}_{1,1} = a_{1,1}\, \bar{s}_{1,1} = \begin{bmatrix} * \\ a_{1,1} \end{bmatrix},$$

where we have taken into account that $\bar{s}_{1,1} = [*\ 1]^{\top}$ (the value of the entry denoted by $*$ has no relevance). If we multiply by blocks, we get

$$\mathtt{D}_1\, \bar{s}_{1,1} = \begin{bmatrix} * \\ d \end{bmatrix},$$

which implies that $a_{1,1} = d = \lambda_1$.

Now, suppose that

$$\mathtt{D}_k\, \bar{s}_{k,j} = \lambda_j\, \bar{s}_{k,j}, \quad 0 \le j \le k,$$

holds for some $k \ge 0$. Recall that

$$\mathtt{D}_{k+1} = \left[\begin{array}{c|c} & 0 \\ & \vdots \\ \mathtt{D}_k & 0 \\ & (k+1)\,k\,c \\ & (k+1)\,(k\,b+e) \\ \hline 0\cdots 0 & (k+1)\,(k\,a+d) \end{array}\right],$$

and that, by Theorem (4.3),

$$\bar{s}_{k+1,j} = \left[\frac{\bar{s}_{k,j}}{0}\right], \quad 0 \le j \le k, \quad \text{and} \quad \bar{s}_{k+1,k+1} = \left[\frac{*}{1}\right],$$

where the values of the entries denoted by $*$ are not relevant here. Multiplying by blocks and using the induction hypothesis, we get

$$\mathtt{D}_{k+1}\, \bar{s}_{k+1,j} = \left[\frac{\mathtt{D}_k\, \bar{s}_{k,j}}{0}\right] = \lambda_j\, \bar{s}_{k+1,j}, \quad 0 \le j \le k.$$

Since $S_{k+1}^{\top}$ is an upper triangular matrix with 1s on its diagonal, its columns constitute a basis for $\mathbb{R}^{k+1}$. Then, we can write

$$\mathtt{D}_{k+1}\, \bar{s}_{k+1,k+1} = \sum_{j=0}^{k+1} a_{k+1,j}\, \bar{s}_{k+1,j},$$

where by the orthogonality of the columns of $S_{k+1}^{\top}$ with respect to $\mathfrak{B}_{k+1}$ and Theorem 5.2, we have

$$\begin{aligned} a_{k+1,j} &= \frac{\mathfrak{B}_{k+1}(\mathtt{D}_{k+1}\, \bar{s}_{k+1,k+1}, \bar{s}_{k+1,j})}{h_j} \\ &= \frac{\mathfrak{B}_{k+1}(\bar{s}_{k+1,k+1}, \mathtt{D}_{k+1}\, \bar{s}_{k+1,j})}{h_j}, \quad 0 \le j \le k+1. \end{aligned}$$

It follows that

$$a_{k+1,j} = \lambda_j\, \frac{\mathfrak{B}_{k+1}(\bar{s}_{k+1,k+1}, \bar{s}_{k+1,j})}{h_j} = 0, \quad 0 \le j \le k,$$

and, consequently,

$$\mathtt{D}_{k+1}\, \bar{s}_{k+1,k+1} = a_{k+1,k+1}\, \bar{s}_{k+1,k+1} = \left[\frac{*}{a_{k+1,k+1}}\right].$$

Moreover, if we multiply by blocks, we get

$$\mathtt{D}_{k+1}\, \bar{s}_{k+1,k+1} = \left[\frac{*}{\lambda_{k+1}}\right],$$

which implies that $a_{k+1,k+1} = \lambda_{k+1}$. Then the sufficient condition is proved by the principle of induction.

Conversely, if $\mathfrak{B}_n$ is the bilinear form defined in Definition 4.1, then

$$\mathfrak{B}_n(\mathtt{D}_n\, \bar{s}_{n,j}, \bar{s}_{n,k}) = \mathfrak{B}_n(\bar{s}_{n,j}, \mathtt{D}_n\, \bar{s}_{n,k}), \quad n \geq 0, \;\; 0 \leq j, \;\; k \leq n.$$

Indeed,

$$\mathfrak{B}_n(\mathtt{D}_n\, \bar{s}_{n,j}, \bar{s}_{n,k}) - \mathfrak{B}_n(\bar{s}_{n,j}, \mathtt{D}_n\, \bar{s}_{n,k}) = (\lambda_j - \lambda_k)\, \mathfrak{B}_n(\bar{s}_{n,j}, \bar{s}_{n,k}) = (\lambda_j - \lambda_k)\, h_j\, \delta_{j,k},$$

which vanishes for all $0 \leq j, k \leq n$. This implies that

$$S_n(\mathtt{D}_n^\top\, G_n - G_n\, \mathtt{D}_n)\, S_n^\top = \mathtt{0}, \quad n \geq 0,$$

or, equivalently,

$$\mathtt{D}_n^\top\, G_n - G_n\, \mathtt{D}_n = \mathtt{0}.$$

Since $\mathtt{D}_n = R_n^\top N_n$, the first column of the above matrix identity reads $N_n^\top\, R_n\, \mathtt{M}_n = \mathtt{0}$, or, equivalently,

$$k[((k-1)\,a + d)\mu_k + ((k-1)\,b + e)\mu_{k-1} + (k-1)\,\mu_{k-2}] = 0,$$

for $n \geq 0$ and $0 \leq k \leq n$. It follows that $\{\mu_n\}_{n\geq 0}$ satisfies (3.1); hence, it is pre-classical. Furthermore, $\{\mu_n\}_{n\geq 0}$ is classical since, otherwise, G_n would not have a Cholesky factorization for some $n \geq 0$. □

The above results can be passed down to the operator $\mathcal{D}$ defined in (5.1). Let $\{\mu_n\}_{n\geq 0}$ be a classical sequence of real numbers, and let $\mathbf{u}$ be the moment functional defined as $\mu_n = \langle \mathbf{u}, x^n \rangle$, $n \geq 0$. Then (5.2) implies that

$$\langle \mathbf{u}, \mathcal{D}[p]q \rangle = \langle \mathbf{u}, p\, \mathcal{D}[q] \rangle, \quad \forall\, p, q \in \Pi.$$

That is, $\mathcal{D}$ is a self-adjoint operator on polynomials. Moreover, for $n \geq 0$, let $S_n^{-1}\, H_n\, S_n^{-\top}$ be the Cholesky factorization of G_n and let $\bar{s}_{n,0}, \bar{s}_{n,1}, \ldots, \bar{s}_{n,n}$ denote the columns of $S_n^\top$. From Theorem 5.3 we deduce that the sequence of polynomials $\{P_n\}_{n\geq 0}$ with

$$P_n(x) = \bar{s}_{n,n}^\top\, \mathtt{X}_n, \quad n \geq 0,$$

are eigenfunctions of the operator $\mathcal{D}$. That is,

$$\mathcal{D}[P_n] = \lambda_n\, P_n, \quad n \geq 0,$$

with $\lambda_n = n\,[(n-1)\,a + d]$. Note that $\{P_n\}_{n\geq 0}$ is a sequence of polynomials orthogonal with respect to $\mathbf{u}$.

5.2. *Hahn-type characterization*

Let $\{\mu_n\}_{n\geq 0}$ be a sequence of real numbers and let $a, b, c \in \mathbb{R}$ such that $|a|+|b|+|c| > 0$. We can define a new sequence $\{\sigma_n\}_{n\geq 0}$ as follows:

$$\sigma_n = a\, \mu_{n+2} + b\, \mu_{n+1} + c\, \mu_n, \quad n \geq 0.$$

Notice that if $\mathbf{u}$ is the moment functional defined as $\mu_n = \langle \mathbf{u}, x^n \rangle$, then $\{\sigma_n\}_{n\geq 0}$ is the sequence of moments of the functional given by $\mathbf{v} = \phi(x)\,\mathbf{u}$ where $\phi(x) = a\,x^2 + b\,x + c$. Indeed, for $n \geq 0$,

$$\langle \mathbf{v}, x^n \rangle = \langle \mathbf{u}, \phi\, x^n \rangle = \langle \mathbf{u}, a\,x^{n+2} + b\,x^{n+1} + c\,x^n \rangle = a\mu_{n+2} + b\mu_{n+1} + c\mu_n = \sigma_n.$$

We denote by $\{G_n^{(1)}\}_{n\geq 0}$ the sequence of $(n+1)\times(n+1)$ matrices with

$$G_0^{(1)} = \sigma_0, \quad \text{and} \quad G_n^{(1)} = \left[\begin{array}{ccc|c} & & & \sigma_n \\ & G_{n-1}^{(1)} & & \vdots \\ & & & \sigma_{2n-1} \\ \hline \sigma_n & \cdots & \sigma_{2n-1} & \sigma_{2n} \end{array}\right], \quad n \geq 1. \tag{5.3}$$

The following theorem shows that the pre-classical character is inherited by $\{\sigma_n\}_{n\geq 0}$.

