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Gradings induced by nilpotent elements ☆

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An element a is nilpotent last-regular if it is nilpotent and its last nonzero power is von Neumann regular. In this paper we show that any nilpotent last-regular element a in an associative algebra R over a ring of scalars Φ gives rise to a complete system of orthogonal idempotents that induces a finite $\mathbb{Z}$-grading on R; we also show that such element gives rise to an $\mathfrak{sl}_2$-triple in R with semisimple adjoint map ad_h, and that the grading of R with respect to the complete system of orthogonal idempotents is a refinement of the Φ-grading induced by the eigenspaces of ad_h. These results can be adapted to nilpotent elements a with all their powers von Neumann regular, in which case the element a can be completed to an $\mathfrak{sl}_2$-triple and a is homogeneous of degree 2 both in the $\mathbb{Z}$-grading of R and in the Φ-grading given by the eigenspaces of ad_h.

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Conversely, when there are enough invertible elements in Φ, we will show that if a nilpotent element a can be completed to an $\mathfrak{sl}_2$-triple, then all powers a^k are von Neumann regular.

1. Introduction

The study of gradings in associative and different nonassociative structures has been of great interest in the last decades (see for example [4], [3], [2], [5], [1] or [24]). We highlight the works of Patera, Zassenhaus, Havlíček and Pelantová ([17], [12] and [13]) who, between 1989 and 2000, performed a systematic study of fine gradings on the finite-dimensional complex simple Lie algebras. We also remark Elduque's and Kochetov's 2013 monograph devoted to gradings in Lie algebras [7]; this book contains a compilation of the results about classifications of gradings by arbitrary groups on simple Lie algebras over algebraically closed fields and unifies the terminology about gradings that had appeared before in the literature.

An important tool in the study of gradings on Lie algebras is the embedding of the Lie algebra under consideration into an associative algebra, and the study of the classification problem in the context of associative algebras (maybe with involution). In this line, Rodrigo-Escudero considered gradings by an abelian group on an associative algebra such that the algebra is graded-simple and satisfies the descending chain condition on graded left ideals, and gave necessary and sufficient conditions for the grading to be fine, see [18]; these results where used as a tool by Elduque, Kochetov and Rodrigo-Escudero to classify fine gradings on all real forms of classical simple Lie algebras up to equivalence, see [8].

In 1997 O. Smirnov showed in [22] that in any unital simple associative algebra R with a finite $\mathbb{Z}$-grading there exists a maximal complete system of orthogonal idempotents $\mathcal{E} = \{e_0, e_1, \ldots, e_n\}$ that induces the grading, i.e., if $R = \bigoplus_{k=-n}^{n} R_k$ then each $R_k = \sum_{i-j=k} e_i R e_j$. More generally, if the hypothesis of being unital is removed, the grading is induced by a maximal complete orthogonal system of modules $\mathcal{H} = \{H_p = R_p R_{-n} R_{n-p} \mid p = 0, \ldots, n\}$. In a subsequent paper [23], the same author studied a $*$-version of these results for associative algebras with involution $*$, and applied them to give a more precise description of associative gradings appearing in E. Zelmanov's classification of simple Lie algebras with a finite $\mathbb{Z}$-grading [25].

Later on, in 2006 Siles Molina revisited Smirnov's results in [20] and showed that every finite $\mathbb{Z}$-grading of a simple non-necessarily unital associative algebra R comes from a Peirce decomposition induced by a complete system of orthogonal idempotents lying in the maximal left quotient algebra of R.

In this paper we will show that as soon as an associative algebra R over a ring of scalars Φ (Φ is a commutative unital ring) contains a nilpotent element whose last nonzero power is von Neumann regular, there exists a complete system of idempotents in the unitization R^1 of R, and R is graded with respect to it. We remark that our results work for general associative algebras without any regularity condition.

Another way of getting a grading in an associative algebra R is the existence of an $\mathfrak{sl}_2$-triple (e, h, f) such that e is nilpotent. In his work of 1958, N. Jacobson showed that when a nilpotent element e can be completed to an $\mathfrak{sl}_2$-triple (e, h, f) and there exist enough invertible elements in Φ, the map $\mathrm{ad}_h : R \to R$ given by $\mathrm{ad}_h(x) := hx - xh$ for every $x \in R$ is semisimple and the decomposition of R into the eigenspaces of ad_h is an associative grading of R, see [14, Lemma 1] and 2.11.

We will prove that the condition of having a nilpotent element with von Neumann regular last nonzero power implies the existence of an $\mathfrak{sl}_2$-triple of R with semisimple adjoint map ad_h, and that the grading of R with respect to the complete system of orthogonal idempotents is a refinement of the grading induced by the eigenspaces of ad_h. Conversely, if a nilpotent element a can be completed to an $\mathfrak{sl}_2$-triple, we will show that a^k is von Neumann regular for every k when there are enough invertible elements in Φ.

Similar techniques involving gradings induced by complete sets of orthogonal idempotents, $\mathfrak{sl}_2$-triples and von Neumann regularity were used by O. Smirnov in [21] to show that the standard conjecture of Lefschetz stating that the adjoint of the Lefschetz operator is induced by an algebraic correspondence follows from Grothendieck's conjecture about the equality of the numerical and homological equivalences.

2. Preliminaries

2.1. Throughout this paper we will deal with non-necessarily unital associative algebras R over a ring of scalars Φ. R^1 will denote the unitization of R, i.e., $R^1 = R + \Phi 1$.

An element $e \in R$ is an idempotent if $e^2 = e$, and two idempotents $e, f \in R$ are orthogonal if $ef = fe = 0$.

Recall that an element $a \in R$ is von Neumann regular if there exists $b \in R$ such that $aba = a$. We say that a nilpotent element $a \in R$ is last-regular if its last nonzero power is von Neumann regular; more precisely, $a \in R$ is nilpotent last-regular of index $t+1$ if a is nilpotent of index $t+1$ and a^t is von Neumann regular in R. Given a nilpotent last-regular element $a \in R$ of index $t+1$, there exists $b \in R$ such that $a^t b a^t = a^t$, $ba^t b = b$ and $ba^k b = 0$ for every $k = 0, \ldots, t-1$ (by abuse of notation $ba^0 b$ means b^2), see [11, Lemma 2.4]. The element b is called a Rus-inverse of a.

Let R be an associative algebra with involution $*$ over a ring of scalars Φ with $\frac{1}{2} \in \Phi$. Given an element $0 \neq a \in H(R,*) \cup \mathrm{Skew}(R,*)$ we define the parity of a, denoted by $|a|$, as

$$|a| = \begin{cases} 0, & \text{if } a \in H(R,*) \\ 1, & \text{if } a \in \mathrm{Skew}(R,*). \end{cases}$$

In the following lemma we will show that when a is a symmetric or skew-symmetric nilpotent last-regular element of index $t+1$, a Rus-inverse of a can be chosen so that its parity coincides with that of a^t.

Lemma 2.2. *Let R be an associative algebra with involution $*$ over a ring of scalars Φ, $\frac{1}{2} \in \Phi$. Let $a \in H(R,*) \cup \mathrm{Skew}(R,*)$ be a nilpotent last-regular element of index $t+1$. Then*

(i) *if $a^t \in H(R,*)$, we can construct a Rus-inverse of a in $H(R,*)$,*
(ii) *if $a^t \in \mathrm{Skew}(R,*)$, we can construct a Rus-inverse of a in $\mathrm{Skew}(R,*)$.*

If b is a Rus-inverse of a and $|a^t| = |b|$, the element b will be called a $*$-Rus-inverse of a.

Proof. When $a \in H(R,^*)$ is a nilpotent last-regular element, the construction of a symmetric Rus-inverse of a was done in [9, Lemma 3.2]. The same argument can be adapted when $a \in \mathrm{Skew}(R,*)$ and $a^t \in \mathrm{Skew}(R,*) \cup H(R,*)$ and we include it here for the sake of completeness.

Let $a \in \mathrm{Skew}(R,*)$ and suppose that $a^t \in \mathrm{Skew}(R,*)$ (respectively, $a^t \in H(R,*)$). Since a^t is von Neumann regular there exists $c \in R$ such that $a^t c a^t = a^t$. Moreover, since $\frac{1}{2} \in \Phi$ we can decompose c as $c = \frac{1}{2}(c - c^*) + \frac{1}{2}(c + c^*)$, and therefore $a^t = a^t c a^t = \frac{1}{2}a^t(c-c^*)a^t + \frac{1}{2}a^t(c+c^*)a^t$ and we can replace c by $\frac{1}{2}(c-c^*) \in \mathrm{Skew}(R,*)$ (respectively, by $\frac{1}{2}(c+c^*) \in H(R,*)$) and assume without loss of generality that $a^t, c \in \mathrm{Skew}(R,*)$ (respectively, that $a^t, c \in H(R,*)$). Now, if we consider $d = ca^tc$ we have

$$da^t d = d \quad \text{and} \quad a^t d a^t = a^t.$$

From now on, the construction of a Rus-inverse follows as in [9, Lemma 3.2]; since it is quite tricky, we include it here for the sake of completeness.

