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# AD-NILPOTENT ELEMENTS OF SKEW INDEX IN SEMIPRIME RINGS WITH INVOLUTION

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Abstract. In this paper we study ad-nilpotent elements of semiprime rings $R$ with involution $*$ whose indices of ad-nilpotence differ on $\mathrm{Skew}(R,*)$ and $R$. The existence of such an ad-nilpotent element $a$ implies the existence of a GPI of $R$, and determines a big part of its structure. When moving to the symmetric Martindale ring of quotients $Q_m^s(R)$ of $R$, $a$ remains ad-nilpotent of the original indices in $\mathrm{Skew}(Q_m^s(R),*)$ and $Q_m^s(R)$. There exists an idempotent $e \in Q_m^s(R)$ that orthogonally decomposes $a = ea + (1-e)a$ and either both $ea$ and $(1-e)a$ are ad-nilpotent of the same index (in this case the index of ad-nilpotence of $a$ in $\mathrm{Skew}(Q_m^s(R),*)$ is congruent with 0 modulo 4), or $ea$ and $(1-e)a$ have different indices of ad-nilpotence (in this case the index of ad-nilpotence of $a$ in $\mathrm{Skew}(Q_m^s(R),*)$ is congruent with 3 modulo 4). Furthermore we show that $Q_m^s(R)$ has a finite $\mathbb{Z}$-grading induced by a $*$-complete family of orthogonal idempotents and that $eQ_m^s(R)e$, which contains $ea$, is isomorphic to a ring of matrices over its extended centroid. All this information is used to produce examples of these types of ad-nilpotent elements for any possible index of ad-nilpotence $n$.

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## 1. Introduction

Let $R$ be a semiprime ring, and let $a \in R$. The map $\mathrm{ad}_a : R \to R$ defined by $\mathrm{ad}_a(x) := ax - xa$ is called an inner derivation of $R$. It is a derivation of the Lie algebra $R^{(-)}$ (over its centroid) with bracket product given by $[x, y] := xy - yx$ for every $x, y \in R$. An element $a \in R$ is ad-nilpotent if the map $\mathrm{ad}_a$ is nilpotent. Suppose that $R$ is a semiprime ring with involution $*$ and let $K$ and $H(R,*)$ respectively denote the sets of skew-symmetric and of symmetric elements of $R$. We say that an element $a \in K$ is ad-nilpotent (of $K$) of index $n$ if $\mathrm{ad}_a^n K = 0$ but $\mathrm{ad}_a^{n-1} K \neq 0$. Since the seminal work [15] by Posner, derivations (or some of their generalizations) forcing a prime or semiprime ring to be PI have been broadly studied (see e.g. [13] or [6]). In this paper we focus on the ad-nilpotent elements of a semiprime ring with involution that produce GPIs.

The study of ad-nilpotent elements of the skew-symmetric elements of a prime ring with involution began in 1991 with the work of Martindale and Miers [14].

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Later on, their result was extended to prime associative superalgebras (see [11]) and to semiprime rings with involution (see [5] and [12]).

Martindale and Miers result in the prime setting separates ad-nilpotent elements of $K$ between those which are ad-nilpotent of $R$ of the same index (this may occur when $n \equiv_4 1, 3$) and those that are nilpotent elements and produce a GPI in the central closure of $R$ (this may happen if $n \equiv_4 0, 3$). A similar phenomenon occurs when $R$ is semiprime under the right torsion constraints (see [5, Proposition 3.4 and Theorem 5.6]): for any ad-nilpotent element $a \in R$ there exists a family of orthogonal idempotents $\epsilon_i$ in the extended centroid of $R$ such that $\hat{R} = \bigoplus \epsilon_i \hat{R}$ (with $\hat{R}$ denoting the central closure of $R$), $a = \sum \epsilon_i a$, each $\epsilon_i a$ is ad-nilpotent of index $n_i$ in $K_i = \mathrm{Skew}(\epsilon_i \hat{R}, *)$, and either

(a) $\epsilon_i a$ is ad-nilpotent in the whole $\epsilon_i \hat{R}$ of the same index $n_i$, or
(b) $\epsilon_i a$ is nilpotent of index $[\frac{n+1}{2}] + 1$, the ideal generated by $a^{[\frac{n+1}{2}]}$ is essential in $\epsilon_i \hat{R}$ and the elements of $\epsilon_i \hat{R}$ satisfy certain GPI involving $a^{[\frac{n+1}{2}]}$.

