Linear Algebra and its Applications 676 (2023) 44–55

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Decompositions of endomorphisms into a sum of roots of the unity and nilpotent endomorphisms of fixed nilpotence

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A R T I C L E I N F O

Article history:
Received 27 February 2023
Received in revised form 5 July 2023
Accepted 5 July 2023
Available online 10 July 2023
Submitted by P. Semrl

MSC:
15A21
15A24
15B99
16U60

Keywords:
Root of the unity
Nilpotent
Square-zero element

A B S T R A C T

For $n \geq 2$ and fixed $k \geq 1$, we study when an endomorphism f of $\mathbb{F}^n$, where $\mathbb{F}$ is an arbitrary field, can be decomposed as $t + m$ where t is a root of the unity endomorphism and m is a nilpotent endomorphism with $m^k = 0$. For fields of prime characteristic, we show that this decomposition holds as soon as the characteristic polynomial of f is algebraic over its base field and the rank of f is at least $\frac{n}{k}$, and we present several examples that show that the decomposition does not hold in general. Furthermore, we completely solve this decomposition problem for $k = 2$ and nilpotent endomorphisms over arbitrary fields (even over division rings). This somewhat continues our recent publications in Linear Multilinear Algebra (2022) and Int.

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[1] The first author was partially supported by the Bulgarian National Science Fund under Grant KP-06 No. 32/1 of December 07, 2019, as well as by the BIDEB 2221 of TÜBÍTAK.
[2] The second author was partially supported by Ayuda Puente 2022, URJC.
[3] The three authors were partially supported by the Junta de Andalucía FQM264.

https://doi.org/10.1016/j.laa.2023.07.005

Weighted directed graph

J. Algebra Comput. (2022) as well as it strengthens results due to Călugăreanu-Lam in J. Algebra Appl. (2016).

0. Introduction and fundamentals

The decomposition of matrices over an arbitrary field into the sum of some special elements, like nilpotent elements, idempotent elements, potent elements, units, etc., was in the focus of many researchers for a long time (see, e.g., [1], [2], [3], [4], [5], [6], [20], [22] and [23] and the bibliography cited therewith). Specifically, concerning our own work on the subject, in [13] we found some necessary (and sufficient) conditions to assure that any square matrix over a field (finite or infinite) is expressible as a sum of a diagonalizable matrix and a nilpotent matrix of index less than or equal to two. In particular, we also obtained some results on the expression of square matrices into the sum of a potent matrix and a square-zero matrix over finite fields. Nevertheless, such a decomposition does not hold for fields of characteristic zero (see [13, Example 4.3]). Further insight in that matter over some special finite rings was achieved by us in [14]. We also refer the interested reader to [11] for some other aspects of the realization of matrices into the sum of specific elements over certain fields.

By combining the notions of invertibility and nilpotence, Călugăreanu and Lam introduced in 2016 the notion of fine rings [7]: those in which every nonzero element can be written as the sum of an invertible element and a nilpotent one, proving in that work that every nonzero square matrix over a division ring is the sum of an invertible matrix and a nilpotent matrix. The rings whose nonzero idempotents are fine turned out to be an interesting class of indecomposable rings and were studied in [8] by Călugăreanu and Zhou. In 2021, the same authors focused on rings in which every nonzero nilpotent element is fine, which they called NF rings, and showed that for a commutative ring R and $n \geq 2$, the matrix ring $\mathbb{M}_n(R)$ is NF if and only if R is a field; see [9]. A slightly more general class of rings than fine rings was defined in [12] under the name nil-good rings (every element a can be expressed as the sum $a = n + u$ where n is nilpotent and u is either zero or a unit); in [18] it is shown that the matrix ring $\mathbb{M}_n(\Delta)$ over a division ring Δ is nil-good. In general, no restriction on the index of nilpotence is required in these decompositions.

In our work [15] we considered an endomorphism f of an n-dimensional vector space over an arbitrary field $\mathbb{F}$, we fixed a bound k for the index of nilpotence, and studied when f could be expressed as the sum of an automorphism u and a nilpotent endomorphism m with $m^k = 0$. Here we will continue our study in this branch by replacing the invertibility condition on u by being a root of the unity. Recall that a root of the unity endomorphism t is the one for which there is a positive integer s such that t^s is the identity. One elementarily sees that such an endomorphism is necessarily invertible as well as that it is $s+1$-potent, i.e., $t^{s+1} = t$. Canonical forms of roots of the unity endomorphisms were studied by D. Sjerve and a full classification over the rational numbers is presented in his paper [21].