Theorem 5.4. *If* $\{\mu_n\}_{n\geq 0}$ *is pre-classical satisfying* (3.1), *then* $\{\sigma_n\}_{n\geq 0}$ *is pre-classical satisfying*

$$(n\,a + d_1)\sigma_{n+1} + (n\,b + e_1)\sigma_n + n\,c\,\sigma_{n-1} = 0, \quad n \geq 0,$$

where $d_1 = d + 2\,a$ *and* $e_1 = e + b$. *Moreover,*

$$\sigma_n = -(n\,a + d)\,\mu_{n+2} - (n\,b + e)\,\mu_{n+1} - n\,c\,\mu_n, \quad n \geq 0. \tag{5.4}$$

Proof. For $n \geq 0$, we compute

$$\begin{aligned}
&(n\,a + d_1)\,\sigma_{n+1} + (n\,b + e_1)\,\sigma_n + n\,c\,\sigma_{n-1} \\
&\quad = a\,\big([(n+2)a + d]\mu_{n+3} + [(n+2)b + e]\mu_{n+2} + (n+2)c\mu_{n+1} \\
&\qquad - b\,\mu_{n+2} - 2\,c\,\mu_{n+1}\big) + b\,\big([(n+1)a + d]\mu_{n+2} + [(n+1)b + e]\mu_{n+1} \\
&\qquad + (n+1)c\,\mu_n + a\,\mu_{n+2} - c\,\mu_n\big) + c\,\big((n\,a + d)\,\mu_{n+1} + (n\,b + e)\mu_n \\
&\qquad + n\,c\,\mu_{n-1} + 2\,a\,\mu_{n+1} + b\,\mu_n\big),
\end{aligned}$$

where we have used $\sigma_n = a\,\mu_{n+2} + b\,\mu_{n+1} + c\,\mu_n$. By (3.1), we have

$$\begin{aligned}
&(n\,a + d_1)\sigma_{n+1} + (n\,b + e_1)\,\sigma_n + n\,c\,\sigma_{n-1} \\
&\quad = -a\,b\,\mu_{n+2} - 2\,a\,c\,\mu_{n+1} + a\,b\,\mu_{n+2} - b\,c\,\mu_n + 2\,a\,c\,\mu_{n+1} + b\,c\,\mu_n = 0.
\end{aligned}$$

Finally, (5.4) follows from the fact that (3.1) can be written as follows:

$$a\,\mu_{n+2} + b\,\mu_{n+1} + c\,\mu_n = -(n\,a + d)\,\mu_{n+2} - (n\,b + e)\,\mu_{n+1} - n\,c\,\mu_n. \qquad \square$$

When $\{\mu_n\}_{n\geq 0}$ is a classical sequence of real numbers, the matrix $G_n^{(1)}$ satisfies an interesting and useful relation involving the matrices G_n and $\mathtt{D}_n$.

Proposition 5.5. *Let* $\{\mu_n\}_{n\geq 0}$ *be a classical sequence of real numbers satisfying* (3.1). *Then, for* $n \geq 0$,

$$N_{n+1}^{\top}\,G_n^{(1)}\,N_{n+1} = -\mathtt{D}_{n+1}^{\top}\,G_{n+1}.$$

Proof. We prove this theorem by induction. For $n = 0$, on one hand we have

$$N_1^\top G_0^{(1)} N_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma_0 \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_0 \end{bmatrix}.$$

On the other hand, we have

$$-\mathtt{D}_1^\top G_1 = \begin{bmatrix} 0 & 0 \\ -e & -d \end{bmatrix} \begin{bmatrix} \mu_0 & \mu_1 \\ \mu_1 & \mu_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -e\,\mu_0 - d\,\mu_1 & -e\,\mu_1 - d\,\mu_2 \end{bmatrix}.$$

Since $\{\mu_n\}_{n\geq 0}$ satisfies (3.1), we have that $-e\,\mu_0 - d\,\mu_1 = 0$ and, by (5.4), $-e\,\mu_1 - d\,\mu_2 = \sigma_0$. Therefore,

$$-\mathtt{D}_1^\top G_1 = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_0 \end{bmatrix},$$

which proves that $N_1^\top G_0^{(1)} N_1 = -\mathtt{D}_1^\top G_1$.

Now, suppose that $N_{k+1}^\top G_k^{(1)} N_{k+1} = -\mathtt{D}_{k+1}^\top G_{k+1}$ holds for $k \geq 0$. On one hand, we compute

$$N_{k+2}^\top\ G_{k+1}^{(1)} N_{k+2} = \left[\begin{array}{c|c} N_{k+1}^\top & \mathtt{0} \\ \hline \mathtt{0} & k+2 \end{array}\right] \left[\begin{array}{c|c} G_k^{(1)} & \begin{matrix} \sigma_{k+1} \\ \vdots \\ \sigma_{2k+1} \end{matrix} \\ \hline \sigma_{k+1} \ \cdots \ \sigma_{2k+1} & \sigma_{2k+2} \end{array}\right] \left[\begin{array}{c|c} N_{k+1} & \mathtt{0} \\ \hline \mathtt{0} & k+2 \end{array}\right].$$

Multiplying by blocks and using the induction hypothesis, we get

$$N_{k+2}^\top G_{k+1}^{(1)} N_{k+2} = \left[\begin{array}{c|c} -\mathtt{D}_{k+1}^\top G_{k+1} & \bar{x}_k \\ \hline \bar{x}_k^\top & (k+2)^2\,\sigma_{2k+2} \end{array}\right],$$

where

$$\bar{x}_k = (k+2)\, N_{k+1}^\top \begin{bmatrix} \sigma_{k+1} \\ \vdots \\ \sigma_{2k+1} \end{bmatrix}.$$

On the other hand,

$$-\mathtt{D}_{k+2}^\top G_{k+2}$$

$$= \left[\begin{array}{c|c} -\mathtt{D}_{k+1}^\top & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 \ -(k+2)\,(k+1)\,c \ -(k+2)\,[(k+1)\,b + e] & -(k+2)\,[(k+1)\,a + d] \end{array}\right]$$

$$\times \left[\begin{array}{c|c} G_{k+1} & \begin{matrix} \mu_{k+2} \\ \vdots \\ \mu_{2k+3} \end{matrix} \\ \hline \mu_{k+2} \ \cdots \ \mu_{2k+3} & \mu_{2k+4} \end{array}\right].$$

After multiplying by blocks, we have

$$-\mathtt{D}_{k+2}^{\top} G_{k+2} = \left[\begin{array}{c|c} -\mathtt{D}_{k+1}^{\top} G_{k+1} & \bar{y}_k \\ \hline \bar{z}_k & w_k \end{array}\right],$$

where

$$\begin{aligned}
\bar{y}_k &= -\mathtt{D}_{k+1}^{\top} \begin{bmatrix} \mu_{k+2} \\ \vdots \\ \mu_{2k+3} \end{bmatrix}, \\
\bar{z}_k &= \begin{bmatrix} 0 \cdots 0 & -(k+2)(k+1)c & -(k+2)[(k+1)b+e] \end{bmatrix} G_{k+1} \\
&\quad - (k+2)[(k+1)a+d] \begin{bmatrix} \mu_{k+2} \cdots \mu_{2k+3} \end{bmatrix}, \\
w_k &= -(k+2)[(k+1)\, c\, \mu_{2k+2} + [(k+1)\, b + e]\, \mu_{2k+3} + [(k+1)\, a + d]\mu_{2k+4}].
\end{aligned}$$

By Theorem 5.2, we have that $\bar{z}_k = \bar{y}_k^{\top}$. Hence, our efforts should focus on showing that $\bar{x}_k = \bar{y}_k$ and $w_k = (k+2)^2 \sigma_{2k+2}$. Let us start by proving the second identity.

Observe that

$$\begin{aligned}
w_k &= -(k+2)[[(2k+2)a+d]\, \mu_{2k+4} + [(2k+2)\, b + e]\mu_{2k+3} + (2k+2)c\, \mu_{2k+2}] \\
&\quad + (k+2)(k+1)(a\, \mu_{2k+4} + b\, \mu_{2k+3} + c\, \mu_{2k+2}).
\end{aligned}$$

From (5.4) and the fact that $\sigma_{2k+2} = a\, \mu_{2k+4} + b\, \mu_{2k+3} + c\, \mu_{2k+2}$, we deduce that

$$w_k = (k+2)\sigma_{2k+2} + (k+2)(k+1)\sigma_{2k+2} = (k+2)^2\, \sigma_{2k+2}.$$

In order to prove that $\bar{x}_k = \bar{y}_k$, we note that

$$\bar{x}_k = (k+2) \left[\begin{array}{c|c} N_k^{\top} & \mathtt{0} \\ \hline \mathtt{0} & k+1 \end{array}\right] \begin{bmatrix} \sigma_{k+1} \\ \vdots \\ \sigma_{2k+1} \end{bmatrix}$$

and

$$\bar{y}_k = \left[\begin{array}{c|c} -\mathtt{D}_k^{\top} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 \; -(k+1)\, k\, c \; -(k+1)\, (k\, b + e) & -(k+1)\, (k\, a + d) \end{array}\right] \begin{bmatrix} \mu_{k+2} \\ \vdots \\ \mu_{2k+3} \end{bmatrix}.$$