Elements of type (a) occur when $n_i \equiv_4 1, 3$ and will be called ad-nilpotent elements of *full index*. Elements of type (b) occur when $n_i \equiv_4 0, 3$ and will be called elements of *skew index*. Notice that ad-nilpotent elements of skew index are also ad-nilpotent elements of $\epsilon_i \hat{R}$, but the indices of ad-nilpotence in $K_i$ and in $\epsilon_i \hat{R}$ differ. The goal of this paper is to describe ad-nilpotent elements of skew index in semiprime rings and to show how they determine a big part of their structure.

The smallest possible index for an element of skew index is $n = 3$. Ad-nilpotent elements of skew index 3 are called *Clifford elements* because associated to them there is a Jordan algebra of Clifford type (see [9] and [4]). Our paper is a natural generalization of the careful study of Clifford elements carried out in [3] (alternatively, see [8, Section 8.4]): If $R$ is a prime ring with involution and $a \in R$ is a Clifford element then it satisfies $a^3 = 0$, $a^2 \neq 0$ and $a^2ka = aka^2$ for all $k \in K$, $a^2$ and $a$ are von Neumann regular elements and there is an element $b \in H(R, *)$ such that $a^2ba^2 = a^2$, $ba^2b = b$ and $b^2 = 0$ (which also has a square root $\sqrt{b} \in K$, $\sqrt{b}^2 = b$, such that $a\sqrt{b}a = a$, $\sqrt{b}a\sqrt{b} = \sqrt{b}$, which is also a Clifford element). The existence of a Clifford element determines much of the structure of the prime ring: it forces $\mathrm{Skew}(C(R), *) = 0$ for the extended centroid $C(R)$, makes $R$ a GPI ring (so $R$ has socle), and the related $*$-orthogonal idempotents $a^2b, ba^2$ induce a 5-grading on $R$ and a compatible 3-grading on $K$ with $a \in K_1$ (and $\sqrt{b} \in K_{-1}$) with $R_{-2}, R_2$ being isomorphic to $C(R)$ as vector spaces and $K_{-1}, K_1$ being Clifford inner ideals of the Lie algebra $K$ (see [2] for details). We generalize these results to ad-nilpotent elements of skew index.

Since they produce GPIs and we are working with semiprime rings with involution, the best setting to study these elements is the symmetric Martindale ring of quotients $Q_m^s(R)$. Accordingly, we show that these elements remain ad-nilpotent of the same index in $\mathcal{K} := \mathrm{Skew}(Q_m^s(R), *)$ and produce a $*$-complete family of orthogonal idempotents in $Q_m^s(R)$ which induces a grading on $Q_m^s(R)$ compatible with the involution. When restricting the grading to $\mathcal{K}$ we obtain a grading with shorter support. We can consider this result an extension of Smirnov's description of simple graded algebras with involution with $\mathrm{Supp}(K) \neq \mathrm{Supp}(R)$, see [16, Theorem 5.4], which deepens Zelmanov's classification of simple Lie algebras with a $\mathbb{Z}$-grading carried out in [17]. Furthermore we show that, given an ad-nilpotent element of skew index, there is an associated set of matrix units making a related subring isomorphic to a ring of matrices, which produces a clear-cut extension of the relevant properties of Clifford elements.

Our last section is devoted to constructing matrix examples of ad-nilpotent elements both of full index and of skew index of all possible ad-nilpotence indices $n$. We highlight that this section completes the work of Martindale and Miers in [14]. In [14, §4.Examples] Martindale and Miers constructed examples of ad-nilpotent elements of skew index in complex matrices with the transpose involution, and they claimed that they were giving examples for both $n \equiv_4 3$ and $n \equiv_4 0$, covering the possibilities of [14, Main Theorem(2b)], but, as it turns out, they actually addressed the case $n \equiv_4 3$ twice: for each $n \equiv_4 0$ they constructed a skew-symmetric matrix $W$ which, as they showed, satisfies $\mathrm{ad}_W^n(K) = 0$; but it is easily checked that it also satisfies $\mathrm{ad}_W^{n-1}(K) = 0$, so that its index of ad-nilpotence is actually $n-1$, which is congruent to 3 modulo 4.