The paper is organized as follows: in the first section we will show that the desired decomposition holds as soon as the characteristic polynomial of f is algebraic over its base field and its rank satisfies a certain bound, and we present several examples that show that the decomposition does not hold in general. In the second section, we focus on nilpotent endomorphisms and deal with the problem of finding a necessary and sufficient condition to decompose such endomorphisms as the sum of a root of the unity and a square-zero endomorphism (fixed nilpotence $k \leq 2$). Since we solve this problem by dealing with the Jordan canonical form of the considered nilpotent endomorphism, our result also holds for nilpotent endomorphisms over division rings.

1. Decomposing endomorphisms into a sum of roots of the unity and endomorphisms of fixed nilpotence

As usual, the letter $\mathbb{F}$ will stand for an arbitrary field unless it is not specified something else. All other notations unexplained explicitly are standard and will be in an agreement with the book [19].

In our paper [15] we showed the following result, which is restated here in terms of endomorphisms instead of matrices. Recall that the rank of an endomorphism of a vector space is the dimension of its image.

Theorem 1.1. *[15, Theorem 2.7] Let $\mathbb{F}$ be a field, consider a vector space V of dimension $n \geq 2$ over $\mathbb{F}$ and let us fix $k \geq 1$. Given a nonzero endomorphism f of V, there exists an automorphism u and a nilpotent endomorphism m with $m^k = 0$ such that $f = u + m$ if and only if the rank of f is greater than or equal to $\frac{n}{k}$.*

Given a vector space V, we say that an endomorphism t of V is a root of the unity if there exists some $s \in \mathbb{N}$ such that t^s is the identity endomorphism. Roots of unity are also called torsion endomorphisms. In this section, we will address the following query:

Problem: *Given a fixed $k \geq 1$, find necessary and sufficient conditions to decompose any non-zero endomorphism of an n-dimensional vector space over a field $\mathbb{F}$ as a sum of a root of the unity and a nilpotent endomorphism m with $m^k = 0$.*

Notice that the proposed Problem is already solved for vector spaces over finite fields by using Theorem 1.1 and the obvious fact that automorphisms over finite fields are always roots of the unity. Nevertheless, the rank condition is not enough to guarantee this decomposition when working over infinite fields. In the rest of this section, we will show some cases when this decomposition holds, and some counterexamples showing that the decomposition does not hold in general.

Remark 1.2. Let f be an endomorphism of an n-dimensional vector space over $\mathbb{F}$. If there exists a nilpotent endomorphism m ($m^k = 0$) such that $t = f - m$ satisfies $t^s = \mathrm{id}$ for some $s \in \mathbb{N}$, then the following three points are fulfilled:

- the trace of f coincides with the trace of t;
- the minimal polynomial of t divides $X^s - 1$ and therefore the eigenvalues of t (in some extension of $\mathbb{F}$) are s-roots of the unity. Moreover, if $X^s - 1$ is separable, t is diagonalizable;
- the trace of t coincides with the sum of its eigenvalues, so it is an algebraic number over $\mathbb{F}$.

For example, an endomorphism f, even of full rank, and whose trace is transcendent over its base field can never be decomposed into the sum $t + m$, where t is a root of the unity and m is nilpotent.

Let $n \geq 2$. Recall that the trace of a polynomial $p(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0 \in \mathbb{F}[x]$ is the scalar $-b_{n-1}$ and coincides with the trace of the companion matrix $C(p(x)) \in \mathbb{M}_n(\mathbb{F})$. Notice that the rank of a companion matrix is always greater than or equal to $n - 1$.

We can now give a partial solution to the Problem proposed above. Concretely, the following statements hold.