Then, the last entry of $\bar{x}_k - \bar{y}_k$ is

$$\begin{aligned}
&(k+2)(k+1)\sigma_{2k+1} + (k+1)[k\, c\, \mu_{2k+1} + (k\, b + e)\mu_{2k+2} + (k\, a + d)\mu_{2k+3}] \\
&\quad = (k+2)(k+1)\sigma_{2k+1} - (k+1)\sigma_{2k+1} - (k+1)^2\, \sigma_{2k+1} = 0,
\end{aligned}$$

where we have used (5.4) and the fact that $\sigma_{2k+1} = a\,\mu_{2k+3} + b\,\mu_{2k+2} + c\,\mu_{2k+1}$. In general, the ith entry of $\bar{x}_k - \bar{y}_k$, with $1 \le i \le k+2$, is

$$\begin{aligned}(i-1)[(k+2)\,\sigma_{k+i-1} + [(i-2)\,c\,\mu_{k+i-1} + [(i-2)\,b + e]\\ \times \mu_{k+i} + [(i-2)\,a + d]\,\mu_{k+1+i}]]\\ = (i-1)[(k+2)\,\sigma_{k+i-1} - \sigma_{k+i-1} - (k+1)\,\sigma_{k+i-1}] = 0,\end{aligned}$$

where we have used (5.4) and the fact that $\sigma_{k+i-1} = a\,\mu_{k+1+i} + b\,\mu_{k+i} + c\,\mu_{k+i-1}$. This proves that $\bar{x}_k - \bar{y}_k = \mathtt{0}$ and, in turn, that $N_{k+2}^{\top}\,G_{k+1}^{(1)}\,N_{k+2} = -\mathtt{D}_{k+2}^{\top}\,G_{k+2}$.

It follows from the Principle of Induction that $N_{n+1}^{\top}\,G_n^{(1)}\,N_{n+1} = -\mathtt{D}_{n+1}^{\top}\,G_{n+1}$ holds for $n \ge 0$. □

Now, let $\{\mu_n\}_{n\ge 0}$ be a classical sequence of real numbers. For $n \ge 1$, let $S_n^{-1}\,H_n\,S_n^{-\top}$ be the Cholesky factorization of G_n and let $\bar{s}_{n,0}, \bar{s}_{n,1}, \ldots, \bar{s}_{n,n}$ denote the columns of $S_n^{\top}$. Consider the set $\{\bar{s}_{n,0}^{(1)}, \ldots, \bar{s}_{n,n}^{(1)}\}$ of vectors in $\mathbb{R}^{n+1}$ with

$$\bar{s}_{n,j}^{(1)} = \frac{1}{j+1}\,N_{n+1}\,\bar{s}_{n+1,j+1}, \quad 0 \le j \le n.$$

Observe that $\{\bar{s}_{n,0}^{(1)}, \ldots, \bar{s}_{n,n}^{(1)}\}$ constitutes a basis for $\mathbb{R}^{n+1}$. Furthermore, since $S_n^{\top}$ is a unit upper triangular matrix, the $(n+1)\times(n+1)$ matrix $S_{n,1}^{\top}$ defined as follows:

$$S_{n,1}^{\top} = [\bar{s}_{n,0}^{(1)}\ \bar{s}_{n,1}^{(1)}\ \cdots\ \bar{s}_{n,n}^{(1)}], \tag{5.5}$$

which is also a unit upper triangular matrix. We show that $G_n^{(1)}$ defined in (5.3) admits a Cholesky factorization with $S_{n,1}^{-1}$ its triangular matrix factor.

Theorem 5.6. *Let $\{\mu_n\}_{n\ge 0}$ be a classical sequence of real numbers satisfying* (3.1)*. For $n \ge 0$, let $S_n^{-1}\,H_n\,S_n^{-\top}$ be the Cholesky factorization of G_n. Then $G_n^{(1)}$ admits the Cholesky factorization given by*

$$G_n^{(1)} = S_{n,1}^{-1}\,H_{n,1}\,S_{n,1}^{-\top},$$

where $S_{n,1}^{\top}$ is the matrix defined in (5.5) *and $H_{n,1} = \mathrm{diag}[h_0^{(1)}, \ldots, h_n^{(1)}]$ with*

$$h_j^{(1)} = -\frac{\lambda_{j+1}}{(j+1)^2}\,h_{j+1}, \quad j \ge 0,$$

and $\lambda_j = j\,[(j-1)\,a + d]$.

Proof. For $0 \le j, k \le n$, we compute

$$(\bar{s}_{n,j}^{(1)})^{\top}\,G_n^{(1)}\,\bar{s}_{n,k}^{(1)} = \frac{1}{(j+1)\,(k+1)}\,\bar{s}_{n+1,j+1}^{\top}\,N_{n+1}^{\top}\,G_n^{(1)}\,N_{n+1}\,\bar{s}_{n+1,k+1}.$$

Using Theorem 5.3 and Proposition 5.5, we obtain

$$\begin{aligned}(\bar{s}_{n,j}^{(1)})^{\top}\,G_n^{(1)}\,\bar{s}_{n,k}^{(1)} &= -\frac{1}{(j+1)\,(k+1)}\,\bar{s}_{n+1,j+1}^{\top}\,\mathtt{D}_{n+1}^{\top}\,G_{n+1}\,\bar{s}_{n+1,k+1}\\ &= -\frac{\lambda_{j+1}}{(j+1)\,(k+1)}\,\bar{s}_{n+1,j+1}^{\top}\,G_{n+1}\,\bar{s}_{n+1,k+1}\end{aligned}$$

$$= -\frac{\lambda_{j+1}}{(j+1)\,(k+1)}\,\mathcal{B}_{n+1}(\bar{s}_{n+1,j+1},\,\bar{s}_{n+1,k+1})$$

$$= -\frac{\lambda_{j+1}}{(j+1)^2}\,h_{j+1}\,\delta_{j,k}.$$

This implies that $S_{n,1}\,G_n^{(1)}\,S_{n,1}^{\top} = H_{n,1}$ and, hence, $G_n^{(1)} = S_{n,1}^{-1}\,H_{n,1}\,S_{n,1}^{-\top}$. □

Corollary 5.7. *If $\{\mu_n\}_{n\geq 0}$ is a classical sequence, then so is $\{\sigma_n\}_{n\geq 0}$.*

Proof. By Theorem (5.4), $\{\sigma_n\}_{n\geq 0}$ is a pre-classical sequence.

Now, we must show that $\det G_n^{(1)} \neq 0$ for $n \geq 0$ (Definition 3.3). From Theorem 5.6 we deduce that

$$\det G_n^{(1)} = \det H_{n,1} = h_0^{(1)} \cdots h_n^{(1)}, \quad n \geq 0,$$

with

$$h_j^{(1)} = -\frac{\lambda_{j+1}}{(j+1)^2}\,h_{j+1}, \quad j \geq 0,$$

and $\lambda_j = j\,[(j-1)\,a + d]$. Since $\{\mu_n\}_{n\geq 0}$ is classical, then we have

$$h_n \neq 0 \quad \text{and} \quad n\,a + d \neq 0, \quad n \geq 0,$$

(see equality (4.2) and Definition 3.1). This implies that $h_j^{(1)} \neq 0$ for $j \geq 0$. Therefore, $\det G_n^{(1)} \neq 0$ for $n \geq 0$ and, thus, $\{\sigma_n\}_{n\geq 0}$ is classical. □

Theorem 5.6 implies that for $n \geq 0$, the columns of $S_{n,1}^{\top}$ constitute an orthogonal basis for $\mathbb{R}^{n+1}$ with respect to the bilinear form associated with $\{\sigma_n\}_{n\geq 0}$ (see Definition 4.1), which we denote by

$$\mathfrak{B}_n^{(1)}(\bar{u},\bar{v}) = \bar{u}^{\top}\,G_n^{(1)}\,\bar{v}, \quad \forall\,\bar{u},\bar{v} \in \mathbb{R}^{n+1}.$$

We are ready for the following characterizations of classical sequences.

Theorem 5.8. *Let $\{\mu_n\}_{n\geq 0}$ be a sequence of real numbers such that $\det G_n \neq 0$ for $n \geq 0$. Let $S_n^{-1}\,H_n\,S_n^{-\top}$ be the Cholesky factorization of G_n and let $\bar{s}_{n,0}, \bar{s}_{n,1}, \ldots, \bar{s}_{n,n}$ denote the columns of $S_n^{\top}$. Then $\{\mu_n\}_{n\geq 0}$ is classical if and only if there are real numbers a, b, c satisfying*

$$|a| + |b| + |c| > 0,$$

such that the set $\{\bar{s}_{n,0}^{(1)}, \ldots, \bar{s}_{n,n}^{(1)}\}$ of vectors in $\mathbb{R}^{n+1}$ with

$$\bar{s}_{n,j}^{(1)} = \frac{1}{j+1}\,N_{n+1}\,\bar{s}_{n+1,j+1}, \quad 0 \leq j \leq n,$$

constitutes an orthogonal basis for $\mathbb{R}^{n+1}$ with respect to the bilinear form associated with $\{\sigma_n\}_{n\geq 0}$, where

$$\sigma_n = a\,\mu_{n+2} + b\,\mu_{n+1} + c\,\mu_n, \quad n \geq 0.$$

Proof. If $\{\mu_n\}_{n\geq 0}$ is classical, then it follows from Theorem 5.6 that

$$S_{n,1}\, G_n^{(1)}\, S_{n,1}^\top = H_{n,1}, \quad n \geq 0,$$

where

$$S_{n,1}^\top = [\bar{s}_{n,0}^{(1)}\ \bar{s}_{n,1}^{(1)}\ \ldots\ \bar{s}_{n,n}^{(1)}],$$

and $H_{n,1} = \mathrm{diag}[h_0^{(1)}, \ldots, h_n^{(1)}]$ with

$$h_j^{(1)} = -\frac{\lambda_{j+1}}{(j+1)^2}\, h_{j+1}, \quad j \geq 0,$$

and $\lambda_j = j\,[(j-1)\,a + d]$. This implies that

$$\mathfrak{B}_n^{(1)}(\bar{s}_{n,i}^{(1)}, \bar{s}_{n,j}^{(1)}) = h_j^{(1)}\, \delta_{i,j}, \quad 0 \leq i, \;\; j \leq n.$$

This proves the necessary condition.