## 2. Preliminaries

**2.1.** In this paper we will deal with semiprime rings $R$ with involution $*$ such that $\frac{1}{2} \in \Gamma(R)$. Recall that the centroid $\Gamma(R)$ of a ring $R$ is the set of additive maps $\gamma : R \to R$ such that $\gamma(xy) = \gamma(x)y = x\gamma(y)$ for all $x, y \in R$. When $R$ is semiprime (indeed, when $\mathrm{Ann}_R(R) = 0$), $\Gamma(R)$ is a commutative ring with the usual sum and composition of homomorphisms and $R$ becomes an associative algebra over $\Gamma(R)$. If $R$ has an involution $*$, this involution can be extended to $\Gamma(R)$.

If we define the bracket product as $[x, y] := xy - yx$ for every $x, y \in R$, $R$ turns into a Lie algebra over $\Gamma(R)$ denoted by $R^{(-)}$. The set of skew-symmetric elements $\{x \in R \mid x^* = -x\}$, which will be denoted by $K$, is a Lie subalgebra of $R^{(-)}$ over $\mathrm{Sym}(\Gamma(R), *)$.

Given a Lie algebra $L$, we say that $a \in L$ is ad-nilpotent of $L$ of index $n$ if $\mathrm{ad}_a^n L = 0$ and $\mathrm{ad}_a^{n-1} L \neq 0$, where $\mathrm{ad}_a$ denotes the usual adjoint map $\mathrm{ad}_a x := [a, x]$ for every $x \in L$. In [5], a deep study of ad-nilpotent elements in semiprime rings with involution was carried out. Following the classification of ad-nilpotent elements obtained in [5, Proposition 3.4 and Theorem 5.6], we introduce the following definitions:

**2.2.** Let $R$ be a semiprime ring with involution $*$. Let $a \in K$.

We say that $a$ is ad-nilpotent of full index if $a$ is ad-nilpotent of $R$ and of $K$ of the same index $n$. By [5, Theorem 5.6], under the adequate torsion requirements, this occurs when $n \equiv_4 1$ or $n \equiv_4 3$.

We say that $a$ is ad-nilpotent of skew index $n$ if it satisfies all the following conditions:

(i) $a$ is ad-nilpotent of $K$ of index $n$ with $n \equiv_4 0$ or $n \equiv_4 3$,
(ii) $a$ is a nilpotent element of index $t+1$ for $t := \left[\frac{n+1}{2}\right]$ (in particular $t$ is even and $a$ is an ad-nilpotent of $R$ of index $n+1$ or $n+2$),
(iii) $a^t$ generates an essential ideal in $R$,
(iv) and
  (iv.1) if $n \equiv_4 0$, $a^t x a^t = 0$ for every $x \in K$.
  (iv.2) if $n \equiv_4 3$, $a^t x a^{t-1} - a^{t-1} x a^t = 0$ for every $x \in K$.

Notice that under the adequate torsion requirements, this last condition follows from [5, Theorem 5.6].

**2.3.** Given ring $R$, we define a permissible map of $R$ as a pair $(I, f)$ where $I$ is an essential ideal of $R$ and $f$ is a homomorphism of right $R$-modules. For permissible maps $(I, f)$ and $(J, g)$ of $R$, define an equivalence relation $\equiv$ by $(I, f) \equiv (J, g)$ if there exists an essential ideal $M$ of $R$, contained in $I \cap J$, such that $f(x) = g(x)$ for all $x \in M$. The quotient set $Q_m^r(R)$ will be called the right Martindale ring of quotients of $R$. Suppose from now on that $R$ is semiprime. Then we can define an addition and a multiplication in $Q_m^r(R)$ coming respectively from the addition and the composition of homomorphisms (see [1, Chapter 2]):

- $[I, f] + [J, g] := [I \cap J, f + g]$,
- $[I, f] \cdot [J, g] := [(I \cap J)^2, f \circ g]$.