Proposition 1.3. *Let $n \geq 2$, let $p(x) \in \mathbb{F}[x]$ be a polynomial of degree n and let $C(p(x)) \in \mathbb{M}_n(\mathbb{F})$ be its companion matrix. If the trace of $p(x)$ can be expressed as the sum of n different roots of the unity in some extension of $\mathbb{F}$, then $C(p(x))$ can be decomposed (in some extension of $\mathbb{F}$) into $T + N$, where T is a root of the unity and $N^2 = 0$. In particular, this always holds if the trace of $p(x)$ is either 1, or -1, or 0.*

Proof. By hypothesis, the trace of $p(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0$ can be expressed as $\alpha_1 + \cdots + \alpha_n$ for some different roots of unity $\alpha_1, \ldots, \alpha_n$ in some extension of $\mathbb{F}$. Let us consider the polynomial $q(x) = (x - \alpha_1) \cdots (x - \alpha_n) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$. Thus, we have

$$C(p(x)) = \begin{pmatrix} 0 & 0 & \ldots & -b_0 \\ 1 & 0 & & \vdots \\ & \ddots & \ddots & \\ 0 & & 1 & -b_{n-1} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & 0 & \dots & -a_0 \\ 1 & 0 & & \vdots \\ & \ddots & \ddots & \\ 0 & & 1 & -a_{n-1} \end{pmatrix}}_{C(q(x))} + \underbrace{\begin{pmatrix} 0 & 0 & \dots & a_0 - b_0 \\ 0 & 0 & & \vdots \\ & \ddots & \ddots & \\ 0 & & 0 & a_{n-1} - b_{n-1} \end{pmatrix}}_{N},$$

where $T := C(q(x))$ is a diagonalizable root of the unity and $N^2 = 0$, because $a_{n-1} - b_{n-1} = 0$, as required. In view of these arguments, the last claim follows now at once. $\square$

As an immediate consequence, we obtain:

Corollary 1.4. *Let f be an endomorphism of an n-dimensional vector space over a field $\mathbb{F}$ and let $p_1(x), \dots, p_k(x)$ be the elementary divisors of f (respectively, the invariant factors of f). If each $p_i(x)$ has degree n_i and its trace is a sum of n_i different roots of the unity in some extension of $\mathbb{F}$, then f can be decomposed (in some extension of $\mathbb{F}$) into $t + m$, where t is a root of the unity and $m^2 = 0$.*

Proof. If $p_1(x), \dots, p_k(x)$ are the elementary divisors (respectively, the invariant factors) of f, then there exists a basis such that the associated matrix A of f is the direct sum of the companion matrices of each $p_i(x)$. Then, utilizing Proposition 1.3, we can express the companion matrix of each $p_i(x)$ as the sum $T_i + N_i$, where T_i is a root of the unity and $N_i^2 = 0$, as needed. $\square$

The next two curious comments are worthwhile.

Remark 1.5. Not every endomorphism of an n-dimensional vector space whose trace is a sum of n roots of the unity can be written as $t + m$, where t is a root of the unity and $m^2 = 0$ (even if those roots are different and f satisfies the rank condition). Indeed, let $n \geq 2$ and let us consider an element $a \in \mathbb{F}$ such that na is a sum of roots of unity, but a itself is not a root of the unity (notice that such an element a always exists and can be easily constructed, so we leave out the details). Then $f = a\,\mathrm{id} \in \mathrm{End}(\mathbb{F}^n)$ cannot be written as a sum $t + m$ where t is a root of the unity and $m^2 = 0$; otherwise there would exist $s \in \mathbb{N}$ such that $t^s = \mathrm{id}$; but then $\mathrm{id} = t^s = (f - m)^s = a^s\,\mathrm{id} - sa^{s-1}m$, so that $(a^s - 1)^2\,\mathrm{id} = ((a^s - 1)\,\mathrm{id})^2 = (sa^{s-1}m)^2 = 0$, a contradiction.

For example, the matrix $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ cannot be expressed as $T + N$ even if its trace is the sum of two (different) 6^{th}-roots of unity: $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

Remark 1.6. Let $\mathbb{F}$ be a field of characteristic 0. When $n \geq 3$, if $p(x)$ is a polynomial of degree n and the trace of $p(x)$ is the sum of n equal roots of unity, then the matrix $C(p(x))$ can never be written as $T + N$ where $T^m = \mathrm{Id}$ for some $m \in \mathbb{N}$ and $N^2 = 0$. Indeed, let α be a root of the unity, and consider a degree n polynomial $p(x)$ whose trace is $n\alpha$. Suppose now that $C(p(x)) = T + N$ where T satisfies $T^m = \mathrm{Id}$ for some $m \in \mathbb{N}$ and $N^2 = 0$. Since the minimal polynomial of T divides $X^m - 1$ and this polynomial has no multiple roots, T is diagonalizable and its eigenvalues are all roots of unity whose sum coincides with the trace of T (which, on the other side, coincides with the trace of $p(x)$), so it is exactly $n\alpha$. The only solution to $n\alpha = \alpha_1 + \cdots + \alpha_n$, α, α_1, …, α_n being roots of unity, is $\alpha = \alpha_1 = \cdots = \alpha_n$. Therefore, the eigenvalues of T are all equal to α and thus $T = \alpha\,\mathrm{Id}$. But then $N = C(p(x)) - \alpha\,\mathrm{Id}$ should have square zero, which is manifestly untrue (the rank of $C(p(x)) - \alpha\,\mathrm{Id}$ is at least $n - 1$).