The proof will follow by descending induction from t to 1: Define $u_t = 1 - \frac{1}{2}ada^{t-1}$ and $b_t = u_t d u_t^* = (1 - \frac{1}{2}ada^{t-1})d(1 - \frac{1}{2}a^{t-1}da)$. Then

- $a^t b_t a^t = a^t(1 - \dfrac{1}{2}ada^{t-1})d(1 - \dfrac{1}{2}a^{t-1}da)a^t = a^t d a^t = a^t,$
- $b_t a^t b_t = (1 - \dfrac{1}{2}ada^{t-1})d(1 - \dfrac{1}{2}a^{t-1}da)a^t(1 - \dfrac{1}{2}ada^{t-1})d(1 - \dfrac{1}{2}a^{t-1}da)$
  $= (1 - \dfrac{1}{2}ada^{t-1})da^t d(1 - \dfrac{1}{2}a^{t-1}da) = (1 - \dfrac{1}{2}ada^{t-1})d(1 - \dfrac{1}{2}a^{t-1}da) = b_t,$
- $b_t a^{t-1} b_t = u_t d(1 - \dfrac{1}{2}a^{t-1}da)a^{t-1}(1 - \dfrac{1}{2}ada^{t-1})du_t^*$
  $= u_t da^{t-1}du_t^* - \dfrac{1}{2}u_t da^t da^{t-1}du_t^* - \dfrac{1}{2}u_t da^{t-1}da^t du_t^* = 0.$

Let us suppose that there exists b_{i+1} such that

$$a^t b_{i+1} a^t = a^t, \quad b_{i+1} a^t b_{i+1} = b_{i+1}, \quad \text{and} \quad b_{i+1} a^s b_{i+1} = 0, \quad \text{for } s = i, \dots, t-1.$$

Let us construct b_i such that

$$a^t b_i a^t = a^t, \quad b_i a^t b_i = b_i, \quad \text{and} \quad b_i a^s b_i = 0, \quad \text{for } s = i-1, \dots, t-1.$$

Define $u_i = 1 - \frac{1}{2} a^{t-i+1} b_{i+1} a^{i-1}$ and $b_i = u_i b_{i+1} u_i^*$. Then for any $s = i, \dots, t-1$

- $a^t b_i a^t = a^t (1 - \frac{1}{2} a^{t-i+1} b_{i+1} a^{i-1}) b_{i+1} (1 - \frac{1}{2} a^{i-1} b_{i+1} a^{t-i+1}) a^t = a^t b_{i+1} a^t = a^t,$
- $b_i a^t b_i = u_i b_{i+1} (1 - \frac{1}{2} a^{i-1} b_{i+1} a^{t-i+1}) a^t (1 - \frac{1}{2} a^{t-i+1} b_{i+1} a^{i-1}) b_{i+1} u_i^*$
  $= u_i b_{i+1} a^t b_{i+1} u_i^* = u_i b_{i+1} u_i^* = b_i,$
- $b_i a^s b_i = u_i b_{i+1} (1 - \frac{1}{2} a^{i-1} b_{i+1} a^{t-i+1}) a^s (1 - \frac{1}{2} a^{t-i+1} b_{i+1} a^{i-1}) b_{i+1} u_i^*$
  $= u_i b_{i+1} a^s b_{i+1} u_i^* - \frac{1}{2} u_i b_{i+1} a^{i-1} b_{i+1} a^{t-i+1+s} b_{i+1} u_i^*$
  $- \frac{1}{2} u_i b_{i+1} a^{t-i+1+s} b_{i+1} a^{i-1} b_{i+1} u_i^*$
  $+ \frac{1}{4} u_i b_{i+1} a^{i-1} b_{i+1} a^{2t-2i+2+s} b_{i+1} a^{i-1} b_{i+1} u_i^* = 0,$
- $b_i a^{i-1} b_i = u_i b_{i+1} (1 - \frac{1}{2} a^{i-1} b_{i+1} a^{t-i+1}) a^{i-1} (1 - \frac{1}{2} a^{t-i+1} b_{i+1} a^{i-1}) b_{i+1} u_i^*$
  $= u_i b_{i+1} (a^{i-1} - \frac{1}{2} a^{i-1} b_{i+1} a^t)(1 - \frac{1}{2} a^{t-i+1} b_{i+1} a^{i-1}) b_{i+1} u_i^*$
  $= u_i b_{i+1} a^{i-1} b_{i+1} u_i^* - \frac{1}{2} u_i b_{i+1} a^{i-1} b_{i+1} a^t b_{i+1} u_i^* - \frac{1}{2} u_i b_{i+1} a^t b_{i+1} a^{i-1} b_{i+1} u_i^* = 0.$

The element b_1 satisfies the claim. □

2.3. Let us recall the notion of local algebra of an associative algebra at an element u (see [10]): let R be an associative algebra over a ring of scalars Φ and let $u \in R$. The Φ-module uRu with product given by

$$uxu \cdot_u uyu := uxuyu$$

is again an associative algebra, denoted by R_u and called the local algebra of R at u. If R has an involution $*$ and $u \in H(R, *) \cup \mathrm{Skew}(R, *)$, the map $\star : R_u \to R_u$ given by $(uxu)^\star = (-1)^{|u|} ux^* u$ is an involution on R_u.

When e is an idempotent of R, the local algebra R_e of R at e coincides with the corner eRe.

In the following proposition we recall that any nilpotent last-regular element a in an associative algebra R gives rise to a family of matrix units and that certain subalgebra of R is isomorphic to a matrix algebra over a local algebra of R at some element, [11, Theorem 2.6] (the construction of the same family of matrix units associated to a nilpotent element with some extra conditions was done by J. Levitzki in [16, Theorem 2.1]); these results can be extended to associative algebras with involution.

Proposition 2.4. *Let R be an associative algebra over a ring of scalars Φ. Let $a \in R$ be a nilpotent last-regular element of index $t+1$ and let b be a Rus-inverse of a. Then*

(1) $\{e_{ij} := a^{i-1}ba^{t+1-j}\}_{i,j=1}^{t+1}$ *is a family of matrix units, i.e., $e_{ij}e_{kl} = \delta_{jk}e_{il}$ (where δ denotes the Kronecker delta). If we denote $e_i = e_{ii}$, then $e_i a = ae_{i-1}$ for $i = 2, \dots, t+1$.*
(2) The idempotent $e = \sum_{i=1}^{t+1} e_i$ satisfies $ea = ae = \sum_{i=1}^{t} e_{i+1,i}$ and the subalgebra eRe is isomorphic to $\mathcal{M}_{t+1}(R_{e_{ij}})$ for any $i, j \in \{1, \dots, t+1\}$.
(3) $(ea)^t = e_{t+1,1} = a^t ba^t = a^t$.

Moreover, if R has an involution $*$, $\frac{1}{2} \in \Phi$, $a \in H(R,*) \cup \mathrm{Skew}(R,*)$ and b is a $*$-Rus inverse of a, then for any r, s such that $e_{rs} \in H(R,*) \cup \mathrm{Skew}(R,*)$ (for instance $e_{t+1,1} = a^t$), eRe and $\mathcal{M}_{t+1}(R_{e_{rs}})$ are $*$-isomorphic under the map

$$\Psi : \mathcal{M}_{t+1}(R_{e_{rs}}) \to eRe \text{ defined by } \Psi(\sum_{ij} x_{ij}E_{ij}) = \sum_{i,j=1}^{t+1} e_{ir}x_{ij}e_{sj}$$

where each $x_{ij} = e_{rs}x_{ij}e_{rs} \in R_{e_{rs}}$, and where the involution $*$ in $\mathcal{M}_{t+1}(R_{e_{rs}})$ is given by

$$A^* = C^{-1}\bar{A}^{\mathrm{tr}}C \text{ for any } A \in \mathcal{M}_{t+1}(R_{e_{rs}})$$

for $C = \sum_{i=1}^{t+1}(-1)^{i|a|}E_{i,t+2-i}$ and $\bar{A}^{\mathrm{tr}} = \sum_{ij}(a_{ij}^\star)E_{ji}$ for every $A = \sum_{ij} a_{ij}E_{ij} \in \mathcal{M}_{t+1}(R_{e_{rs}})$.

The idempotent e will be called Rus-idempotent (respectively, $*$-Rus-idempotent) associated to a and its Rus-inverse b (respectively, to a and its $*$-Rus-inverse b).