Conversely, for $n \geq 0$, on one hand we have

$$\mathfrak{B}_n^{(1)}(\bar{s}_{n,k}^{(1)}, \bar{s}_{n,0}^{(1)}) = h_0^{(1)}\, \delta_{k,0}$$

or, equivalently,

$$(\bar{s}_{n,k}^{(1)})^\top\, G_n^{(1)}\, \bar{s}_{n,0}^{(1)} = \frac{1}{k+1}\, \bar{s}_{n+1,k+1}^\top\, N_{n+1}^\top\, G_n^{(1)}\, \bar{s}_{n,0}^{(1)} = -\lambda_1\, h_1\, \delta_{k,0}.$$

Using the fact that $s_{n,0}^{(1)} = \bar{e}_0$ where, recall, $\bar{e}_0$ is the first column of the identity matrix of order $n+1$, we write

$$\bar{s}_{n+1,k+1}^\top\, N_{n+1}^\top \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_n \end{bmatrix} = -(k+1)\,\lambda_1\, h_1\, \delta_{k,0}.$$

On the other hand

$$\mathfrak{B}_{n+1}(\bar{s}_{n+1,k+1}, \bar{s}_{n+1,1}) = \bar{s}_{n+1,k+1}^\top\, G_{n+1}\, \bar{s}_{n+1,1} = h_1\, \delta_{k,0} = h_1\,(k+1)\, \delta_{k,0}.$$

Therefore,

$$\bar{s}_{n+1,k+1}^\top\, N_{n+1}^\top \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_n \end{bmatrix} = -\lambda_1\, \bar{s}_{n+1,k+1}^\top\, G_{n+1}\, \bar{s}_{n+1,1},$$

and, thus,

$$\bar{s}_{n+1,k+1}^\top \left(N_{n+1}^\top \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_n \end{bmatrix} + \lambda_1\, G_{n+1}\, \bar{s}_{n+1,1} \right) = 0, \quad 0 \leq k \leq n.$$

Since the first row of $N_{n+1}^{\top}$ is $[0\ 0\cdots 0]$ and $\bar{s}_{n+1,0}^{\top}\, G_{n+1}\, \bar{s}_{n+1,1} = 0$, we have

$$S_{n+1}^{\top}\left(N_{n+1}^{\top} \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_n \end{bmatrix} + \lambda_1\, G_{n+1}\, \bar{s}_{n+1,1} \right) = \mathtt{0},$$

and, consequently,

$$N_{n+1}^{\top} \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_n \end{bmatrix} + \lambda_1\, G_{n+1}\, \bar{s}_{n+1,1} = \mathtt{0}, \tag{5.6}$$

where we have used the fact that $S_n^{\top}$ is an invertible matrix. If

$$\bar{s}_{n+1,1} = [s_{1,0}\ 1\ 0\cdots 0]^{\top},$$

then let $d = \lambda_1$ and $e = \lambda_1\, s_{1,0}$. Observe that by Theorem 4.3, $s_{1,0}$ is independent of n. In this way, the entries of (5.6) read

$$d\,\mu_1 + e\,\mu_0 = 0,$$
$$k\,\sigma_{k-1} + d\,\mu_{k+1} + e\,\mu_k = (k\,a + d)\,\mu_{k+1} + (k\,b + e)\,\mu_k + k\,c\,\mu_{k-1} = 0, \quad 1 \le k \le n,$$

which proves that $\{\mu_n\}_{n\ge 0}$ satisfies (3.1) and, thus, $\{\mu_n\}_{n\ge 0}$ is classical. □

Corollary 5.7 shows that the classical character of $\{\mu_n\}_{n\ge 0}$ is inherited by $\{\sigma_n\}_{n\ge 0}$ allowing us to apply all of our previous results about classical sequences to $\{\sigma_n\}_{n\ge 0}$ and $\{G_n^{(1)}\}_{n\ge 0}$. Therefore, we can define a new sequence $\{\sigma_n^{(2)}\}_{n\ge 0}$ as follows:

$$\sigma_n^{(2)} = a\,\sigma_{n+2} + b\,\sigma_{n+1} + c\,\sigma_n, \quad n \ge 0.$$

We denote by $\{G_n^{(2)}\}_{n\ge 0}$ the sequence of $(n+1)\times(n+1)$ matrices with

$$G_0^{(2)} = \sigma_0^{(2)}, \quad \text{and} \quad G_n^{(2)} = \left[\begin{array}{ccc|c} & & & \sigma_n^{(2)} \\ & G_{n-1}^{(2)} & & \vdots \\ & & & \sigma_{2n-1}^{(2)} \\ \hline \sigma_n^{(2)} & \cdots & \sigma_{2n-1}^{(2)} & \sigma_{2n}^{(2)} \end{array}\right], \quad n \ge 1.$$

By Theorem 5.4, $\{\sigma_n^{(2)}\}_{n\ge 0}$ is pre-classical satisfying

$$(n\,a + d_2)\sigma_{n+1}^{(2)} + (n\,b + e_2)\,\sigma_n^{(2)} + n\,c\,\sigma_{n-1}^{(2)} = 0, \quad n \ge 0,$$

where $d_2 = d + 4\,a$ and $e_2 = e + 2\,b$. Moreover, if $\det G_n^{(2)} \ne 0$ for $n \ge 0$, then $\{\sigma_n^{(2)}\}_{n\ge 0}$ is classical and, by Theorem 5.8, the set $\{\bar{s}_{n,0}^{(2)}, \ldots, \bar{s}_{n,n}^{(2)}\}$ of vectors in

$\mathbb{R}^{n+1}$ with

$$\bar{s}^{(2)}_{n,j} = \frac{1}{j+1}\, N_{n+1}\, \bar{s}^{(1)}_{n+1,j+1}, \quad 0 \le j \le n,$$

constitutes an orthogonal basis for $\mathbb{R}^{n+1}$ with respect to the bilinear form $\mathfrak{B}^{(2)}_n$ associated with $\{\sigma^{(2)}_n\}_{n\ge 0}$, that is,

$$\mathfrak{B}^{(2)}_n(\bar{s}^{(2)}_{n,j}, \bar{s}^{(2)}_{n,k}) = h^{(2)}_j\, \delta_{j,k}, \quad 0 \le j, \quad k \le n,$$

where

$$h^{(2)}_j = -\frac{\lambda^{(1)}_{j+1}}{(j+1)^2}\, h^{(2)}_{j+1}, \quad j \ge 0,$$

and $\lambda^{(1)}_j = j\,[(j-1)\,a + d_1]$.

Iterating this idea, we obtain the following result.

Corollary 5.9. *Let $\{\mu_n\}_{n\ge 0}$ be a classical sequence of real numbers. Let $S^{-1}_n\, H_n\, S^{-\top}_n$ be the Cholesky factorization of G_n and let $\bar{s}_{n,0}, \bar{s}_{n,1}, \ldots, \bar{s}_{n,n}$ denote the columns of $S^{\top}_n$. For each $k \ge 1$, define the sequence of real numbers $\{\sigma^{(k)}_n\}_{n\ge 0}$ by*

$$\sigma^{(k)}_n = a\,\sigma^{(k-1)}_{n+2} + b\,\sigma^{(k-1)}_{n+1} + c\,\sigma^{(k-1)}_n, \quad n \ge 0,$$

where $\sigma^{(0)}_n = \mu_n$ for $n \ge 0$. Then $\{\sigma^{(k)}_n\}_{n\ge 0}$ is classical satisfying

$$(n\,a + d_k)\,\sigma^{(k)}_{n+1} + (n\,b + e_k)\,\sigma^{(k)}_n + n\,c\,\sigma^{(k)}_{n-1} = 0, \quad n \ge 0,$$

where $d_k = d + 2\,k\,a$ and $e_k = e + k\,b$. Moreover, the set $\{\bar{s}^{(k)}_{n,0}, \ldots, \bar{s}^{(k)}_{n,n}\}$ of vectors in $\mathbb{R}^{n+1}$ with

$$\bar{s}^{(k)}_{n,j} = \frac{1}{j+1}\, N_{n+1}\, \bar{s}^{(k-1)}_{n+1,j+1}, \quad 0 \le j \le n,$$

where $s^{(0)}_{n,j} = \bar{s}_{n,j}$ constitutes an orthogonal basis for $\mathbb{R}^{n+1}$ with respect to the bilinear form associated with $\{\sigma^{(k)}_n\}_{n\ge 0}$.