For example, the matrix

$$C((x-1)^3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{pmatrix}$$

cannot be expressed as $T + N$, where T is a root of the unity matrix and $N^2 = 0$.

Nevertheless, when we focus on fields of prime characteristic, we can partially solve the Problem. Let $\mathbb{F}$ be a field of prime characteristic p and let us denote by $\mathbb{F}_p$ its base field.

Lemma 1.7. *Let $n \geq 2$. If $\mathbb{F}$ is a field of characteristic p and we fix $k \geq 1$, for every endomorphism f of rank greater than or equal to $\frac{n}{k}$ and having an associated matrix $A \in \mathbb{M}_n(\mathbb{F})$ with respect to some basis whose entries are algebraic over $\mathbb{F}_p$ there exists a nilpotent endomorphism m with $m^k = 0$ such that $f - m$ is a root of the unity.*

Proof. The subfield $\mathbb{K}$ of $\mathbb{F}$ generated by the base field $\mathbb{F}_p$ and by the entries of the matrix A is a finite field, and $A \in \mathbb{M}_n(\mathbb{K})$. Apply Theorem 1.1 to decompose A as $U + N$, where $U \in \mathbb{M}_n(\mathbb{K})$ is invertible and $N \in \mathbb{M}_n(\mathbb{K})$ satisfies $N^k = 0$. Since U is an invertible matrix over a finite field, being invertible is equivalent to being a root of the unity matrix, as wanted. □

The conditions on the entries of the matrix can be translated to the coefficients of the characteristic polynomial of the endomorphism f. We will say that a polynomial is algebraic over $\mathbb{F}_p$ if all its coefficients are algebraic over $\mathbb{F}_p$.

Theorem 1.8. *Let $\mathbb{F}$ be a field of characteristic p, let us fix an index of nilpotence $k \geq 1$ and let f be an endomorphism of $\mathbb{F}^n$ of rank greater than or equal to $\frac{n}{k}$. If the characteristic polynomial of f is algebraic over $\mathbb{F}_p$, then f can be written as $t + m$, where t is a root of the unity and $m^k = 0$. In particular, this decomposition always holds for nilpotent endomorphisms of rank greater than or equal to $\frac{n}{k}$.*

Proof. Let us consider the primary rational canonical form C of f, whose characteristic polynomial is algebraic over $\mathbb{F}_p$. The eigenvalues of C (roots in some extension of $\mathbb{F}$ of the characteristic polynomial) are algebraic over $\mathbb{F}_p$ and, therefore, all the elementary divisors of f are algebraic polynomials over $\mathbb{F}_p$. Consequently, the entries of C are all algebraic over $\mathbb{F}_p$ and we can apply Lemma 1.7 to get the proof. $\square$

Open Question: Given a fixed index of nilpotence $k \geq 1$, find a suitable criterion for the decomposition of an arbitrary endomorphism of $\mathbb{F}^n$ where $\mathbb{F}$ is a field of characteristic zero into the sum of a root of the unity and a nilpotent endomorphism of index of nilpotence $\leq k$.

In the following section we will answer this question for $k \leq 2$ and nilpotent endomorphisms of rank at least $\frac{n}{2}$. Since our arguments are quite technical, we leave open the question of decomposing nilpotent endomorphisms of rank at least $\frac{n}{k}$ into roots of the unity and nilpotent endomorphisms of index less than or equal to k.

2. Decomposing nilpotent endomorphisms into a sum of roots of the unity and square-zero endomorphisms

Let V be a left n-dimensional vector-space over an arbitrary division ring Δ. Recall that an endomorphism u of V is nilpotent if $u^k = 0$ for some k. It is well known (see, for example, [10], [16] or [17]) that every nilpotent endomorphism over a division ring is similar to its Jordan form, i.e., it is a direct sum of Jordan cells, all of them associated to the (left) eigenvalue 0.