Proof. Items (1), (3) and (2) in the particular case of the corner $e_{11}Re_{11}$ were shown in [11, Theorem 2.6]. Notice that the corner $e_{11}Re_{11}$, the local algebra $R_{e_{11}}$ and the local algebras $R_{e_{ij}}$ at each of the matrix units e_{ij} are all isomorphic (under the isomorphism $\varphi : R_{e_{ij}} \to e_{11}Re_{11}$ given by $\varphi(e_{ij}xe_{ij}) = e_{11}e_{1i}e_{ij}xe_{ij}e_{j1}e_{11}$), so (2) holds in general.

Suppose that R is an associative algebra with involution $*$, let $a \in H(R,*) \cup \mathrm{Skew}(R,*)$, let b be a $*$-Rus inverse of a and let us compute e_{ij}^*:

- If $a \in \mathrm{Skew}(R,*)$, $e_{ij}^* = (-1)^{i-j}e_{t+2-j,t+2-i}$: indeed, $e_{ij}^* = (a^{i-1}ba^{t+1-j})^* = (-1)^{t+1-j+i-1+|b|}a^{t+1-j}ba^{i-1} = (-1)^{t+i-j+|b|}e_{t+2-j,t+2-i}$ where $|b| = 0$ if $b \in$

$H(R,*)$ and $|b| = 1$ if $b \in \mathrm{Skew}(R,*)$. Since $b \in H(R,*)$ when t is even and $b \in \mathrm{Skew}(R,*)$ when t is odd, we conclude that $e_{ij}^* = (-1)^{i-j} e_{t+2-j,t+2-i}$.
- If $a \in H(R,*)$ then $e_{ij}^* = e_{t+2-j,t+2-i}$.

Summarizing, if $a \in H(R,*) \cup \mathrm{Skew}(R,*)$ then $e_{ij}^* = (-1)^{(i-j)|a|} e_{t+2-j,t+2-i}$.

Let e_{rs} be a symmetric or skew-symmetric matrix unit (in particular, $r+s = t+2$ and $(-1)^{(r-s)|a|} = (-1)^{|e_{rs}|}$) and consider the local algebra $R_{e_{rs}}$ of R at e_{rs}. Since $\{e_{ij}\}_{i,j=1}^{t+1}$ is a family of matrix units, the map

$$(e_{rs}xe_{rs})E_{ij} \mapsto e_{ir}(e_{rs}xe_{rs})e_{sj} \in eRe$$

defines a $*$-isomorphism between $\mathcal{M}_{t+1}(R_{e_{rs}})$ and eRe. Moreover, since

$$\begin{aligned}
(e_{ir}(e_{rs}xe_{rs})e_{sj})^* &= e_{sj}^* e_{rs} x^* e_{rs} e_{ir}^* \\
&= (-1)^{(s-j+i-r)|a|} e_{t+2-j,t+2-s}(e_{rs}x^*e_{rs})e_{t+2-r,t+2-i} \\
&= (-1)^{(s-j+i-r)|a|} e_{t+2-j,r}(e_{rs}x^*e_{rs})e_{s,t+2-i} \\
&= (-1)^{(i-j)|a|} e_{t+2-j,r}((-1)^{(s-r)|a|} e_{rs}x^*e_{rs})e_{s,t+2-i} \\
&= (-1)^{(i-j)|a|} e_{t+2-j,r}(e_{rs}xe_{rs})^\star e_{s,t+2-i},
\end{aligned}$$

we have that

$$\Psi(((e_{rs}xe_{rs})E_{ij})^*) = \Psi((-1)^{(i-j)|a|}(e_{rs}xe_{rs})^\star E_{t+2-j,t+2-i})$$

so the involution in $\mathcal{M}_{t+1}(R_{e_{rs}})$ is given by

$$A^* = C^{-1}\bar{A}^{\mathrm{tr}}C$$

where $C = \sum_{i=1}^{t+1} (-1)^{i|a|} E_{i,t+2-i}$ and

$$\bar{A}^{\mathrm{tr}} = \sum_{ij} a_{ij}^\star E_{ji} \text{ for every } A = \sum_{ij} a_{ij}E_{ij} \in \mathcal{M}_{t+1}(R_{e_{rs}}). \quad \square$$

Remark 2.5. The $*$-isomorphism described in Proposition 2.4 has already appeared in [6, 3.1] in a particular situation. In that case t was even and eRe was shown to be $*$-isomorphic to $\mathcal{M}_{t+1}(S)$ where $S = e_{\frac{t+2}{2}} R e_{\frac{t+2}{2}}$ (when t is even, $e_{\frac{t+2}{2}} \in H(R,*)$).

2.6. Suppose that $a \in R$ is nilpotent of index $t+1$ and all the powers a^k, $k \leq t$, are von Neumann regular. By [11, 2.10] there exists a family of nonzero orthogonal idempotents $\{e^{(i)}\}_{i=1}^m$ that commute with a and such that $a = \sum_{i=1}^m e^{(i)}a$ and the elements $e^{(i)}a$ are nilpotent last-regular of decreasing indices $t_i + 1$, $t_1 = t > t_2 > \cdots > t_m$, each $e^{(i)}a \in e^{(i)}Re^{(i)}$ and $e^{(i)}$ is a Rus-idempotent for $e^{(i)}a$.

2.7. Let G be a group. A G-grading on R is a family $\Gamma = \{R_g\}_{g\in G}$ of Φ-modules, called homogeneous components, such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_{g'} \subset R_{gg'}$ for all $g, g' \in G$. We denote by $\Gamma : R = \bigoplus_{g\in G} R_g$ and call (R, Γ) a G-graded algebra. The support, denoted Supp Γ, is the set $\{g \in G \mid R_g \neq \{0\}\}$. We say that a grading is finite if it has finite support. The most frequently encountered gradings are finite gradings by the group $\mathbb{Z}$, and will be called finite $\mathbb{Z}$-gradings. Any group homomorphism $\alpha : G \to H$ gives a functor from G-graded algebras to H-graded ones: for $R = \bigoplus_{g\in G} R_g$ we define the H-graded algebra R where R is the same Φ-algebra but equipped with the H-grading $R = \bigoplus_{h\in H} R'_h$ where $R'_h = \bigoplus_{g\in\alpha^{-1}(h)} R_g$. If the G-grading on R is denoted by Γ, we will write ${}^{\alpha}\Gamma$ for the corresponding H-grading on R. A G-grading $\Gamma : R = \bigoplus_{g\in G} R_g$ is said to be a refinement of an H-grading $\Gamma' : R = \bigoplus_{h\in H} R'_h$ (or Γ' a coarsening of Γ) if, for any $g \in G$, there exists $h \in H$ such that $R_g \subset R'_h$. If the inclusion is proper for at least one $g \in$ Supp Γ, the refinement (or the coarsening) is called proper. For example, if $\alpha : G \to H$ is a group homomorphism, then ${}^{\alpha}\Gamma$ is a coarsening of Γ, which is proper if and only if α is not injective on the support of Γ.

A complete finite family of orthogonal idempotents in R^1 is a finite family $\mathcal{E} = \{e_0, \ldots, e_n\}$ of elements of R^1 such that $e_i e_j = \delta_{ij} e_i$, $i, j = 0, \ldots, n$, (where δ denotes the Kronecker delta) and such that $\sum_{i=0}^{n} e_i = 1$. Notice that every complete finite family of orthogonal idempotents $\mathcal{E}$ in R^1 induces a finite $\mathbb{Z}$-grading $\Gamma_{\mathcal{E}}$ on R whose homogeneous submodules are given by $R_k = \sum_{i-j=k} e_i R e_j$, $k = 0, \pm 1, \ldots, \pm n$.

2.8. Let R be an associative algebra over a ring of scalars Φ. An $\mathfrak{sl}_2$-triple $(\hat{e}, \hat{h}, \hat{f})$ of R consists on three elements $\hat{e}, \hat{h}, \hat{f} \in R$ such that

$$[\hat{e}, \hat{f}] = \hat{h}, \quad [\hat{h}, \hat{e}] = 2\hat{e} \quad \text{and} \quad [\hat{h}, \hat{f}] = -2\hat{f}$$

where $[x, y] := xy - yx$ for every $x, y \in R$. Given $x \in R^1$, the map $\mathrm{ad}_x : R \to R$ is given by $\mathrm{ad}_x(y) = [x, y]$, $y \in R$. We say that $\hat{e} \in R$ can be completed to an $\mathfrak{sl}_2$-triple if there exists $\hat{f} \in R$ such that $(\hat{e}, \hat{h}, \hat{f})$ is an $\mathfrak{sl}_2$-triple (recall that $\hat{h} = [\hat{e}, \hat{f}]$).