Observe that the vectors $\{\bar{s}^{(k)}_{n,0}, \ldots, \bar{s}^{(k)}_{n,n}\}$ in Corollary 5.9 can be written in terms of the vectors $\bar{s}_{n,0}, \bar{s}_{n,1}, \ldots, \bar{s}_{n,n}$ as follows: for each $k \ge 1$,

$$\bar{s}^{(k)}_{n,j} = \frac{1}{(j+1)_k} N_{n+1}\, N_{n+2} \cdots N_{n+k}\, \bar{s}_{n+k,j+k}, \quad 0 \le j \le n, \tag{5.7}$$

where $(\nu)_k = \nu\,(\nu+1)\cdots(\nu+k-1)$, $(\nu)_0 = 1$, denotes the Pochhammer symbol. If $\{G^{(k)}_n\}_{n\ge 0}$ denotes the sequence of $(n+1)\times(n+1)$ matrices with

$$G^{(k)}_0 = \sigma^{(k)}_0, \quad \text{and} \quad G^{(k)}_n = \left[\begin{array}{ccc|c} & & & \sigma^{(k)}_n \\ & G^{(k)}_{n-1} & & \vdots \\ & & & \sigma^{(k)}_{2n-1} \\ \hline \sigma^{(k)}_n & \cdots & \sigma^{(k)}_{2n-1} & \sigma^{(k)}_{2n} \end{array}\right], \quad n \ge 1, \tag{5.8}$$

then the orthogonality of $\{\bar{s}_{n,0}^{(k)}, \ldots, \bar{s}_{n,n}^{(k)}\}$ with respect to the bilinear form $\mathfrak{B}_n^{(k)}$ associated with $\{\sigma_n^{(k)}\}_{n\geq 0}$ is given by

$$\mathfrak{B}_n^{(k)}(\bar{s}_{n,j}^{(k)}, \bar{s}_{n,i}^{(k)}) = (\bar{s}_{n,j}^{(k)})^\top G_n^{(k)} \, \bar{s}_{n,i}^{(k)} = h_j^{(k)} \, \delta_{j,i}, \quad 0 \leq i, \ j \leq n, \tag{5.9}$$

where

$$h_j^{(k)} = -\frac{\lambda_{j+1}^{(k-1)}}{(j+1)^2} \, h_{j+1}^{(k-1)}, \quad j \geq 0,$$

and $\lambda_j^{(k)} = j\,[(j-1)\,a + d_k]$. Note that we can write

$$h_j^{(k)} = (-1)^k \frac{\prod_{i=0}^{k-1} \lambda_{j+k-i}^{(i)}}{[(j+1)_k]^2} \, h_{j+k}, \quad j \geq 0.$$

Moreover, (5.9) implies the Cholesky factorization of $G_n^{(k)}$:

$$S_{n,k} \, G_n^{(k)} \, S_{n,k}^\top = H_{n,k}, \quad n \geq 0,$$

where

$$S_{n,k}^\top = [\bar{s}_{n,0}^{(k)} \ \bar{s}_{n,1}^{(k)} \ \cdots \ \bar{s}_{n,n}^{(k)}],$$

and $H_{n,k} = \mathrm{diag}[h_0^{(k)}, \ldots, h_n^{(k)}]$.

Let us reformulate the above results in terms of polynomials and moment functionals. Let $\{\mu_n\}_{n\geq 0}$ be a classical sequence of real numbers satisfying (3.1), and let $\mathbf{u}$ be the moment functional defined as $\mu_n = \langle \mathbf{u}, x^n \rangle$, $n \geq 0$. For each $k \geq 1$, the sequence of real numbers $\{\sigma_n^{(k)}\}_{n\geq 0}$ is defined by

$$\sigma_n^{(k)} = a\,\sigma_{n+2}^{(k-1)} + b\,\sigma_{n+1}^{(k-1)} + c\,\sigma_n^{(k-1)}, \quad n \geq 0,$$

where $\sigma_n^{(0)} = \mu_n$ for $n \geq 0$ is the sequence of moments of the functional given by $\mathbf{v}_k = \phi^k \, \mathbf{u}$ where $\phi(x) = a\,x^2 + b\,x + c$. Moreover, for $n \geq 0$, let $S_n^{-1} \, H_n \, S_n^{-\top}$ be the Cholesky factorization of G_n and let $\bar{s}_{n,0}, \bar{s}_{n,1}, \ldots, \bar{s}_{n,n}$ denote the columns of $S_n^\top$. For $n \geq 0$, Corollary 5.9 implies that the polynomials $\{Q_{0,k}(x), Q_{1,k}(x), \ldots, Q_{n,k}(x)\}$ given by

$$Q_{j,k}(x) = (\bar{s}_{n,j}^{(k)})^\top \, \mathtt{X}_n, \quad 0 \leq j \leq n,$$

satisfy $\langle \mathbf{v}_k, Q_{i,k} \, Q_{j,k} \rangle = h_j^{(k)} \, \delta_{i,j}$. Therefore, $\{Q_{0,k}(x), Q_{1,k}(x), \ldots, Q_{n,k}(x)\}$ constitutes an orthogonal basis for Π_n with respect to $\mathbf{v}_k$. Furthermore, Theorem 4.3 allows us to write

$$Q_{n,k}(x) = (\bar{s}_{n,n}^{(k)})^\top \, \mathtt{X}_n, \quad n \geq 0,$$

and, in this way, $\{Q_{n,k}(x)\}_{n\geq 0}$ is an MOPS associated with $\mathbf{v}_k$. We note that if

$$P_n(x) = \bar{s}_{n,n}^\top \, \mathtt{X}_n, \quad n \geq 0,$$

then, by (5.7), we have

$$Q_{n,k}(x) = \frac{P^{(k)}_{n+k}(x)}{(n+1)_k}, \quad n \geq 0,$$

where $P_n^{(k)}(x)$ denotes the kth-order derivative of $P_n(x)$.

5.3. *First structure relation*

For $n \geq 1$ and given real numbers $a, b,$ and c, we define the $(n+3) \times (n+1)$ matrices

$$\Phi_n = \left[\begin{array}{c|c} \Phi_{n-1} & \begin{matrix} 0 \\ c \\ b \end{matrix} \\ \hline 0 & a \end{array}\right], \quad \Phi_0 = \begin{bmatrix} c \\ b \\ a \end{bmatrix}. \tag{5.10}$$

Lemma 5.10. *Let $\{\mu_n\}_{n\geq 0}$ and $\{\sigma_n\}_{n\geq 0}$ be sequences of real numbers satisfying*

$$\sigma_n = a\,\mu_{n+2} + b\,\mu_{n+1} + c\,\mu_n, \quad n \geq 0,$$

where $a, b, c \in \mathbb{R}$. Then, for $n \geq 2$,

$$G_n^{(1)} \begin{bmatrix} \bar{v} \\ 0 \\ 0 \end{bmatrix} = G_n\,\Phi_{n-2}\,\bar{v}, \quad \forall\, \bar{v} \in \mathbb{R}^{n-1}. \tag{5.11}$$

Proof. We use induction to prove this result. For $n = 2$, observe that, for all $v \in \mathbb{R}$,

$$G_2^{(1)} \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \end{bmatrix} v = a\,v \begin{bmatrix} \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} + b\,v \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} + c\,v \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \end{bmatrix} = G_2\,\Phi_0\,v.$$

This proves the base case.

Suppose that (5.11) holds for some $k \geq 2$. Let

$$\bar{v} = \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_k \end{bmatrix} \in \mathbb{R}^k.$$

We compute

$$G_{k+1}^{(1)} \begin{bmatrix} \bar{v} \\ 0 \\ 0 \end{bmatrix} = G_{k+1}^{(1)} \left(\begin{bmatrix} \nu_1 \\ \vdots \\ \nu_{k-1} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \nu_k \\ 0 \\ 0 \end{bmatrix} \right).$$

Multiplying by blocks and using the induction hypothesis, we get

$$G_{k+1}^{(1)} \begin{bmatrix} \bar{v} \\ 0 \\ 0 \end{bmatrix} = \left[\begin{array}{ccc|c} & & & \sigma_{k+1} \\ & G_k^{(1)} & & \vdots \\ & & & \sigma_{2k+1} \\ \hline \sigma_{k+1} & \cdots & \sigma_{2k+1} & \sigma_{2k+2} \end{array}\right] \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_{k-1} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \nu_k \begin{bmatrix} \sigma_{k-1} \\ \vdots \\ \sigma_{2k-1} \\ \sigma_{2k} \end{bmatrix}$$

$$= \left[\begin{array}{ccc} & G_k\,\Phi_{k-2} & \\ \hline \sigma_{k+1} & \cdots & \sigma_{2k-1} \end{array}\right] \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_{k-1} \end{bmatrix} + \nu_k \begin{bmatrix} \sigma_{k-1} \\ \vdots \\ \sigma_{2k-1} \\ \sigma_{2k} \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} & & & \sigma_{k-1} \\ & G_k\,\Phi_{k-2} & & \vdots \\ & & & \sigma_{2k-1} \\ \hline \sigma_{k-1} & \cdots & \sigma_{2k-1} & \sigma_{2k} \end{array}\right] \bar{v}.$$

Then, it is straightforward to verify that

$$\left[\begin{array}{ccc|c} & & & \sigma_{k-1} \\ & G_k\,\Phi_{k-2} & & \vdots \\ & & & \sigma_{2k-1} \\ \hline \sigma_{k-1} & \cdots & \sigma_{2k-1} & \sigma_{2k} \end{array}\right] = \left[\begin{array}{ccc|c} & & & \mu_{k+1} \\ & G_k & & \vdots \\ & & & \mu_{2k+1} \\ \hline \mu_{k+1} & \cdots & \mu_{2k+1} & \mu_{2k+2} \end{array}\right] \left[\begin{array}{c|c} & 0 \\ \Phi_{k-2} & c \\ & b \\ \hline 0 & a \end{array}\right].$$

This proves that

$$G_{k+1}^{(1)} \begin{bmatrix} \bar{v} \\ 0 \\ 0 \end{bmatrix} = G_{k+1}\,\Phi_{k-1}\,\bar{v},$$

and, thus, (5.11) holds for $n \geq 2$. □

We have the following characterization of classical sequences of real numbers.