The main goal of this section is to show that every nilpotent endomorphism u can be written as the sum of a root of the unity in $\mathrm{End}_\Delta(V)$ and a square-zero endomorphism if and only if the rank of u is greater than or equal to $\frac{n}{2}$.

Let us first deal with a particular situation: a nilpotent endomorphism u consisting of one big Jordan cell of size $k > 1$ followed by s Jordan cells of size 1, $0 \leq s \leq k-2$. The proof of the following result, much shorter and clearer than the original one, was suggested by the anonymous referee.

Proposition 2.1. *Let V be an n-dimensional left vector space over a division ring Δ and let us suppose that u is a nilpotent endomorphism of V consisting of one big Jordan cell of size k and s Jordan cells of size 1, with $0 \leq s \leq k-2$. Then $u = v + m$ where $v \in \mathrm{End}_\Delta(V)$ is a root of the unity and $m^2 = 0$.*

Proof. The key idea of the proof consists in representing endomorphisms in a basis by weighted directed graphs, where the vertices of the graph are the elements of the basis, there is at most one arrow from a given vertex to another, and the weights of the arrows are elements of Δ. When an arrow is unlabeled, it is understood that its weight is one. The image of a given vertex v_i by the endomorphism is $\sum \alpha_{ij} v_j$ where v_j are all the vertices with incoming arrows from v_i to v_j and α_{ij} is the weight of such edges.

If u has no Jordan cells of size 1, then v consists of one big Jordan cell of size n and it can be represented by the following directed graph:

$$\begin{tikzcd} e_1 \arrow[r] & e_2 \arrow[r] & \cdots \arrow[r] & e_n \end{tikzcd}$$

Let us consider the square-zero endomorphism m given by $m(e_i) = 0$, $i = 1, \ldots, n-1$, and $m(e_n) = -e_1$. Clearly, $v = u - m$ is a root of the unity (it is a cycle of length n):

$$\begin{tikzcd} e_1 \arrow[r] & e_2 \arrow[r] & \cdots \arrow[r] & e_n \arrow[lll, bend right] \end{tikzcd}$$

Suppose from now on that u has $s \geq 1$ Jordan cells of size 1. Our proof consists of several steps.

1. Let $0 < r \leq n - 2s - 1$ be an arbitrary parameter and define $t = n - 2s - r$. We claim that there exists a basis $\mathcal{B}$ of V

$$\mathcal{B} = \{e_1, \ldots, e_s\} \cup \{f_1, \ldots, f_s\} \cup \{g_1, \ldots, g_r\} \cup \{h_1, \ldots, h_t\}$$

such that u can be represented by the following directed weighted graph:

$$\begin{tikzcd} & & & e_1 \arrow[dr] & e_2 \arrow[dr] & \cdots \arrow[dr] & e_s \arrow[dr] & & & \\ g_1 \arrow[r] & \cdots \arrow[r] & g_r \arrow[r] & f_1 \arrow[r] & f_2 \arrow[r] & \cdots \arrow[r] & f_s \arrow[r] & h_1 \arrow[r] & \cdots \arrow[r] & h_t \end{tikzcd}$$

Indeed, the chain

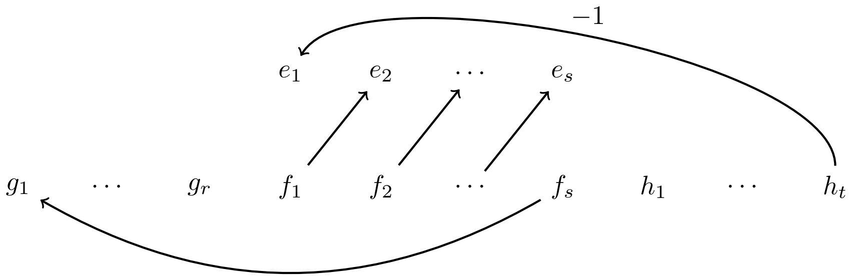
$$\begin{tikzcd} g_1 \arrow[r] & \cdots \arrow[r] & g_r \arrow[r] & f_1 \arrow[r] & f_2 \arrow[r] & \cdots \arrow[r] & f_s \arrow[r] & h_1 \arrow[r] & \cdots \arrow[r] & h_t \end{tikzcd}$$

represents a Jordan cell of size $r + s + t = n - s = k$, while each $\{e_i - f_i\}$, $i = 1, \ldots, s$, corresponds to a Jordan cell of size one (it is easy to see that the image of each $e_i - f_i$ in the graph is always 0). Therefore, seen in the basis