The map $i : R \hookrightarrow Q_m^r(R)$ defined by $i(r) := [R, L_r]$, where $L_r : R \to R$ is the left multiplication map $L_r(x) := rx$, is a monomorphism of rings (called the usual embedding of $R$ into $Q_m^r(R)$), i.e., $R$ can be considered as a subring of its right Martindale ring of quotients. Moreover, given any $0 \neq q := [I, f] \in Q_m^r(R)$ we have that $0 \neq qI \subseteq R$. Therefore every subring $S$ of $Q_m^r(R)$ which contains $R$ is semiprime because every nonzero ideal of $S$ has nonzero intersection with $R$. We also recall the following useful property: for every $q \in Q_m^r(R)$ and every essential ideal $J$ of $R$, $qJ = 0$ or $Jq = 0$ imply $q = 0$.

The symmetric Martindale ring of quotients of $R$ is defined as

$$Q_m^s(R) := \{q \in Q_m^r(R) | \ \exists \text{ an essential ideal } I \text{ of } R \text{ such that } qI + Iq \subseteq R\}.$$

Since $R \subseteq Q_m^s(R) \subseteq Q_m^r(R)$, $Q_m^s(R)$ is again semiprime. When $R$ has an involution, the involution is uniquely extended to $Q_m^s(R)$ ([1, Proposition 2.5.4]).

**2.4.** The extended centroid $C(R)$ of a semiprime ring $R$ is defined as the center of $Q_m^s(R)$. It is commutative and unital von Neumann regular.

The central closure of $R$, denoted by $\hat{R}$, is defined as the subalgebra of $Q_m^s(R)$ generated by $R$ and $C(R)$, i.e., $\hat{R} := C(R) + C(R)R$; so the elements of $R$ can be identified with elements in its central closure. The algebra $\hat{R}$ is semiprime since $R \subseteq \hat{R} \subseteq Q_m^r(R)$, and it is centrally closed, meaning that $\hat{R}$ coincides with its central closure.

Since the extended centroid $C(R)$ of a semiprime $R$ is von Neumann regular, given an element $\lambda \in C(R)$ there exists $\lambda' \in C(R)$ such that $\lambda\lambda'\lambda = \lambda$ and $\lambda' = \lambda'\lambda\lambda'$. Let us define $\epsilon_\lambda := \lambda\lambda'$. Then $\epsilon_\lambda$ is an idempotent of $C(R)$ such that $\epsilon_\lambda\lambda = \lambda$.

Moreover, if $R$ is semiprime with involution $*$ and $\lambda \in \mathrm{Skew}(C(R), *)$, then $-\lambda = \lambda^* = (\lambda\lambda'\lambda)^* = \lambda\lambda'^*\lambda$, which implies that $\lambda'$ can be taken in $\mathrm{Skew}(C(R), *)$ (replace $\lambda'$ by $\frac{1}{2}(\lambda' - \lambda'^*)$). In this case $\epsilon_\lambda = \lambda\lambda' \in H(C(R), *)$ is a symmetric idempotent of $C(R)$.

The following result relates the extended centroid and the center of the local ring at an idempotent element, and can be easily deduced from [1, Corollary 2.3.12].

**Lemma 2.5.** *Let $R$ be a semiprime centrally closed ring and let $e$ be an idempotent of $R$ such that the ideal generated by $e$ in $R$ is essential. Then $C(R) \cong Z(eRe)$.*

*Proof.* The homomorphism $\varphi : C(R) \to Z(eRe)$ defined by $\varphi(\lambda) = \lambda e = e\lambda e$ is an isomorphism: by [1, Corollary 2.3.12], $\varphi$ is surjective; moreover, if $\varphi(\lambda) = 0$, then the ideals $\lambda R$ and $ReR$ are orthogonal, which implies that $\lambda = 0$ because $ReR$ is an essential ideal. $\square$

The following technical lemma, which collects two results about $*$-identities, was proved in [5, Lemma 5.1].