In Lie theory, $\mathfrak{sl}_2$-triples are usually denoted with the letters e, h and f. We have denoted the elements of an $\mathfrak{sl}_2$-triple by $\hat{e}$, $\hat{h}$ and $\hat{f}$ to avoid confusions with the notation of idempotents in the previous results, which naturally arises from the usual notation e_{ij} of matrix units and the usual ring theory notation of idempotents with the letters e and f.

Lemma 2.9. *Let R be an associative algebra over Φ. Let $\mathcal{F} = \{f_i \mid i \in \Delta \subset \mathbb{Z}\}$ be a complete finite family of orthogonal idempotents in R^1 and let $h = \sum_{i\in\Delta} i f_i \in R^1$. Then $\mathrm{ad}_h : R \to R$ is semisimple and induces a Φ-grading Γ_h on R given by the eigenspaces of $\mathrm{ad}_h : R \to R$, which is a coarsening of the finite $\mathbb{Z}$-grading $\Gamma_{\mathcal{F}}$ induced by $\mathcal{F}$ under the natural homomorphism $\varphi : \mathbb{Z} \to \Phi$.*

Proof. We can easily show that every homogeneous submodule of the $\mathbb{Z}$-grading induced by the family $\mathcal{F}$ is contained in an eigenspace of ad_h. Indeed, for every $x \in R$

$$\begin{aligned}\mathrm{ad}_h(f_jxf_k) &= hf_jxf_k - f_jxf_kh = (\sum_{i\in\Delta} if_i)f_jxf_k - f_jxf_k(\sum_{i\in\Delta} if_i)\\ &= jf_jxf_k - kf_jxf_k = (j-k)f_jxf_k,\end{aligned}$$

so f_jRf_k is contained in the eigenspace of ad_h associated to the eigenvalue $j-k$. □

Lemma 2.10. *Let R be an associative algebra over Φ and suppose that $(2t)!$ is invertible in Φ. Let $\{e_k \mid k=-t,\dots,t\}$ and $\{f_k \mid k=-t,\dots,t\}$ be two families of orthogonal idempotents and suppose that $\sum_{k=-t}^{t} ke_k = \sum_{k=-t}^{t} kf_k$. Then $f_k = e_k$ for every $k \neq 0$.*

Proof. Multiply $\sum_{k=-t}^{t} ke_k = \sum_{k=-t}^{t} kf_k$ on the left by e_i, so $ie_i = \sum_k ke_if_k$. Then multiply on the right by f_s: $ie_if_s = se_if_s$, i.e., $(i-s)e_if_s = 0$. If $i \neq s$ then $e_if_s = 0$ because $i-s$ is invertible in Φ. Hence $ie_i = \sum_k ke_if_k = ie_if_i$, and when $i \neq 0$ this means that $e_i = e_if_i$.

Multiplying $\sum_{k=-t}^{t} ke_k = \sum_{k=-t}^{t} kf_k$ on the right by e_i we get $ie_i = \sum_{k=-t}^{t} kf_ke_i$, and multiplying on the left by f_s, $if_se_i = sf_se_i$. As before, if $i \neq s$, $f_se_i = 0$, so $ie_i = if_ie_i$, i.e., $e_i = f_ie_i$ when $i \neq 0$.

Exchanging the roles of e_i and f_i in the argument above we get $f_i = e_if_i = f_ie_i$. Thus $e_i = f_i$, for every $i \neq 0$. □

2.11. When $(t+1)!$ is invertible in Φ, N. Jacobson showed in [14, Lemma 1] that if a nilpotent element $\hat{e} \in R$ of index $t+1$ can be completed to an $\mathfrak{sl}_2$-triple $(\hat{e}, \hat{h}, \hat{f})$, then $\hat{h}$ satisfies the polynomial

$$f(X) := \prod_{j=-t}^{t} (X-j) \in \Phi[X].$$

If $(2t)!$ is invertible in Φ, for every $i,j \in \{-t,\dots,0,\dots,t\}$, $i \neq j$, the monomials $X-i$ and $X-j$ are coprime. Let us define $q_k(X) := f(X)/(X-k)$, $k = 0, \pm1, \dots, \pm t$. Then there exist $r_k(X) \in \Phi[X]$, $k = 0, \pm1, \dots, \pm t$, such that $\sum_{k=-t}^{t} r_k(X)q_k(X) = 1$ (recall that if $a(X)$ is coprime to $b(X)$ and to $c(X)$, then $a(X)$ is coprime to $b(X)c(X)$). Let

$$\mathcal{F} = \{f_k := r_k(\hat{h})q_k(\hat{h}) \mid k = -t,\dots,t\},$$

which is a complete finite family of orthogonal idempotents in R^1. Since $(\hat{h}-k)q_k(\hat{h}) = 0$, $\hat{h}f_k = kf_k$, so $\hat{h} = \hat{h}(\sum_{k=-t}^{t} f_k) = \sum_{k=-t}^{t} kf_k$. By Lemma 2.9, the map $\mathrm{ad}_{\hat{h}} : R \to R$ is semisimple and the Φ-grading $\Gamma_{\hat{h}}$ on R given by the eigenspaces of $\mathrm{ad}_{\hat{h}} : R \to R$ is a coarsening (under the natural homomorphism) of the $\mathbb{Z}$-grading $\Gamma_{\mathcal{F}}$ induced by $\mathcal{F}$ on R.

3. Main

3.1. Let $a \in R$ be a nilpotent last-regular element of index $t+1$, and let b be a Rus-inverse of a. Consider the family of matrix units (defined in Proposition 2.4(1))

$$\{e_{ij} := a^{i-1}ba^{t+1-j}\}_{i,j=1}^{t+1}$$

associated to a and b. Denote $e_i := e_{ii}$, $i = 1, \dots, t+1$. Let $e = e_1 + \cdots + e_{t+1}$ be the Rus-idempotent associated to a and its Rus-inverse b. For convenience, let us rename the idempotents $e_1, \dots, e_{t+1}$ and consider the following family of orthogonal idempotents:

$$\mathcal{F}_{ea} = \{f_{-t+2(i-1)} := e_i \mid i = 1, \dots, t+1\} = \{f_{-t}, f_{-t+2}, \dots, f_{t-2}, f_t\}. \tag{3.1}$$

If $\mathcal{F}_{ea}$ is not a complete family of orthogonal idempotents in R^1 we complete this family defining a new f_0 as follows: If t is even, replace f_0 by $f_0 + (1-e)$ and if t is odd, define f_0 as $(1-e)$. Let us denote by $\hat{\mathcal{F}}_{ea}$ this complete family of orthogonal idempotents in R^1:

$$\begin{array}{ll} \hat{\mathcal{F}}_{ea} = \mathcal{F}_{ea} \cup \{f_0 = 1-e\}, & \text{if } t \text{ is odd;} \\ \hat{\mathcal{F}}_{ea} = \{f_{-t+2(i-1)} \in \mathcal{F}_{ea} \mid i \neq \frac{t}{2}+1\} \cup \{f_0 = 1 - e + e_{\frac{t}{2}+1}\}, & \text{otherwise.} \end{array} \tag{3.2}$$

Proposition 3.2. *Let R be an associative algebra over Φ and let $a \in R$ be a nilpotent last-regular element of index $t+1$. Let e be a Rus-idempotent associated to a and its Rus-inverse b, and let $\mathcal{F}_{ea}$ be the finite family of idempotents given in 3.1. Then ea can be completed to an $\mathfrak{sl}_2$-triple $(ea, \hat{h}, \hat{f})$ of R with $\hat{h} = \sum k f_k$ where the f_k's are the idempotents of $\mathcal{F}_{ea}$ (and also of $\hat{\mathcal{F}}_{ea}$).*

Proof. Let b be a Rus-inverse associated to the nilpotent last-regular element a. Let $\{e_{ij} := a^{i-1}ba^{t+1-j}\}_{i,j=1}^{t+1}$ be the family of matrix units associated to the nilpotent last-regular element a and its Rus-inverse b defined in Proposition 2.4(1), and let $e_i := e_{ii}$, $i = 1, \dots, t+1$. Define $e = \sum_{i=1}^{t+1} e_i$ (the Rus-idempotent associated to a and b) and

$$\hat{f} := \sum_{i=1}^{t} i(t+1-i)e_{i,i+1}, \tag{3.3}$$

and let us see that $\hat{h} := [ea, \hat{f}]$ satisfies $[\hat{h}, ea] = 2ea$ and $[\hat{h}, \hat{f}] = -2\hat{f}$. Firstly, let us compute $\hat{h}$ in terms matrix units:

$$\begin{aligned} \hat{h} = [ea, \hat{f}] &= \sum_{i=1}^{t} i(t+1-i)e_{i+1} - \sum_{i=1}^{t} i(t+1-i)e_i \\ &= \sum_{i=2}^{t+1} (i-1)(t+2-i)e_i - \sum_{i=1}^{t} i(t+1-i)e_i \\ &= -te_1 + \sum_{i=2}^{t} ((i-1)(t+2-i) - i(t+1-i))e_i + te_{t+1} \\ &= -te_1 + \sum_{i=2}^{t} (2i-t-2)e_i + te_{t+1} = \sum_{i=1}^{t+1} (-t+2(i-1))e_i. \end{aligned} \tag{3.4}$$