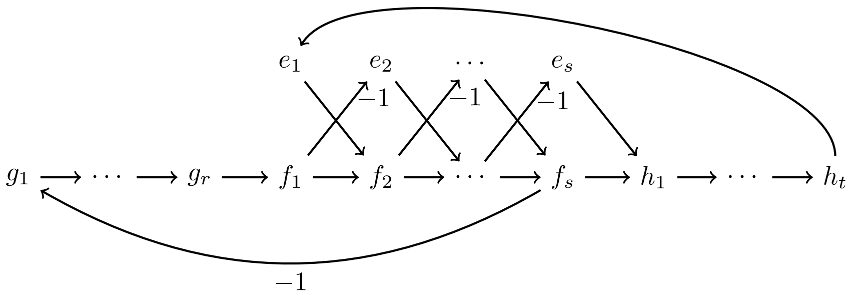
$$\mathcal{B}' = \{g_1, \ldots, g_r\} \cup \{f_1, \ldots, f_s\} \cup \{h_1, \ldots, h_t\} \cup \{e_1 - f_1, \ldots, e_s - f_s\}$$

the previous graph represents an endomorphism consisting of a Jordan cell of size k and s Jordan cells of size 1, i.e., it is similar to the endomorphism u.

2. Let us define in the previous basis $\mathcal{B}$ the endomorphism m represented by the following weighted directed graph:



Since there are no directed paths of length greater than or equal to 2, $m^2 = 0$.
3. Let us consider the endomorphism $v := u - m$, which is represented on the basis $\mathcal{B}$ by the following weighted directed graph:



We claim that we can choose the parameter r so that v is a root of the unity. In order to study the dynamics in this graph, let us isolate three parts in it:

- the *tunnel* is the central part, with vertices $e_1, \dots, e_s, f_1, \dots, f_s$ and the corresponding arrows;
- the *first backloop* is made of $g_1, \dots, g_r$ and the corresponding arrows;
- the *second backloop* is made of $h_1, \dots, h_t$ and the corresponding arrows.

The behavior of the tunnel is the following: for each $i \in \{1, \dots, s-1\}$, v maps $\mathrm{Span}(e_i, f_i)$ to $\mathrm{Span}(e_{i+1}, f_{i+1})$ and it is a cycle of order 6 on the bases $\{e_i, f_i\}$ and $\{e_{i+1}, f_{i+1}\}$. In particular, v maps $\{\pm e_i, \pm f_i, \pm(e_i - f_i)\}$ into $\{\pm e_{i+1}, \pm f_{i+1}, \pm(e_{i+1} - f_{i+1})\}$ and v^{s-1} maps $\{\pm e_1, \pm f_1\}$ into $\{\pm e_s, \pm f_s, \pm(e_s - f_s)\}$, with the signs depending on the congruence of s modulo 6.

There are essentially three cases: $s \equiv 0 \mod 3$, $s \equiv 1 \mod 3$, and $s \equiv 2 \mod 3$ depending on the behavior of $\{\pm e_1, \pm f_1\}$ in the tunnel.

Case 1 ($s \equiv 0 \mod 3$): if we begin with e_1, $v^{s-1}(e_1) = \mu(e_s - f_s)$ (where $\mu = 1$ if $s \equiv 0 \mod 6$, and $\mu = -1$ if $s \equiv 3 \mod 6$); then $v^s(e_1) = \mu g_1$, and following the first backloop we arrive at $v^{s+r}(e_1) = \mu f_1$; going again through the tunnel we get $v^{2s+r}(e_1) = h_1$, and then the vector is routed through the second backloop obtaining $v^{2s+r+t}(e_1) = e_1$. We have obtained a cycle of length n starting with e_1, so v is annihilated by the polynomial $X^n - 1$.

Case 2 ($s \equiv 1 \mod 3$): if we begin with e_1, $v^{s-1}(e_1) = \mu e_s$ (where $\mu = 1$ if $s \equiv 1 \mod 6$, and $\mu = -1$ if $s \equiv 4 \mod 6$), so $v^s(e_1) = \mu h_1$, and following the second backloop we end up with $v^{s+t}(e_1) = \mu e_1$; now we mod out the subspace $\Delta[v]e_1$ and we obtain an endomorphism of the quotient space $V/\Delta[v]e_1$ that, starting with $\overline{f_1}$, produces a cycle whose minimal polynomial is $X^{s+r} + \mu$ (notice that, after going through the tunnel and the first backloop, $v^{s+r}(\overline{f_1}) = -\mu\overline{f_1}$). Then v is annihilated by the polynomial $(X^{s+t} - \mu)(X^{s+r} + \mu)$.