**Lemma 2.6.** *Let $R$ be a semiprime ring with involution $*$ and suppose that $\frac{1}{2}\in\Gamma(R)$. Let $k\in K$ and $h\in H(R,*)$. Then:*

*(1) $hKh=0$ implies $hRh\subseteq H(C(R),*)h$. Moreover, $R$ satisfies $hxhyh=hyhxh$ for every $x,y\in R$, and if $\mathrm{Id}_R(h)$ is essential this identity is a strict GPI in $R$ and $\mathrm{Skew}(C(R),*)=0$.*

*(2) $hKh=0$ and $hKk=0$ imply $hRk=0$.*

*(3) $kKk=0$ implies $k=0$.*

**Corollary 2.7.** *Let $R$ be as in the previous lemma and let $a\in K$ be ad-nilpotent of skew index $n$. Then $t:=\left[\frac{n+1}{2}\right]$ is even, $a$ is ad-nilpotent in $R$ of index $n+1$ or $n+2$, $\mathrm{Skew}(C(R),*)=0$, and $R$ satisfies a strict GPI.*

*Proof.* By the definition 2.2(ii) of element of skew index, $t$ is even (so $a^t\in H(R,*)$) and $a$ is ad-nilpotent in $R$ of index $n+1$ or $n+2$. In addition $a^tKa^t=0$ by 2.2(iv) and $\mathrm{Id}_R(a^t)$ is essential by 2.2(iii), hence Lemma 2.6(1) shows that $\mathrm{Skew}(C(R),*)=0$ and $R$ satisfies the strict GPI $a^txa^tya^t=a^tya^txa^t$ for every $x,y\in R$. □

## 3. Main

**3.1.** Let $R$ be a semiprime ring with $\frac{1}{2}\in\Gamma(R)$. Let $a\in K$ be a nilpotent element of index $t+1$ such that $a^t\in H(R,*)$ is von Neumann regular –as occurs when $a$ is an ad-nilpotent element of skew index, see Theorem 3.5 below. In this situation we can associate a $*$-Rus inverse to $a$, i.e., an element $b\in H(R,*)$ satisfying $a^tba^t=a^t$, $ba^tb=b$ and $ba^sb=0$ for every $s<t$, see [10, Lemma 2.4] and [7, Lemma 3.2] (which works also when $a\in K$). Define $e_{ij}:=a^{i-1}ba^{t+1-j}$, $e_i:=e_{ii}$ for every $i,j=1,\dots,t+1$, and $e:=\sum_{i=1}^{t+1}e_i$. The element $e$ is a symmetric idempotent which we call a $*$-Rus idempotent associated to $a$. It satisfies $ea=ae=\sum_{i=2}^{t+1}e_{i,i-1}$, $ea^t=a^t$ and $eb=b=be$. The set $\{e_{ij}\}_{i,j=1}^{t+1}$ is a set of matrix units for $eRe$. Notice that $e_{\frac{t+2}{2}}\in H(R,*)$ and let $S:=e_{\frac{t+2}{2}}Re_{\frac{t+2}{2}}$. Then the subring $eRe$ and $\mathcal{M}_{t+1}(S)$ are $*$-isomorphic under the isomorphism

$$\Psi:\mathcal{M}_{t+1}(S)\to eRe \text{ defined by } \Psi((x_{ij})_{i,j=1}^{t+1}):=\sum_{i,j=1}^{t+1}e_{i,\frac{t+2}{2}}x_{ij}e_{\frac{t+2}{2},j}$$

where each $x_{ij}=e_{\frac{t+2}{2}}x_{ij}e_{\frac{t+2}{2}}\in e_{\frac{t+2}{2}}Re_{\frac{t+2}{2}}$, and the involution in $\mathcal{M}_{t+1}(S)$ is given by

$$A^*:=D^{-1}\bar{A}^{\mathrm{tr}}D,\quad \text{for every } A=(x_{ij})_{i,j=1}^{t+1}\in\mathcal{M}_{t+1}(S),$$

where $\bar{A}^{\mathrm{tr}}=(y_{ij})_{i,j=1}^{t+1}$ with $y_{ij}=x_{ji}^*$, and where $D=(d_{ij})_{i,j=1}^{t+1}$, with $d_{ij}=(-1)^i$ when $j=t+2-i$ and $d_{ij}=0$ otherwise, satisfies $D=\bar{D}^{\mathrm{tr}}=D^{-1}$.