In particular, since $\mathcal{F}_{ea}$ is just a translation in the indices of the elements e_i (see equation (3.1)), we get that $\hat{h} = \sum_{f_k \in \mathcal{F}_{ea}} k f_k$. This sum also holds taking the idempotents f_k in $\hat{\mathcal{F}}_{ea}$ because the completion of the family $\mathcal{F}_{ea}$ to $\hat{\mathcal{F}}_{ea}$ only affects the idempotent f_0, which is multiplied by the coefficient 0 in the expression of $\hat{h}$ as a sum of idempotents. Moreover,

$$\begin{aligned}[\hat{h}, ea] &= \sum_{i=2}^{t+1}(-t+2(i-1))e_{i,i-1} - \sum_{i=1}^{t}(-t+2(i-1))e_{i+1,i} \\ &= \sum_{i=1}^{t}(-t+2i)e_{i+1,i} - \sum_{i=1}^{t}(-t+2(i-1))e_{i+1,i} \\ &= \sum_{i=1}^{t} 2e_{i+1,i} = 2ea.\end{aligned}$$

Similarly,

$$\begin{aligned}[\hat{h}, \hat{f}] &= \sum_{i=1}^{t}(-t+2(i-1))i(t+1-i)e_{i,i+1} - \sum_{i=1}^{t} i(t+1-i)(-t+2i)e_{i,i+1} \\ &= -2\sum_{i=1}^{t} i(t+1-i)e_{i,i+1} = -2\hat{f},\end{aligned}$$

i.e., $(ea, \hat{h}, \hat{f})$ is an $\mathfrak{sl}_2$-triple of R. $\square$

Remark 3.3. The family $\hat{\mathcal{F}}_{ea}$ induces the following finite $\mathbb{Z}$-grading in R

$$\Gamma_{ea} : R = R_{-2t} \oplus \cdots \oplus R_0 \oplus \cdots \oplus R_{2t} \tag{3.5}$$

where each $R_k := \sum_{i-j=k} f_i R f_j$. With respect to this grading

$$ea \in R_2, \ (1-e)a \in R_0 \text{ and } a^t \in R_{2t}$$

because, by Proposition 2.4,

- $ea = \sum_{i=1}^{t} e_{i+1,i} = \sum_{i=1}^{t} f_{-t+2i}\, e_{i+1,i} f_{-t+2(i-1)} \in R_2,$
- $(1-e)a = f_0(1-e)af_0 \in R_0,$
- $a^t = e_{t+1,1} = f_t e_{t+1,1} f_{-t} \in R_{2t}.$

This grading (R, Γ_{ea}) will be called the finite $\mathbb{Z}$-grading of R induced by the nilpotent last-regular element a and its Rus-inverse b.

If R has an involution $*$ and $a \in \mathrm{Skew}(R,*) \cup H(R,*)$ we will assume that $\frac{1}{2} \in \Phi$ and that b is a $*$-Rus-inverse of a. Then $f_i^* = f_{-i}$ for every i, the idempotent e is symmetric, $\hat{\mathcal{F}}_{ea}$ is a $*$-complete family of orthogonal idempotents in R^1 and the above grading (R, Γ_{ea}) is compatible with the involution (for each k, $R_k^* \subset R_k$).

Example 3.4. Suppose that a is a nilpotent last-regular element of index 3. This means that $a^3 = 0$ and a^2 is von Neumann regular. Let b be a Rus-inverse of a. In this case $e_1 = ba^2$, $e_2 = aba$ and $e_3 = a^2 b$. Let $e = e_1 + e_2 + e_3$. Then

$$\begin{aligned} \hat{\mathcal{F}}_{ea} &= \{f_{-2} = e_1, f_0 = e_2, f_2 = e_3\}, \text{ and} \\ \hat{\mathcal{F}}_{ea} &= \{f_{-2}, f_0 = 1 - e + e_2, f_2\} \end{aligned}$$

is a complete system of orthogonal idempotents in R^1 and induces the grading

$$\Gamma_{ea} : R = R_{-4} \oplus R_{-2} \oplus R_0 \oplus R_2 \oplus R_4$$

with $R_k = \sum_{i-j=k} f_i R f_j$, $k = 0, \pm 2, \pm 4$. Clearly, $ea \in R_2$, $a^2 \in R_4$ and $(1-e)a \in R_0$.

More generally, if we have a nilpotent last-regular element of odd index $t+1$, we get a grading Γ_{ea} on R from R_{-2t} to R_{2t} with

$$\mathrm{Sup}\ \Gamma_{ea} = \{\pm 2i \mid i = 0, \dots, t\}$$

by Proposition 2.4(2) because the subalgebra eRe has nonzero elements in each submodule of $R_{\pm 2i}$, $i = 0, \dots, t$.

Example 3.5. Suppose that a is a nilpotent last-regular element of index 4. This means that $a^4 = 0$ and a^3 is von Neumann regular. Let b be a Rus-inverse of a. In this case $e_1 = ba^3$, $e_2 = aba^2$, $e_3 = a^2 ba$ and $e_4 = a^3 b$. Let $e = e_1 + e_2 + e_3 + e_4$. Then

$$\begin{aligned} \mathcal{F}_{ea} &= \{f_{-3} = e_1, f_{-1} = e_2, f_1 = e_3, f_3 = e_4\}, \text{ and} \\ \hat{\mathcal{F}}_{ea} &= \{f_{-3}, f_{-1}, f_0 = 1 - e, f_1, f_3\} \end{aligned}$$

is a complete system of orthogonal idempotents in R^1 and induces the grading

$$\Gamma_{ea} : R = R_{-6} \oplus R_{-4} \oplus R_{-3} \oplus R_{-2} \oplus R_{-1} \oplus R_0 \oplus R_1 \oplus R_2 \oplus R_3 \oplus R_4 \oplus R_6$$

with $R_k = \sum_{i-j=k} f_i R f_j$, $k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 6$. Notice that R_5 and R_{-5} do not appear in this grading. Clearly, $ea \in R_2$, $a^3 \in R_6$ and $(1-e)a \in R_0$.

More generally, if we have a nilpotent last-regular element of even index $t+1$, we get a grading Γ_{ea} on R from R_{-2t} to R_{2t} with

$$\mathrm{Sup}\ \Gamma_{ea} \subset \{\pm 2i \mid i = 0, \dots, t\} \cup \{\pm j \mid j = 1, \dots, t\},$$

and also $\{\pm 2i \mid i = 0, \ldots, t\} \subset \operatorname{Sup} \Gamma_{ea}$ by Proposition 2.4(2) because the subalgebra eRe has nonzero elements in each submodule of $R_{\pm 2i}$, $i = 0, \ldots, t$.

3.6. Let a be a nilpotent element of index $t+1$ such that a^k is regular for every $k \in \{1, ..., t\}$. By 2.6 there exists a family of nonzero orthogonal idempotents $\{e^{(i)}\}_{i=1}^m$ that commute with a and such that $a = \sum_{i=1}^m e^{(i)}a$ and the elements $e^{(i)}a$ are nilpotent last-regular of decreasing indices $t_i + 1$, $t_1 = t > t_2 > \cdots > t_m$. For each $e^{(i)}a$, let us denote by $\mathcal{F}^{(i)}_{e^{(i)}a}$ the non-necessarily complete family of idempotents given in 3.1.