Case 3 ($s \equiv 2 \mod 3$): if we begin with f_1, $v^{s-1}(f_1) = \mu(f_s - e_s)$ (where $\mu = 1$ if $s \equiv 2 \mod 6$, and $\mu = -1$ if $s \equiv 5 \mod 6$); then $v^s(f_1) = -\mu g_1$, and following the first backloop we end up with $v^{s+r}(f_1) = -\mu f_1$; we mod out $\Delta[v]f_1$ and we obtain a cyclic endomorphism beginning in $\overline{e_1}$ in the quotient space $V/\Delta[v]f_1$ with minimal polynomial equal to $X^{s+t} - \mu$ (notice that, after going through the tunnel and the second backloop, we get $v^{s+t}(\overline{e_1}) = \mu\overline{e_1}$). Then v is annihilated by the polynomial $(X^{s+r} + \mu)(X^{s+t} - \mu)$.

When n is even, in Cases 2 and 3 we choose $r = t = \frac{n}{2} - s$ and obtain that an annihilating polynomial of v $(X^{\frac{n}{2}} - 1)(X^{\frac{n}{2}} + 1) = X^n - 1$. Therefore, if we are in Case 1 (for any n) or if we are in Cases 2 or 3 and an even n we obtain an annihilating polynomial of the form $X^n - 1$, which implies that $v^n = \mathrm{id}$.

In general, v is annihilated by polynomial either of the form $X^n - 1$ or $(X^{s+r} + \mu)(X^{s+t} - \mu)$, $\mu = \pm 1$. When the characteristic of Δ is prime, v is an invertible endomorphism that is annihilated by a polynomial whose roots are all roots of the unity, implying that v is a root of the unity itself.

We still need to manage Cases 2 and 3 when the characteristic of Δ is zero and n is odd. The idea is to adequately choose r so that v is annihilated by a polynomial with no multiple roots. Let us distinguish two cases:

- if $n = 4k + 1$ for some $k \geq 0$, we choose
$$r, t \in \{\frac{n-1}{2} - s, \frac{n+1}{2} - s\} = \{2k - s, 2k + 1 - s\}, \quad r \neq t,$$
such that an annihilating polynomial of v is $(X^{2k} + 1)(X^{2k+1} - 1)$;
- if $n = 4k + 3$ for some $k \geq 0$, we choose
$$r, t \in \{\frac{n-1}{2} - s, \frac{n+1}{2} - s\} = \{2k + 1 - s, 2k + 2 - s\}, \quad r \neq t,$$
such that an annihilating polynomial of v is $(X^{2k+2} + 1)(X^{2k+1} - 1)$.

In both cases we have obtained an annihilating polynomial of v with no multiple roots, so v is diagonalizable and all its eigenvalues are (different) roots of the unity hence there exists some $k \in \mathbb{N}$ with $v^k = \mathrm{id}$. $\square$

Now we can prove the main theorem of this section.

Theorem 2.2. *Let V be a left Δ-vector space of dimension n and let u be a nilpotent endomorphism of V. Then u can be written as the sum of a root of the unity in* $\mathrm{End}_\Delta(V)$ *and a square-zero endomorphism if and only if the rank of u is greater than or equal to $\frac{n}{2}$.*

Proof. Since every root of the unity endomorphism is invertible (it has full rank) and square-zero endomorphisms have rank at most 2, the necessity condition is obvious.

In order to prove the sufficiency, let us suppose that, under an appropriate change of basis, u is written in its Jordan form. Since the rank of u is greater than or equal to $\frac{n}{2}$ we can organize its Jordan cells such that its Jordan cells of size 1, if any, always appear after big Jordan cells of bigger size. In particular, we can suppose that each big Jordan cell of size $k > 1$ is followed by s Jordan cells of size 1, $0 \leq s \leq k - 2$. Then the decomposition follows by Proposition 2.1. □

Funding

The first-named author (Peter V. Danchev) was supported in part by the Bulgarian National Science Fund under Grant KP-06 No. 32/1 of December 07, 2019, as well as by the BIDEB 2221 of TÜBÍTAK, the second-named author (Esther García) was partially supported by Ayuda Puente 2022, URJC. The three authors were partially supported by the Junta de Andalucía FQM264.