Theorem 5.11. *Let $\{\mu_n\}_{n\geq 0}$ be a sequence of real numbers such that* $\det G_n \neq 0$ *for $n \geq 0$. Let $S_n^{-1}\,H_n\,S_n^{-\top}$ be the Cholesky factorization of G_n and let $\bar{s}_{n,0}, \bar{s}_{n,1}, \ldots, \bar{s}_{n,n}$ denote the columns of $S_n^{\top}$. Then $\{\mu_n\}_{n\geq 0}$ is classical if and only* *if there are real numbers* a, b, c *satisfying*

$$|a| + |b| + |c| > 0,$$

and real numbers a_j, b_j, c_j, $j \geq 0$, *with* $c_j \neq 0$, *such that the set* $\{\bar{s}^{(1)}_{n,0}, \ldots, \bar{s}^{(1)}_{n,n}\}$ *of vectors in* $\mathbb{R}^{n+1}$ *with*

$$\bar{s}^{(1)}_{n,j} = \frac{1}{j+1} N_{n+1}\, \bar{s}_{n+1,j+1}, \quad 0 \leq j \leq n,$$

satisfy

$$\Phi_{n-2}\, \bar{s}^{(1)}_{n-2,j} = a_j\, \bar{s}_{n,j+2} + b_j\, \bar{s}_{n,j+1} + c_j\, \bar{s}_{n,j}, \quad 0 \leq j \leq n-2, \tag{5.12}$$

with Φ_n *as defined in* (5.10)

Proof. We will use the fact that

$$S^{\top}_{n,1} = [\bar{s}^{(1)}_{n,0}\ \bar{s}^{(1)}_{n,1}\ \cdots\ \bar{s}^{(1)}_{n,n}]$$

is a unit upper triangular matrix.

Suppose that $\{\mu_n\}_{n\geq 0}$ is a classical sequence satisfying (3.1). Since $\Phi_{n-2}\, \bar{s}^{(1)}_{n-2,j} \in \mathbb{R}^{n+1}$ and $\{\bar{s}_{n,0}, \bar{s}_{n,1}, \ldots, \bar{s}_{n,n}\}$ constitutes an orthogonal basis for $\mathbb{R}^{n+1}$, we can write

$$\Phi_{n-2}\, \bar{s}^{(1)}_{n-2,j} = \sum_{k=0}^{n} a_{j,k}\, \bar{s}_{n,k},$$

with

$$a_{j,k}\, h_k = \bar{s}^{\top}_{n,k}\, G_n\, \Phi_{n-2}\, \bar{s}^{(1)}_{n-2,j}, \quad 0 \leq k \leq n.$$

The absence of a subindex indicating some dependence on n is justified by Theorems 4.3 and 5.6, which imply that the expression for $\bar{s}_{n,k}$ and $\bar{s}^{(1)}_{n-2,j}$ are independent of n.

By (5.11) and since

$$S^{\top}_{n,1} = \left[\begin{array}{c|cc} S^{\top}_{n-2,1} & \multicolumn{2}{c}{*} \\ \hline \multirow{2}{*}{0} & 1 & * \\ & 0 & 1 \end{array}\right],$$

we have

$$a_{j,k}\, h_k = \bar{s}^{\top}_{n,k}\, G^{(1)}_n \begin{bmatrix} \bar{s}^{(1)}_{n-2,j} \\ 0 \\ 0 \end{bmatrix} = \bar{s}^{\top}_{n,k}\, G^{(1)}_n\, \bar{s}^{(1)}_{n,j}, \quad 0 \leq k \leq n,\ \ 0 \leq j \leq n-2.$$

On one hand, by Corollary 5.9, the set $\{\bar{s}^{(1)}_{n,0}, \ldots, \bar{s}^{(1)}_{n,n}\}$ is an orthogonal basis for $\mathbb{R}^{n+1}$, and since $\bar{s}_{n,k} = \bar{s}^{(1)}_{n,k} + \alpha_{k,k-1}\, \bar{s}^{(1)}_{n,k-1} + \cdots + \alpha_{k,0}\, \bar{s}^{(1)}_{n,0}$, we get that $a_{j,k} = 0$

for $0 \leq k \leq j-1$, and

$$a_{j,j}\, h_j = \bar{s}_{n,j}^{\top}\, G_n^{(1)}\, \bar{s}_{n,j}^{(1)} = (\bar{s}_{n,j}^{(1)})^{\top}\, G_n^{(1)}\, \bar{s}_{n,j}^{(1)} = h_j^{(1)}.$$

Therefore,

$$a_{j,j} = \frac{h_j^{(1)}}{h_j} = -\frac{\lambda_{j+1}}{(j+1)^2} \neq 0.$$

On the other hand, since $S_{n-2,1}^{\top}$ is upper triangular, we have

$$\Phi_{n-2}\, \bar{s}_{n-2,j}^{(1)} = \begin{bmatrix} * \\ a \\ \mathsf{0} \end{bmatrix} \in \mathbb{R}^{n+1}, \quad 0 \leq j \leq n-2,$$

where the last $n-j-2$ entries are zero. This implies that $a_{j,k} = 0$ for $k \geq j+3$. Therefore, (5.12) holds with $a_j = a_{j,j+2}$, $b_j = a_{j,j+1}$ and $c_j = a_{j,j} \neq 0$.

Conversely, suppose that (5.12) holds with the entries of Φ_{n-2} are real numbers a, b, c satisfying $|a| + |b| + |c| > 0$. On one hand, multiplying both sides of (5.12) by $\bar{s}_{n,0}^{\top}\, G_n$, we obtain for $0 \leq j \leq n-2$,

$$\bar{s}_{n,0}^{\top}\, G_n(a_j\, \bar{s}_{n,j+2} + b_j\, \bar{s}_{n,j+1} + c_j\, \bar{s}_{n,j}) = \bar{s}_{n,0}^{\top}\, G_n \Phi_{n-2}\, \bar{s}_{n-2,j}^{(1)}.$$

By the orthogonality of $\{\bar{s}_{n,0}, \bar{s}_{n,1}, \ldots, \bar{s}_{n,n}\}$, we have

$$\bar{s}_{n,0}^{\top}\, G_n \Phi_{n-2}\, N_{n-1}\, \bar{s}_{n-1,j+1} = (j+1) c_0\, h_0\, \delta_{j,0}.$$

On the other hand, we have

$$\bar{s}_{n-1,1}^{\top}\, G_{n-1}\, \bar{s}_{n-1,j+1} = h_1\, \delta_{j,0} = (j+1)\, h_1\, \delta_{j,0}, \quad 0 \leq j \leq n-2.$$

Therefore,

$$\bar{s}_{n,0}^{\top}\, G_n \Phi_{n-2}\, N_{n-1}\, \bar{s}_{n-1,j+1} = c_0\, \frac{h_0}{h_1}\, \bar{s}_{n-1,1}^{\top}\, G_{n-1}\, \bar{s}_{n-1,j+1},$$

or, equivalently,

$$\left(\bar{s}_{n,0}^{\top}\, G_n\, \Phi_{n-2}\, N_{n-1} - c_0\, \frac{h_0}{h_1}\, \bar{s}_{n-1,1}^{\top}\, G_{n-1}\right) \bar{s}_{n-1,j+1} = 0, \quad 0 \leq j \leq n-2.$$