Let us consider the following family of idempotents

$$\mathcal{F}_a = \{f_j := \sum_{i=1}^m f_j^{(i)} \mid j = -t, \ldots, t \text{ where } f_j^{(i)} \in \mathcal{F}^{(i)}_{e^{(i)}a} \text{ and } f_j^{(i)} = 0 \text{ if } |j| > t_i\}. \tag{3.6}$$

If this family $\mathcal{F}_a$ is not complete in R^1, let us complete it by defining a new f_0 as follows: If $f_0 \in \mathcal{F}_a$ we replace f_0 by $f_0 + (1 - e^{(1)} - \cdots - e^{(m)})$ and if $f_0 \notin \mathcal{F}_a$ (this happens if all t_i's are odd) we define $f_0 := 1 - e^{(1)} - \cdots - e^{(m)}$. Let us denote by $\hat{\mathcal{F}}_a$ this complete family of orthogonal idempotents, i.e.,

$$\begin{array}{ll} \hat{\mathcal{F}}_a = \mathcal{F}_a \cup \{f_0 := 1 - e\}, & \text{if } f_0 \notin \mathcal{F}_a; \\ \hat{\mathcal{F}}_a = \{f_j \in \mathcal{F}_a \mid j = -t, \ldots, t,\ j \neq 0\} \cup \{f_0 + 1 - e\}, & \text{if } f_0 \in \mathcal{F}_a \end{array} \tag{3.7}$$

where $e := e^{(1)} + \cdots + e^{(m)}$.

Proposition 3.7. *Let R be an associative algebra over Φ and let $a \in R$ be a nilpotent element of index $t+1$ such that all the powers a^k, $k = 1, \ldots, t$, are von Neumann regular. Then the element a can be completed to an $\mathfrak{sl}_2$-triple $(a, \hat{h}, \hat{f})$ with $\hat{h} = \sum_j j f_j$ where the f_j's are the idempotents of the family $\mathcal{F}_a$ (and also of the family $\hat{\mathcal{F}}_a$).*

Proof. Let us suppose that a^k is von Neumann regular for every $k = 1, \ldots, t$. By 2.6 there exists a family $\{e^{(i)}\}_{i=1}^m$ of nonzero orthogonal idempotents that commute with a and such that $a = \sum_{i=1}^m e^{(i)}a$, and every $e^{(i)}a$ is a nilpotent last-regular element of index $t_i + 1$ with $t = t_1 > t_2 \cdots > t_m$. For each i, by Proposition 3.2 there exist $\hat{h}^{(i)}$ and $\hat{f}^{(i)}$ such that each $e^{(i)}a$ is part of an $\mathfrak{sl}_2$-triple $(e^{(i)}a, \hat{h}^{(i)}, \hat{f}^{(i)})$. If we define $\hat{h} = \sum_{i=1}^m \hat{h}^{(i)}$ and $\hat{f} = \sum_{i=1}^m \hat{f}^{(i)}$ we have that $(a, \hat{h}, \hat{f})$ is an $\mathfrak{sl}_2$-triple of R.

Moreover, by Proposition 3.2, each $\hat{h}^{(i)} = \sum_{f_j^{(i)} \in \mathcal{F}^{(i)}_{e^{(i)}a}} j f_j^{(i)}$, so

$$\hat{h} = \sum_{i=1}^m \hat{h}^{(i)} = \sum_{i=1}^m \sum_{f_j^{(i)} \in \mathcal{F}^{(i)}_{e^{(i)}a}} j f_j^{(i)} = \sum_{f_j \in \mathcal{F}_a} j f_j.$$

This sum also holds taking the idempotents f_k in $\hat{\mathcal{F}}_a$ because the completion of the family $\mathcal{F}_a$ to $\hat{\mathcal{F}}_a$ only affects the idempotent f_0, which is multiplied by the coefficient 0 in the expression of $\hat{h}$ as a sum of idempotents. $\square$

Remark 3.8. The family $\hat{\mathcal{F}}_a$ induces the following finite $\mathbb{Z}$-grading on R:

$$\Gamma_a : R = R_{-2t} \oplus \cdots \oplus R_0 \oplus \cdots \oplus R_{2t} \tag{3.8}$$

where each $R_k := \sum_{i-j=k} f_i R f_j$. With respect to this grading Γ_a,

$$a \in R_2 \text{ and } a^t \in R_{2t}.$$

If R has an involution $*$ and $a \in \mathrm{Skew}(R,*) \cup H(R,*)$ we will assume that $\frac{1}{2} \in \Phi$ and that the Rus-inverses $b_1, \ldots, b_k$ associated to $e^{(1)}a, \ldots, e^{(m)}a$ are $*$-Rus-inverses. Then $f_j^* = f_{-j}$ for every j, the idempotents $e^{(i)}$ are all symmetric, $\hat{\mathcal{F}}_a$ is a $*$-complete family of orthogonal idempotents in R^1, and the above grading Γ_a in R is compatible with the involution (for each k, $R_k^* \subset R_k$).

Remark 3.9. In 3.3 and 3.8 we have built complete finite families of orthogonal idempotents in R^1 and the finite $\mathbb{Z}$-gradings Γ_{ea} and Γ_a induced by them on R. We have also built the $\mathfrak{sl}_2$-triples $(\hat{e}a, \hat{h}, \hat{f})$ (Proposition 3.2) and $(\hat{a}, \hat{h}, \hat{f})$ (Proposition 3.7) where $\hat{h}$ has a precise form. Lemma 2.9 can be applied to compare the $\mathbb{Z}$-grading Γ_{ea} (Remark 3.3) with the Φ-grading Γ_h of the $\mathfrak{sl}_2$-triple $(\hat{e}a, \hat{h}, \hat{f})$ of Proposition 3.2; similarly, it can be applied to compare the $\mathbb{Z}$-grading Γ_a (Remark 3.8) with the Φ-grading Γ_h of the $\mathfrak{sl}_2$-triple $(\hat{a}, \hat{h}, \hat{f})$ of Proposition 3.7.

In the rest of the section we will prove the converse of Proposition 3.7. Let us begin with a technical result.

Lemma 3.10. *Let R be an associative algebra over a ring of scalars Φ and let G be a group. Suppose that R is graded by G and let $a \in R_g$ be a homogeneous nilpotent last-regular element of index $t+1$. Then we can take a homogeneous Rus-inverse b of a in $R_{g^{-t}}$. In particular, the associated Rus-idempotent $e = \sum_{i=1}^{t+1} a^{i-1} b a^{t+1-i}$ belongs to R_u where $u \in G$ denotes the identity of the group G.*

Proof. The construction of a homogeneous Rus-inverse of a follows [11, Lemma 2.4] but takes into account that R has a G-grading and that $a \in R_g$: Since a^t is von Neumann regular there exists $b \in R$ such that $a^t b a^t = a^t$. By grading, $a^t \in R_{g^t}$ implies that b can be taken in $R_{g^{-t}}$. Clearly $b' = b a^t b \in R_{g^{-t}}$ satisfies

$$a^t b' a^t = a^t \qquad b' a^t b' = b'.$$

We use a recursive argument: by decreasing induction on $s = t-1, \ldots, 0$ suppose that there exists $b \in R_{g^{-t}}$ such that for every $k = s+1, \ldots, t-1$ we have that

$$a^t b a^t = a^t, \quad b a^t b = b \quad \text{and} \quad b a^k b = 0.$$

Denote $d := 1 - a^{t-s}ba^s \in R^1$ and define $c := db \in R$. Then $c = b - (a^{t-s}ba^s)b \in R_{g^{-t}}$ because $a^{t-s}ba^s \in R_u$, and satisfies $a^k c = a^k b$, $k = s+1, \ldots, t$, since $a^{t+1} = 0$, and

- $a^t c a^t = a^t b a^t = a^t$,
- $ca^t c = ca^t b = dba^t b = db = c$,
- $ca^k c = ca^k b = dba^k b = 0$, and
- $ca^s c = ca^s(1 - a^{t-s}ba^s)b = ca^s b - ca^s a^{t-s}ba^s b$
  $= dba^s b - dba^t ba^s b = 0.$

The result follows by recursion. $\square$

Theorem 3.11. *Let R be an associative algebra over a ring of scalars Φ with $(3t)!$ invertible in Φ and let $a \in R$ be a nilpotent element of index $t+1$. If a can be completed to an $\mathfrak{sl}_2$-triple $(a, \hat{h}, \hat{f})$ of R, then a^k is von Neumann regular for every $k \leq t$.*