Declaration of competing interest

None declared.

Data availability

No data was used for the research described in the article.

Acknowledgements

We are deeply indebted to the anonymous referee for the use of weighted directed graphs and their dynamics in the proof of Proposition 2.1. This combinatorial viewpoint has simplified the arguments and made a much clearer and more understandable article.

References

[1] A.N. Abyzov, I.I. Mukhametgaliev, On some matrix analogues of the little Fermat theorem, Mat. Zametki 101 (2) (2017) 163–168.

[2] A.N. Abyzov, D.T. Tapkin, Rings over which every matrices are sums of idempotent and q-potent matrices, Sib. Math. J. 62 (1) (2021) 1–13.

[3] A.N. Abyzov, D.T. Tapkin, When is every matrix over a ring the sum of two tripotents?, Linear Algebra Appl. 630 (2021) 316–325.

[4] S. Breaz, Matrices over finite fields as sums of periodic and nilpotent elements, Linear Algebra Appl. 555 (2018) 92–97.

[5] S. Breaz, G. Călugăreanu, P. Danchev, T. Micu, Nil-clean matrix rings, Linear Algebra Appl. 439 (2013) 3115–3119.

[6] S. Breaz, S. Megiesan, Nonderogatory matrices as sums of idempotent and nilpotent matrices, Linear Algebra Appl. 605 (2020) 239–248.

[7] G. Călugăreanu, T.Y. Lam, Fine rings: a new class of simple rings, J. Algebra Appl. 15 (9) (2016).

[8] G. Călugăreanu, Y. Zhou, Rings with fine idempotents, J. Algebra Appl. 21 (1) (2022) 2250013.

[9] G. Călugăreanu, Y. Zhou, Rings with fine nilpotents, Ann. Univ. Ferrara, Sez. 7: Sci. Mat. 67 (2021) 231–241.

[10] P.M. Cohn, The similarity reduction of matrices over a skew field, Math. Z. 132 (1973) 151–163.

[11] P. Danchev, On some decompositions of matrices over algebraically closed and finite fields, J. Sib. Fed. Univ. Math. Phys. 14 (5) (2021) 547–553.

[12] P.V. Danchev, Nil-good unital rings, Int. J. Algebra 10 (5) (2016) 239–252.

[13] P. Danchev, E. García, M. Gómez Lozano, Decompositions of matrices into diagonalizable and square-zero matrices, Linear Multilinear Algebra 70 (19) (2022) 4056–4070.

[14] P. Danchev, E. García, M.G. Lozano, Decompositions of matrices into potent and square-zero matrices, Int. J. Algebra Comput. 32 (2) (2022) 251–263.

[15] P. Danchev, E. García, M. Gómez Lozano, Decompositions of matrices into a sum of invertible matrices and matrices of fixed nilpotence, arXiv:2302.12751.

[16] E. García, M. Gómez Lozano, R. Muñoz Alcázar, G. Vera de Salas, A Jordan canonical form for nilpotent elements in an arbitrary ring, Linear Algebra Appl. 581 (2019) 1–12.

[17] E. García, M. Gómez Lozano, R. Muñoz Alcázar, G. Vera de Salas, Gradings induced by nilpotent elements, Linear Algebra Appl. 656 (2023) 1–20.

[18] A.B. Gorman, W.Y. Shiao, Nil-good and nil-good clean matrix rings, arXiv:1512.04640v1.

[19] T.W. Hungerford, Algebra, Graduate Texts in Mathematics, vol. 73, Springer-Verlag, New York-Berlin, 1980 (reprint of the 1974 original).

[20] Y. Shitov, The ring $\mathbb{M}_{8k+4}(\mathbb{Z}_2)$ is nil-clean of index four, Indag. Math. 30 (2019) 1077–1078.

[21] D. Sjerve, Canonical forms for torsion matrices, J. Pure Appl. Algebra 22 (1981) 103–111.

[22] J. Šter, On expressing matrices over $\mathbb{Z}_2$ as the sum of an idempotent and a nilpotent, Linear Algebra Appl. 544 (2018) 339–349.

[23] J. Šter, Nil-clean index of $\mathbb{M}_n(\mathbb{F}_2)$, Linear Algebra Appl. 632 (1) (2022) 294–307.