When considering the following $*$-complete family of orthogonal idempotents

$$\{f_i:=e_{i+1},\ i=0,\dots,t, i\neq\frac{t}{2}\}\cup\{f_{\frac{t}{2}}:=1-e+e_{\frac{t+2}{2}}\}$$

($f_{\frac{t}{2}}$ belonging to the unitization of $R$), which satisfy $f_i^*=f_{t-i}$ for every $i$, we obtain a grading in $R$ which is compatible with the involution:

$$R=R_{-t}\oplus\cdots\oplus R_0\oplus\cdots\oplus R_t$$

where $R_j := \sum_{k-l=j} f_k R f_l$ (notice that $R_j^* = R_j$ for each $j$). With respect to this grading we have

$$ea \in R_1,\ (1-e)a \in R_0,\ a^t \in R_t \text{ and } b \in R_{-t}.$$

This grading is called the grading of $R$ induced by $a$ and its $*$-Rus inverse $b$.

In the above argument, the element $a$ can be replaced by $ea$ without changing the grading in $R$: the element $b = eb$ is also a $*$-Rus inverse for $ea$ and gives rise to the same set of matrix units

$$e_{ij} = a^{i-1}ba^{t+1-j} = a^{i-1}ebea^{t+1-j} = (ea)^{i-1}b(ea)^{t+1-j},$$

so the grading in $R$ induced by $ea$ and its $*$-Rus inverse $b$ coincides with the grading of $R$ induced by $a$ and $b$.

When $a$ is an ad-nilpotent element of $K$ of skew index, the GPIs satisfied in $R$ allow a more precise description of this grading, as we will show in the following theorem.

**Theorem 3.2.** *Let $R$ be a semiprime ring with involution $*$ with $\frac{1}{2} \in \Gamma(R)$, let $K := \mathrm{Skew}(R,*)$ and let $a \in K$ be an ad-nilpotent element of skew index $n$. Let $t := [\frac{n+1}{2}]$ and suppose that $a^t$ is von Neumann regular. Let us consider the grading in $R$*

$$R = R_{-t} \oplus \cdots \oplus R_0 \oplus \cdots \oplus R_t \qquad (\star)$$

*induced by $a$ and its $*$-Rus inverse $b$. Let $e$ be a $*$-Rus idempotent associated to $a$. Then:*

1. *The grading $(\star)$ restricted to $K$ has $K_{-t} = 0 = K_t$.*
2. *$S$ is a semiprime commutative ring with identity involution. In particular, the involution in $eRe \cong \mathcal{M}_{t+1}(S)$ under this isomorphism is given by*
$$A^* = D^{-1}A^{\mathrm{tr}}D \text{ for any } A \in \mathcal{M}_{t+1}(S).$$
3. *As additive groups, both $R_t$ and $R_{-t}$ are isomorphic to $S$.*
4. *If $t > 2$, both $K_{-(t-1)}$ and $K_{t-1}$ are isomorphic to $S$.*

*Moreover, if $R$ is centrally closed, $S \cong C(R)$.*

*Proof.* Since the grading $(\star)$ is compatible with the involution, we can restrict it to $K$,

$$K = K_{-t} \oplus K_{-t+1} \oplus \cdots \oplus K_0 \oplus \cdots \oplus K_{t-1} \oplus K_t.$$

(1) Let us show that $K_{-t} = 0 = K_t$: if $x \in K_{-t} = R_{-t} \cap K$ then $x = f_0 k f_t = e_1 k e_{t+1}$ for some $k \in K$, so $x = ba^t k a^t b \in ba^t K a^t b = 0$. Similarly, if $x \in K_t = R_t \cap K$ then $x = f_t k f_0 = e_{t+1} k e_1$ for some $k \in K$, so $x = a^t bkba^t \in a^t K a^t = 0$.