Proof. Suppose that $(a, \hat{h}, \hat{f})$ is an $\mathfrak{sl}_2$-triple (recall $\hat{h} = [a, \hat{f}] = a\hat{f} - \hat{f}a$). We will show that a is nilpotent last regular of index $t+1$. To do so, let us first prove by induction that for every $k \in \{1, 2, \ldots, t\}$ we have that

$$a^k \hat{f}^k a^t = k!\, \hat{h}(\hat{h} - 1) \cdots (\hat{h} - k + 1)a^t.$$

Indeed,

- $a\hat{f}a^t = \hat{h}a^t$,
- $a\hat{f}a\hat{f}a^t = a\hat{f}\hat{h}a^t = a\hat{f}[\hat{h}, a^t] + a\hat{f}a^t\hat{h} = 2a\hat{f}a^t + \hat{h}a^t\hat{h} = 2\hat{h}a^t + \hat{h}[a^t, \hat{h}] + \hat{h}^2 a^t$
  $= 2\hat{h}a^t - 2\hat{h}a^t + \hat{h}^2 a^t = \hat{h}^2 a^t$,
- $a^2\hat{f}^2 a^t = a(a\hat{f})\hat{f}a^t = a[a, \hat{f}]\hat{f}a^t + a\hat{f}a\hat{f}a^t = [a, [a, \hat{f}]]\hat{f}a^t + [a, \hat{f}]a\hat{f}a^t$
  $+ a\hat{f}a\hat{f}a^t = -2a\hat{f}a^t + \hat{h}^2 a^t + \hat{h}^2 a^t = -2\hat{h}a^t + 2\hat{h}^2 a^t = 2\hat{h}(\hat{h} - 1)a^t$,
- $[a, \hat{f}^k] = \displaystyle\sum_{i=0}^{k-1} \hat{f}^i [a, \hat{f}] \hat{f}^{k-1-i}$ (because ad_a acts as a derivation on $\hat{f}^k$),
- $\displaystyle\sum_{i=0}^{k-1} [a^k \hat{f}^{i+1}, [a, \hat{f}]] = \sum_{i=0}^{k-1} [a^k \hat{f}^{i+1}, \hat{h}]$
  $= \displaystyle\sum_{i=0}^{k-1} (-2(k - i - 1))a^k \hat{f}^{i+1}$ (because $\mathrm{ad}_{\hat{h}}$ acts as a derivation),
- $a^k\hat{f}a\hat{f}^k a^t = a^k\hat{f}[a, \hat{f}^k]a^t = a^k\hat{f}\displaystyle\sum_{i=0}^{k-1} \hat{f}^i [a, \hat{f}]\hat{f}^{k-1-i}a^t = \sum_{i=0}^{k-1} [a^k \hat{f}^{i+1}, [a, \hat{f}]]\hat{f}^{k-1-i}a^t$

$$+\sum_{i=0}^{k-1}[a,\hat f]a^k\hat f^k a^t=\sum_{i=0}^{k-1}(-2(k-i-1))a^k\hat f^k a^t+k\hat h a^k\hat f^k a^t$$
$$=-k(k-1)a^k\hat f^k a^t+k\hat h a^k\hat f^k a^t.$$

If we suppose that $a^k\hat f^k a^t=k!\,\hat h(\hat h-1)\cdots(\hat h-k+1)a^t$, then:

$$\begin{aligned}a^{k+1}\hat f^{k+1}a^t&=a^k(a\hat f)\hat f^k a^t\\&=a^k[a,\hat f]\hat f^k a^t+a^k\hat f a\hat f^k a^t=[a^k,[a,\hat f]]\hat f^k a^t+[a,\hat f]a^k\hat f^k a^t+a^k\hat f a\hat f^k a^t\\&=-2ka^k\hat f^k a^t+\hat h a^k\hat f^k a^t-k(k-1)a^k\hat f^k a^t+k\hat h a^k\hat f^k a^t\\&=-k(k+1)a^k\hat f^k a^t+(k+1)\hat h a^k\hat f^k a^t=(k+1)(\hat h-k)a^k\hat f^k a^t\\&=(k+1)!\hat h(\hat h-1)\cdots(\hat h-k+1)(\hat h-k)a^t.\end{aligned}$$

In particular, if we define the polynomial $q(X)=\prod_{i=0}^{t-1}(X-i)\in\Phi[X]$, we have that

$$\frac{1}{t!}a^t\hat f^t a^t=\hat h(\hat h-1)\cdots(\hat h-(t-1))a^t=q(\hat h)a^t.$$

Since for any polynomial $\varphi(X)\in\Phi[X]$ we have that $\varphi(\hat h)a^k=a^k\varphi(\hat h+2k)$ for every $k\in\mathbb{N}$ (see [14, §2.(3)]), we get that $q(\hat h)a^t=a^tq(\hat h+2t)$.

On the other hand, by 2.11, $\hat h$ satisfies the polynomial $p(X)=\prod_{j=-t}^{t}(X-j)$.

In general, given three polynomials $a(X),b(X)$ and $c(X)\in\Phi[X]$ such that $\lambda_1a(X)+\mu_1b(X)=1$ and $\lambda_2a(X)+\mu_2c(X)=1$ for some $\lambda_1,\lambda_2,\mu_1,\mu_2\in\Phi[X]$, then

$$\begin{aligned}1&=\lambda_1\lambda_2a(X)^2+\lambda_1\mu_2a(X)c(X)+\mu_1\lambda_2b(X)a(X)+\mu_1\mu_2b(X)c(X)\\&=(\lambda_1\lambda_2a(X)+\lambda_1\mu_2c(X)+\mu_1\lambda_2b(X))\,a(X)+\mu_1\mu_2\,b(X)c(X),\end{aligned}$$

i.e., $a(X)$ and $b(X)c(X)$ satisfy a Bezout-like identity in $\Phi[X]$.

Since $(3t)!$ is invertible in Φ, for every $i,j\in\{-2t,\dots,0,\dots,t\}$, $i\neq j$, the monomials $X-i$ and $X-j$ satisfy a Bezout-like identity: $\frac{1}{j-i}(X-i)+\frac{1}{i-j}(X-j)=1\in\Phi[X]$. Thus the polynomials $q(X+2t)$ and $p(X)$ satisfy a Bezout-like identity, and there exist $r(X),s(X)\in\Phi[X]$ such that $r(X)q(X+2t)+s(X)p(X)=1$. Since $r(\hat h-2t)a^t=a^tr(\hat h)$,

$$\begin{aligned}\frac{1}{t!}a^t\hat f^t r(\hat h-2t)a^t=\frac{1}{t!}a^t\hat f^t a^t r(\hat h)&=q(\hat h)a^tr(\hat h)\\&=a^tq(\hat h+2t)r(\hat h)=a^t(q(\hat h+2t)r(\hat h)+s(\hat h)p(\hat h))\\&=a^t,\ \text{ i.e., } a^t(\frac{\hat f^t r(\hat h-2t)}{t!})a^t=a^t.\end{aligned}$$

Therefore, a^t is a von Neumann regular element of R and a is nilpotent last-regular of index $t+1$.

By 2.11, $\mathrm{ad}_{\hat{h}}: R \to R$ is semisimple and R has a finite Φ-grading with respect to the eigenspaces of $\mathrm{ad}_{\hat{h}}$. In particular, $a \in R_{(2)}$ and $\hat{h} \in R_{(0)}$. By Lemma 3.10 there exists a Rus inverse $b \in R_{(-2t)}$ of a and $e = \sum_{i=1}^{t+1} a^{i-1} b a^{t+1-i} \in R_{(0)}$. Since $ea \in eRe \cong \mathcal{M}_{t+1}(e_{k_0} R e_{k_0})$ for any fixed $k_0 \in \{1, \dots, t+1\}$ we get that every power $(ea)^k$ is von Neumann regular in eRe for each $k \in \mathbb{N}$.

The element $(1-e)a$ is nilpotent of index $t_2+1 < t+1$ by 2.6. Let $a' = (1-e)a(1-e) \in R_{(2)}$, $h' = (1-e)\hat{h}(1-e) \in R_{(0)}$ and $f' = (1-e)\hat{f}(1-e) \in R_{(-2)}$. Then (a', h', f') is an $\mathfrak{sl}_2$ triple:

$$\begin{aligned}
h' &= (1-e)[a, \hat{f}](1-e) = (1-e)(a\hat{f} - \hat{f}a)(1-e) \\
&= (1-e)a(1-e)\hat{f}(1-e) - (1-e)\hat{f}(1-e)a(1-e) = [a', f'], \\
[h', a'] &= (1-e)\hat{h}(1-e)a(1-e) - (1-e)a(1-e)\hat{h}(1-e) \\
&= (1-e)[\hat{h}, (1-e)a(1-e)](1-e) = 2a', \\
[h', f'] &= (1-e)\hat{h}(1-e)\hat{f}(1-e) - (1-e)\hat{f}(1-e)\hat{h}(1-e) \\
&= (1-e)[\hat{h}, (1-e)\hat{f}(1-e)](1-e) = -2f'.
\end{aligned}$$

With the same argument as above we can show that $(1-e)a$ is nilpotent last-regular of index t_2+1 and there exists an idempotent $e^{(2)}$, $e^{(2)}e = 0 = ee^{(2)}$, such that $e^{(2)}a = ae^{(2)} \in e^{(2)}Re^{(2)} \cong \mathcal{M}_{t_2+1}(S)$ for certain corner S of R, and therefore each power of $e^{(2)}a$ is von Neumann regular in $e^{(2)}Re^{(2)}$. Repeating this process we can find a family of orthogonal idempotents $e^{(1)} = e$, $e^{(2)}, \dots, e^{(m)}$ such that $a = e^{(1)}a + e^{(2)}a + \cdots + e^{(m)}a$ where the $e^{(i)}a's$ are nilpotent of decreasing indices $t_i + 1$ and all the powers of each $e^{(i)}a$ are von Neumann regular in the subalgebra $e^{(i)}Re^{(i)}$. In particular, all the powers a^k of a are von Neumann regular in R. $\square$

In the hypothesis of the last theorem we have two complete families of idempotents: the one arising from the $\mathfrak{sl}_2$-triple (see 2.11) and the one arising from a and all its regular powers (see 3.6). By Lemma 2.10 they coincide.