Since $N_{n-1} \bar{s}_{n-1,0} = \mathsf{0}$ and $\bar{s}_{n-1,1}^{\top}\, G_{n-1}\, \bar{s}_{n-1,0} = 0$, we have

$$\left(\bar{s}_{n,0}^{\top}\, G_n\, \Phi_{n-2}\, N_{n-1} - c_0\, \frac{h_0}{h_1}\, \bar{s}_{n-1,1}^{\top}\, G_{n-1}\right) S_{n-1}^{\top} = \mathsf{0},$$

and, therefore,

$$\bar{s}_{n,0}^{\top}\, G_n\, \Phi_{n-2}\, N_{n-1} - c_0\, \frac{h_0}{h_1}\, \bar{s}_{n-1,1}^{\top}\, G_{n-1} = \mathsf{0}, \tag{5.13}$$

where we have used the fact that $S_{n-1}^{\top}$ is an invertible matrix. If

$$\bar{s}_{n-1,1}^{\top} = [s_{1,0}\ 1\ 0 \cdots 0],$$

then let $d = -c_0 \frac{h_0}{h_1}$ and $e = d\, s_{1,0}$. In this way, the entries of (5.13) read

$$d\,\mu_1 + e\,\mu_0 = 0,$$

$$(k\,a + d)\,\mu_{k+1} + (k\,b + e)\,\mu_k + k\,c\,\mu_{k-1} = 0, \quad 1 \leq k \leq n-1,$$

which proves that $\{\mu_n\}_{n\geq 0}$ satisfies (3.1) and, thus, $\{\mu_n\}_{n\geq 0}$ is classical. □

We briefly recast Theorem 5.11 in terms of polynomials. Let $\{\mu_n\}_{n\geq 0}$ be a classical sequence of real numbers, and let $\mathbf{u}$ be the moment functional defined as $\mu_n = \langle \mathbf{u}, x^n \rangle$, $n \geq 0$. Let $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ be sequences of polynomials with

$$P_n(x) = \bar{s}_{n,n}^{\top}\, \mathtt{X}_n \quad \text{and} \quad Q_n(x) = (\bar{s}_{n,n}^{(1)})^{\top}\, \mathtt{X}_n \quad n \geq 0.$$

Then $\{P_n(x)\}_{n\geq 0}$ is an OPS associated with $\mathbf{u}$, and

$$Q_j(x) = \frac{P'_{j+1}(x)}{j+1}, \quad 0 \leq j \leq n.$$

For $a, b, c \in \mathbb{R}$ such that $|a| + |b| + |c| > 0$, and $n \geq 0$, the matrix Φ_n is the matrix representation of the linear mapping from Π_n to Π_{n+2} defined by $p(x) \mapsto \phi(x)p(x)$ with $\phi(x) = a\,x^2 + b\,x + c$. Therefore,

$$(\Phi_n\, \bar{s}_{n,n}^{(1)})^{\top}\, \mathtt{X}_{n+2} = \phi(x)\, Q_n(x).$$

In this way, by Theorem 5.11, $\mathbf{u}$ is a classical moment functional if and only if there is a nonzero polynomial $\phi(x)$ with $\deg \phi(x) \leq 2$, and real numbers a_n, b_n, c_n, $n \geq 0$, with $c_n \neq 0$,

$$\phi(x) Q_n(x) = a_n\, p_{n+2}(x) + b_n\, p_{n+2}(x) + c_n\, p_n(x), \quad n \geq 0.$$

5.4. *Second structure relation*

The following characterization is similar to (5.12) but it has a dual flavor in the sense that the roles of $\{\bar{s}_{n,j}\}_{j=0}^{n}$ and $\{\bar{s}_{n,j}^{(1)}\}_{j=0}^{n}$ are interchanged.

Theorem 5.12. *Let $\{\mu_n\}_{n\geq 0}$ be a sequence of real numbers such that $\det G_n \neq 0$ for $n \geq 0$. Let $S_n^{-1}\, H_n\, S_n^{-\top}$ be the Cholesky factorization of G_n and let $\bar{s}_{n,0}, \bar{s}_{n,1}, \ldots, \bar{s}_{n,n}$ denote the columns of $S_n^{\top}$. Then $\{\mu_n\}_{n\geq 0}$ is classical if and only if there are real numbers κ_j, ξ_j, $j \geq 0$, such that*

$$\bar{s}_{n,j} = \bar{s}_{n,j}^{(1)} + \kappa_j\, \bar{s}_{n,j-1}^{(1)} + \xi_j\, \bar{s}_{n,j-2}^{(1)}, \quad 0 \leq j \leq n, \tag{5.14}$$

where, by convention, we set $\bar{s}_{n,-2}^{(1)} = \bar{s}_{n,-1}^{(1)} = \mathtt{0}$.

Proof. We will use the fact that

$$S_{n,1}^{\top} = [\bar{s}_{n,0}^{(1)}\ \bar{s}_{n,1}^{(1)}\ \cdots\ \bar{s}_{n,n}^{(1)}]$$

is a unit upper triangular matrix.

Suppose that $\{\mu_n\}_{n\geq 0}$ is a classical sequence. By Corollary 5.9, $\{\bar{s}^{(1)}_{n,0},\ldots,\bar{s}^{(1)}_{n,n}\}$ is an orthogonal basis for $\mathbb{R}^{n+1}$. Since $S^\top_{n,1}$ is a unit upper triangular matrix, we can write

$$\bar{s}_{n,j}=\bar{s}^{(1)}_{n,j}+\sum_{k=0}^{j-1} a_{j,k}\,\bar{s}^{(1)}_{n,k},\quad 0\leq j\leq n,$$

where

$$a_{j,k}\,h^{(1)}_k=(\bar{s}^{(1)}_{n,k})^\top\,G^{(1)}_n\,\bar{s}_{n,j},\quad 0\leq k\leq j-1.$$

As before, the absence of a subindex indicating some dependence on n is justified by Theorems 4.3 and 5.6, which imply that the expression for $\bar{s}_{n,k}$ and $\bar{s}^{(1)}_{n-2,j}$ are independent of n. Using (5.11) and Theorem 5.11, we get

$$a_{j,k}\,h^{(1)}_k=(a_k\,\bar{s}_{n,k+2}+b_k\,\bar{s}_{n,k+1}+c_k\,\bar{s}_{n,k})^\top\,G_n\,\bar{s}_{n,j},\quad 0\leq k\leq n-2.$$

From the orthogonality of $\{\bar{s}_{n,0},\ldots,\bar{s}_{n,n}\}$, we obtain $a_{j,k}=0$ for $k\leq j-3$. Therefore, (5.14) holds with $\kappa_j=a_{j,j-1}$ and $\xi_j=a_{j,j-2}$.

Conversely, suppose that there are real numbers κ_j,ξ_j, $j\geq 0$, such that (5.14) holds. For $n\geq 0$, define the $(n+1)\times(n+1)$ unit upper triangular matrices

$$U_0=1,\quad U_1=\begin{bmatrix}1 & \kappa_1\\ 0 & 1\end{bmatrix},\quad U_n=\left[\begin{array}{c|c} & 0\\ U_{n-1} & \xi_n\\ & \kappa_n\\ \hline 0 & 1\end{array}\right],\quad n\geq 2.$$

Then (5.14) can be written as follows:

$$S^\top_n=S^\top_{n,1}\,U_n.$$

Using $H_n=S_n\,G_n\,S^\top_n$, we get

$$U_n=U_n\,H^{-1}_n\,S_n\,G_n\,S^\top_n=U_n\,H^{-1}_n\,S_n\,G_n\,S^\top_{n,1}\,U_n,$$

which implies

$$I_{n+1}=U_n\,H^{-1}_n\,S_n\,G_n\,S^\top_{n,1}.$$

From this equality we get

$$S^{-\top}_{n,1}=U_n\,H^{-1}_n\,S_n\,G_n. \tag{5.15}$$

On one hand, observe that

$$\begin{aligned} N_{n+1}\,S^\top_{n+1}&=\begin{bmatrix}0 & N_{n+1}\,\bar{s}_{n+1,1} & N_{n+1}\,\bar{s}_{n+1,2} & \cdots & N_{n+1}\,\bar{s}_{n+1,n+1}\end{bmatrix}\\ &=\begin{bmatrix}0 & \bar{s}^{(1)}_{n,0} & 2\,\bar{s}^{(1)}_{n,1} & \cdots & (n+1)\,\bar{s}^{(1)}_{n,n}\end{bmatrix}\\ &=S^\top_{n,1}\,N_{n+1}. \end{aligned} \tag{5.16}$$

Combining (5.15) and (5.16), we get

$$N_{n+1} = U_n\, H_n^{-1}\, S_n\, G_n\, N_{n+1}\, S_{n+1}^{\top}.$$

On the other hand, we have

$$\frac{1}{h_1}\, \bar{s}_{n+1,1}^{\top}\, G_{n+1}\, S_{n+1}^{\top} = \bar{s}_{n,0}^{\top}\, N_{n+1}.$$

Therefore,

$$\frac{1}{h_1}\, \bar{s}_{n+1,1}^{\top}\, G_{n+1} = \bar{s}_{n,0}^{\top}\, U_n\, H_n^{-1}\, S_n\, G_n\, N_{n+1}. \tag{5.17}$$

If we let

$$\begin{bmatrix} c \\ b \\ a \\ \mathtt{0} \end{bmatrix} = \frac{\xi_2}{h_2}\, \bar{s}_{n,2} + \frac{\kappa_1}{h_1}\, \bar{s}_{n,1} + \frac{1}{h_0}\, \bar{s}_{n,0} \quad \text{and} \quad \begin{bmatrix} e \\ d \\ \mathtt{0} \end{bmatrix} = -\frac{1}{h_1}\, \bar{s}_{n+1,1}, \tag{5.18}$$

then the entries of (5.17) read

$$-e\, \mu_0 - d\, \mu_1 = 0,$$
$$-e\, \mu_k - d\, \mu_{k+1} = k\, c\, \mu_{k-1} + k\, b\, \mu_k + k\, a\, \mu_{k+1}, \quad 1 \le k \le n+1,$$

which proves that $\{\mu_n\}_{n\ge 0}$ satisfies (3.1) and, thus, $\{\mu_n\}_{n\ge 0}$ is classical. □