(2) We claim that $S = e_{\frac{t+2}{2}} R e_{\frac{t+2}{2}}$ does not contain skew-symmetric elements: let $k := \frac{t+2}{2}$; if $x = -x^* \in e_k R e_k$ then $x = e_k x e_k = e_{k,t+1}(e_{t+1,k} x e_{k,1}) e_{1,k}$, but $e_{t+1,k} x e_{k,1} = e_{t+1} e_{t+1,k} x e_{k,1} e_1$ is a skew-symmetric element of $R_t$, so it is zero by (1). Therefore $x = 0$, the involution in $S$ is the identity and hence $S$ is commutative.

(3) Replacing $eRe$ by $\mathcal{M}_{t+1}(S)$, by (2), we have

$$R_t = f_t R f_0 = e_{t+1}(eRe)e_1 = e_{t+1}\mathcal{M}_{t+1}(S)e_1 = e_{t+1,1}\mathcal{M}_{t+1}(S)e_{t+1,1}.$$

Now it is clear that the map $s \to se_{t+1,1}$ is an isomorphism from $S$ to $R_t$ as abelian groups. Analogously for $R_{-t}$.

(4) Since $t > 2$, $R_{-(t-1)} = \sum_{k-l=-(t-1)} f_k R f_l = e_1 R e_t + e_2 R e_{t+1} \subseteq eRe \cong \mathcal{M}_{t+1}(S)$, and under this isomorphism the elements of $R_{-(t-1)}$ are of the form

$$x = \lambda e_{1,t} + \mu e_{2,t+1}, \qquad \lambda, \mu \in S,$$

whence $x = \frac{\lambda+\mu}{2} u + \frac{\lambda-\mu}{2} v$ for $u := e_{1,t} + e_{2,t+1} \in H(R,*)$ and $v := e_{1,t} - e_{2,t+1} \in K$. Therefore $K_{-(t-1)} \subseteq Sv$. A similar argument applies to $K_{t-1}$.

Moreover, if $R$ is centrally closed, by Lemma 2.5, since the ideal of $R$ generated by $e_{\frac{t+2}{2}}$ is essential because it contains $a^t = a^{\frac{t}{2}} e_{\frac{t+2}{2}} a^{\frac{t}{2}}$, we get $S = e_{\frac{t+2}{2}} R e_{\frac{t+2}{2}} = Z(e_{\frac{t+2}{2}} R e_{\frac{t+2}{2}}) \cong C(R)$ as rings. □

The theorem above allows to describe $ea \in eRe \cong \mathcal{M}_{t+1}(S)$ in detail. Now we show how is $a$ related to $ea$.

**Theorem 3.3.** *Let $R$ be a semiprime ring with involution $*$ with $\frac{1}{2} \in \Gamma(R)$ , let $K := \mathrm{Skew}(R,*)$ and let $a \in K$ be an ad-nilpotent element of skew index $n$. Set $t := [\frac{n+1}{2}]$ and suppose that $R$ is free of $\binom{2t-2}{t-1}$-torsion and $a^t$ is von Neumann regular. Then for any $*$-Rus-idempotent $e \in R$ associated to $a$, $a = ea + (1-e)a$ with $ea = ae$, and*

*(1) if $n \equiv_4 0$:*
- *$ea$ is nilpotent of index $t+1$ and ad-nilpotent of skew index $n-1$ in $K$.*
- *$(1-e)a$ is nilpotent of index $t$ and ad-nilpotent of full index $n-1$ in $K$.*

*(2) if $n \equiv_4 3$:*
- *$ea$ is nilpotent of index $t+1$ and ad-nilpotent of skew index $n$ in $K$.*
- *$(1-e)a$ is nilpotent of index $\leq t-1$ and ad-nilpotent in $K$ of index $\leq n-2$.*
- *$ea^{t-1} = a^{t-1}$.*

*Proof.* Let $b \in H(R,*)$ be a $*$-Rus-inverse of $a$ and let $e$ be the associated $*$-Rus idempotent. Since $a$ is nilpotent of index $t+1$ and $(ea)^t = a^t \neq 0$ we have that $ea = ae$ is again nilpotent of index $t+1$.