The following corollary is a generalized version of the well-known Jacobson-Movoroz lemma, which has appeared in the literature with different restrictive hypothesis such as finite-dimension, zero characteristic or index of ad-nilpotence less than or equal to three (see for example [19, Lemma V.8.2]). Jacobson's original proof [15, pag. 99], which refers to an argument of Morozov, holds in a more general context. We combine here Jacobson's original argument, replacing the hypothesis of finite-dimensionality by the ad-nilpotence of the element $\hat{e}$ and relate the torsion of the ring of scalars Φ with the index of ad-nilpotence of e.

First, we need a technical lemma.

Lemma 3.12. *Let R be an associative algebra over a ring of scalars Φ. Let $\hat{e} \in R$ be a nilpotent element of index $t+1$ and suppose that $(2t+1)!$ is invertible in Φ. If $(\hat{e}, \hat{h}, \hat{f})$ is an $\mathfrak{sl}_2$-triple, then $\hat{f}^{t+1} = 0$.*

Proof. Let $(\hat{e}, \hat{h}, \hat{f})$ be an $\mathfrak{sl}_2$-triple. As we have shown in 2.11, there exists a complete family of idempotents $\mathcal{F} = \{f_k := r_k(\hat{h})q_k(\hat{h}) \mid k = -t, \dots, t\}$ in R^1 that induces a finite $\mathbb{Z}$-grading $\Gamma_{\mathcal{F}}$ on R with support from $-2t$ to $2t$.

The invertibility of $(2t+1)!$ in Φ assures that $\hat{f} \in R_{-2}$ in this $\mathbb{Z}$-grading: Indeed, since $\sum f_k = 1$ then $\hat{f} = (\sum_k f_k)\hat{f}(\sum_k f_k) = \sum_{kl} f_k \hat{f} f_l$. On the other hand, from $\hat{h} = \sum_{k=-t}^{t} k f_k$ and $[\hat{h}, \hat{f}] = \hat{h}\hat{f} - \hat{f}\hat{h} = -2\hat{f}$ we get

$$\hat{h}\hat{f} = \sum_{k=-t}^{t} k f_k \hat{f} \qquad \text{and} \qquad \hat{f}\hat{h} = \sum_{k=-t}^{t} k \hat{f} f_k, \text{ so}$$

$$[\hat{h}, \hat{f}] = \sum_{r=-t}^{t} r f_r \hat{f} - \sum_{s=-t}^{t} s \hat{f} f_s = -2 \sum_{rs} f_r \hat{f} f_s.$$

Multiplying on the left by f_i and on the right by f_j $(i, j \in \{0, \pm 1, \dots, \pm t\})$ we get

$$i f_i \hat{f} f_j - j f_i \hat{f} f_j = (i-j) f_i \hat{f} f_j = -2 f_i \hat{f} f_j.$$

If $0 \neq i - j + 2$ we have that $f_i \hat{f} f_j = 0$ because $i - j + 2$ is invertible in Φ, and therefore

$$\hat{f} = \sum_{i-j=-2} f_i \hat{f} f_j \in \bigoplus_{i-j=-2} f_i R f_j = R_{-2}.$$

In particular $(\hat{f})^{t+1} = 0$. $\square$

Corollary 3.13. *Let L be a Lie algebra over a ring of scalars Φ and let $\hat{e} \in L$ with $(\mathrm{ad}_{\hat{e}})^{t+1} = 0$. Suppose that $(t+1)!$ is invertible in Φ and there exists $z \in L$ such that $[[\hat{e}, z], \hat{e}] = 2\hat{e}$. Then there exists $\hat{f} \in L$ such that $(\hat{e}, \hat{h}, \hat{f})$ is an $\mathfrak{sl}_2$-triple of L. Moreover, if $(2t+1)!$ is invertible in Φ, $(\mathrm{ad}_{\hat{f}})^{t+1} = 0$.*

Proof. Let $\hat{h} = [\hat{e}, z]$. In the first part of this proof we will reproduce Jacobson's argument of [15, pag. 99] and show that there exists $\hat{f} \in L$ with $[\hat{e}, \hat{f}] = h$, $[h, \hat{e}] = 2\hat{e}$ and $[h, \hat{f}] = -2\hat{f}$: Let us work in the associative algebra $\operatorname{End} L$, and let us denote by capital letters the adjoint maps of elements of L: $E := \mathrm{ad}_{\hat{e}}$, $H := \mathrm{ad}_h$ and $Z := \mathrm{ad}_z$. Then

$$[E, Z] = [\mathrm{ad}_{\hat{e}}, \mathrm{ad}_z] = \mathrm{ad}_{[\hat{e},z]} = H \text{ and}$$
$$[H, E] = [\mathrm{ad}_{[\hat{e},z]}, \mathrm{ad}_{\hat{e}}] = \mathrm{ad}_{[[\hat{e},z],\hat{e}]} = 2E.$$

For every $i \in \mathbb{N}$

$$[E^i, Z] = i(H - (i-1)\,\mathrm{id})E^{i-1}. \tag{$*$}$$

Indeed,

$$\begin{aligned}[E^i, Z] &= E^{i-1}[E,Z] + E^{i-2}[E,Z]E + \cdots + [E,Z]E^{i-1} \\ &= E^{i-1}H + E^{i-2}HE + \cdots + HE^{i-1} \\ &= iHE^{i-1} - 2((i-1) + (i-2) + \cdots + 1)E^{i-1} \\ &= i(H - (i-1)\,\mathrm{id})E^{i-1}\end{aligned}$$

because $E^k H = HE^k - 2kE^k$ for each $k = 1, \ldots, i-1$. Define the family of Φ-submodules $S_i = \operatorname{Ker} E \cap E^i(L)$, $i = 0, \ldots, t+1$. From the formula $(*)$ we get that $(H - (i-1)\,\mathrm{id})(S_{i-1}) \subset S_i$ if i is invertible in Φ: if $b = E^{i-1}(a) \in \operatorname{Ker} E$ then $(H - (i-1)\,\mathrm{id})(b) \in \operatorname{Ker} E$ and

$$i(H - (i-1)\,\mathrm{id})(b) = i(H - (i-1)\,\mathrm{id})E^{i-1}(a) = [E^i, Z](a) = E^i Z(a) \in E^i(L).$$

Thus

$$\begin{aligned}&(H - t\,\mathrm{id}) \ldots (H - \mathrm{id})H(\operatorname{Ker} E) \subset (H - t\,\mathrm{id}) \ldots (H - \mathrm{id})(S_1) \\ &\subset \cdots \subset (H - t\,\mathrm{id})(S_t) \subset S_{t+1} = 0\end{aligned}$$

because $E_0 = \operatorname{Ker} E$ and $E^{t+1} = 0$. Up to this point we have only needed the invertibility of $(t+1)!$ in Φ. In particular, the map $H : \operatorname{Ker} E \to \operatorname{Ker} E$ has eigenvalues in the set $\{0, 1, \ldots, t\}$ and therefore the map $H + 2\,\mathrm{id} : \operatorname{Ker} E \to \operatorname{Ker} E$ is a Φ-module automorphism. Take any $v \in \operatorname{Ker} E$ such that $(H + 2\,\mathrm{id})(v) = (H + 2\,\mathrm{id})(z)$ and define $\hat{f} = z - v$. Then $[\hat{e}, \hat{f}] = \hat{h}$ and $[\hat{h}, \hat{f}] = -2\hat{f}$.

Let us denote $F := \mathrm{ad}_{\hat{f}}$, so (E, H, F) is an $\mathfrak{sl}_2$-triple of the associative algebra $\operatorname{End} L$ and $E^{t+1} = 0$. By 3.12, since $(2t+1)!$ is invertible in Φ, $F^{t+1} = 0$. $\square$

Remark 3.14. In the particular case $t + 1 = 3$, we recover the well-known result of Seligman [19, Lemma V.8.2], which requires $\frac{1}{2}, \frac{1}{3}, \frac{1}{5} \in \Phi$.

Declaration of competing interest

There is no competing interest.

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