For a classical sequence $\{\mu_n\}_{n\ge 0}$, let $\mathbf{u}$ be the moment functional defined as $\mu_n = \langle \mathbf{u}, x^n \rangle$, $n \ge 0$, and let $\{P_n(x)\}_{n\ge 0}$ and $\{Q_n(x)\}_{n\ge 0}$ be sequences of polynomials with

$$P_n(x) = \bar{s}_{n,n}^{\top}\, \mathtt{X}_n \quad \text{and} \quad Q_n(x) = (\bar{s}_{n,n}^{(1)})^{\top}\, \mathtt{X}_n \quad n \ge 0.$$

Theorem 5.12 implies that there are real numbers κ_n, ξ_n, $n \ge 0$, such that

$$P_n(x) = Q_n(x) + \kappa_n\, Q_{n-1}(x) + \xi_n\, Q_{n-2}(x), \quad n \ge 0,$$

where, by convention, $Q_{-2}(x) = Q_{-1}(x) = 0$. Moreover, we deduce from (5.18) and Theorem 3.2 that $\mathbf{u}$ satisfies $D(\phi\, \mathbf{u}) = \psi\, \mathbf{u}$ with

$$\phi(x) = \frac{\xi_2}{h_2} P_2(x) + \frac{\kappa_1}{h_1} P_1(x) + \frac{1}{h_0} P_0(x) \quad \text{and} \quad \psi(x) = -\frac{1}{h_1} P_1(x).$$

5.5. *Rodrigues-type formula*

Recall that classical sequences of real numbers $\{\mu_n\}_{n\ge 0}$ satisfy the three-term recurrence relation (3.1) which can be written in matrix form as follows:

$$N_{n+1}^{\top}\, G_n^{(1)}\, \bar{s}_{n,0} + G_n \begin{bmatrix} e \\ d \\ \mathtt{0} \end{bmatrix} = \mathtt{0}, \quad n \ge 1,$$

with $G_n^{(1)}$ as defined in (5.3). The following characterization shows that classical sequences satisfy higher-order recurrence relations which can be written in matrix form as well.

Theorem 5.13. *Let $\{\mu_n\}_{n\geq 0}$ be a sequence of real numbers such that $\det G_n \neq 0$, $n \geq 0$. Let $S_n^{-1}\, H_n\, S_n^{-\top}$ be the Cholesky factorization of G_n and let $\bar{s}_{n,0}, \bar{s}_{n,1}, \ldots, \bar{s}_{n,n}$ denote the columns of $S_n^{\top}$. Then $\{\mu_n\}_{n\geq 0}$ is classical if and only if there are $a, b, c \in \mathbb{R}$ such that $|a| + |b| + |c| > 0$, and nonzero real numbers ϖ_k, $k \geq 1$, such that*

$$N_{n+k}^{\top} \cdots N_{n+1}^{\top}\, G_n^{(k)}\, \bar{s}_{n,0} = \varpi_k\; G_{n+k}\, \bar{s}_{n+k,k}, \quad n \geq 1, \;\; 1 \leq k \leq n, \tag{5.19}$$

with $G_n^{(k)}$ as defined in (5.8).

Proof. Suppose that $\{\mu_n\}_{n\geq 0}$ is classical satisfying (3.1). For $k \geq 0$, by Corollary 5.9, the set $\{\bar{s}_{n,0}^{(k)}, \ldots, \bar{s}_{n,n}^{(k)}\}$ of vectors in $\mathbb{R}^{n+1}$ with

$$\bar{s}_{n,j}^{(k)} = \frac{1}{j+1}\, N_{n+1}\, \bar{s}_{n+1,j+1}^{(k-1)}, \quad 0 \leq j \leq n,$$

where $s_{n,j}^{(0)} = \bar{s}_{n,j}$ constitutes an orthogonal basis for $\mathbb{R}^{n+1}$ with respect to the bilinear form associated with $\{\sigma_n^{(k)}\}_{n\geq 0}$. Let

$$S_{n,k}^{\top} = [\bar{s}_{n,0}^{(k)}\; \bar{s}_{n,1}^{(k)} \cdots \bar{s}_{n,n}^{(k)}].$$

On one hand, observe that for $k \geq 1$,

$$\begin{aligned} N_{n+1}\, S_{n+1,k-1}^{\top} &= [\mathsf{0}\;\, N_{n+1}\, \bar{s}_{n+1,1}^{(k-1)}\;\, N_{n+1}\, \bar{s}_{n+1,2}^{(k-1)} \cdots N_{n+1}\, \bar{s}_{n+1,n+1}^{(k-1)}] \\ &= [\mathsf{0}\;\, \bar{s}_{n,0}^{(k)}\;\, 2\, \bar{s}_{n,1}^{(k)} \cdots (n+1)\, \bar{s}_{n,n}^{(k)}] \\ &= S_{n,k}^{\top}\, N_{n+1}. \end{aligned}$$

Then, since $\bar{s}_{n,0}^{(k)} = \bar{s}_{n,0}$ for $k \geq 1$,

$$\begin{aligned} S_{n+k}\, N_{n+k}^{\top} \cdots N_{n+1}^{\top}\, G_n^{(k)}\, \bar{s}_{n,0} &= N_{n+k}^{\top} \cdots N_{n+1}^{\top}\, S_{n,k}\, G_n^{(k)}\, \bar{s}_{n,0}^{(k)} \\ &= k!\, h_0^{(k)}\, \bar{e}_k \in \mathbb{R}^{n+k+1}, \end{aligned}$$

where $\bar{e}_k$ denotes the $(k+1)$th column of the identity matrix I_{n+k+1}. On the other hand,

$$\frac{1}{h_k}\, S_{n+k}\, G_{n+k}\, \bar{s}_{n+k,k} = \bar{e}_k.$$

Since S_{n+k} is invertible, it follows that

$$N_{n+k}^{\top} \cdots N_{n+1}^{\top}\, G_n^{(k)}\, \bar{s}_{n,0} = k!\, \frac{h_0^{(k)}}{h_k}\; G_{n+k}\, \bar{s}_{n+k,k}, \quad k \geq 1.$$

Hence, (5.19) holds with $\varpi_k = k!\, \frac{h_0^{(k)}}{h_k}$.

Conversely, suppose that there are $a, b, c \in \mathbb{R}$ such that $|a| + |b| + |c| > 0$, and nonzero real numbers ϖ_k, $k \geq 1$, such that (5.19) holds. In particular, for $k = 1$, we have

$$N_{n+1}^{\top} G_n^{(1)} \bar{s}_{n,0} = \varpi_1 \, G_{n+1} \, \bar{s}_{n+1,1}, \quad n \geq 0. \tag{5.20}$$

If we let

$$\begin{bmatrix} e \\ d \end{bmatrix} = -\varpi_1 \, \bar{s}_{1,1},$$

and since $\bar{s}_{n+1,1} = \begin{bmatrix} \bar{s}_{1,1} \\ 0 \end{bmatrix}$, then the entries of (5.20) read

$$-e \, \mu_0 - d \, \mu_1 = 0,$$

$$-e \, \mu_k - d \, \mu_{k+1} = k \, c \, \mu_{k-1} + k \, b \, \mu_k + k \, a \, \mu_{k+1}, \quad 1 \leq k \leq n+2,$$

which proves that $\{\mu_n\}_{n \geq 0}$ satisfies (3.1) and, thus, $\{\mu_n\}_{n \geq 0}$ is classical. □

We remark that if $\{\mu_n\}_{n \geq 0}$ is a classical sequence of real numbers, and $\mathbf{u}$ be the moment functional defined as $\mu_n = \langle \mathbf{u}, x^n \rangle$, $n \geq 0$, then the entries of (5.20) can be written as follows:

$$\langle D(\phi \, \mathbf{u}), x^k \rangle = \langle \psi \, \mathbf{u}, x^k \rangle, \quad 0 \leq k \leq n+2,$$

where $\phi(x) = a\,x^2 + b\,x + c$ and $\psi(x) = d\,x + e$, which holds for all $n \geq 0$. Hence, $\mathbf{u}$ satisfies (2.1). Moreover, it is straightforward, but tedious, to verify that the entries of (5.19) can be written as follows:

$$\langle D^k(\phi^k \, \mathbf{u}), x^j \rangle = \langle \varpi_k \, P_k \, \mathbf{u}, x^j \rangle, \quad 0 \leq j \leq n+k+1,$$

where $P_k(x) = \bar{s}_{n+k,k}^{\top} \, \mathtt{X}_{n+k}$, $k \geq 1$, which holds for $n \geq 1$. Hence, $\mathbf{u}$ satisfies $D^k(\phi^k \, \mathbf{u}) = \varpi_k \, P_k \, \mathbf{u}$ for $k \geq 1$ (and holds for $k = 0$ with $\varpi_0 = 1$).

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