(1) Suppose that $n \equiv_4 0$. Let us see that $ea$ is ad-nilpotent of index $n-1$ in $K$: for every $k \in K$,

$$\begin{aligned}
\mathrm{ad}_{ea}^{n-1} k = \mathrm{ad}_{ea}^{2t-1} k &= \binom{n-1}{t-1}(ea^{t-1}kea^t - ea^tkea^{t-1}) = \\
&= \binom{n-1}{t-1}(ea^{t-1}ka^t - a^tkea^{t-1}) = \\
&= \binom{n-1}{t-1}((a^tba^{t-1} + a^{t-1}ba^t)ka^t - a^tk(a^tba^{t-1} + a^{t-1}ba^t)) = \\
&= \binom{n-1}{t-1}(a^t(ba^{t-1}k)a^t - a^t(ka^{t-1}b)a^t) = \\
&= \binom{n-1}{t-1}a^t((ba^{t-1}k) - (ba^{t-1}k)^*)a^t = 0
\end{aligned}$$

because $(ba^{t-1}k)^* = ka^{t-1}b$ and $a^tKa^t = 0$. Thus $ea$ is ad-nilpotent of index $\leq n-1$. Let us see that its index of ad-nilpotence is $n-1$. Suppose on the contrary that $\mathrm{ad}_{ea}^{n-2} K = 0$. Then for every $k \in K$,

$$0 = a \cdot \mathrm{ad}_{ea}^{n-2} k = \binom{2t-2}{t} ea^{t-1}ka^t - \binom{2t-2}{t-1} a^t kea^{t-1}.$$

Since $ea^{t-1} = a^t ba^{t-1} + a^{t-1}ba^t$ and $a^t ka^t = 0$ we obtain

$$\binom{2t-2}{t} a^t ba^{t-1}ka^t - \binom{2t-2}{t-1} a^t ka^{t-1}ba^t = 0,$$

and since $a^t x^* a^t = a^t x a^t$ for all $x \in R$ and $(ba^{t-1}k)^* = ka^{t-1}b$ we get

$$\left(\binom{2t-2}{t-1} - \binom{2t-2}{t}\right) a^t ka^{t-1}ba^t = 0.$$

Now, again from $a^t ka^t = 0$ and $ea^{t-1} = a^t ba^{t-1} + a^{t-1}ba^t$, we find

$$\begin{aligned}\left(\binom{2t-2}{t-1} - \binom{2t-2}{t}\right) (a^t ka^{t-1}ba^t + a^t ka^t ba^{t-1}) = \\ = \left(\binom{2t-2}{t-1} - \binom{2t-2}{t}\right) a^t kea^{t-1} = 0.\end{aligned}$$

Since $\binom{2t-2}{t-1} - \binom{2t-2}{t}$ divides $\binom{2t-2}{t-1}$ and $R$ is $\binom{2t-2}{t-1}$-torsion free we have $a^t Kea^{t-1} = 0$, so by Lemma 2.6(2) we get $a^t Rea^{t-1} = 0$ with $a^t$ generating an essential ideal of $R$, and thus $ea^{t-1} = 0$, a contradiction. Thus $ea$ is ad-nilpotent of index $n-1$ in $K$.

Since $ea^t = a^t$, $(1-e)a$ is nilpotent of index less than or equal to $t$. Let us see that its index of nilpotence is $t$. Suppose on the contrary that $ea^{t-1} = a^{t-1}$. Then, for every $k \in K$,

$$\begin{aligned}\mathrm{ad}_a^{n-1} k = \mathrm{ad}_a^{2t-1} k = \binom{2t-1}{t-1} (-1)^t (a^{t-1}ka^t - a^t ka^{t-1}) = \\ = \binom{2t-1}{t-1} (-1)^t (ea^{t-1}ka^t - a^t kea^{t-1}) = \mathrm{ad}_{ea}^{2t-1} k = \mathrm{ad}_{ea}^{n-1} k = 0\end{aligned}$$

would mean that $a$ has index of ad-nilpotence $\leq n-1$ in $K$, a contradiction. Hence $(1-e)a^{t-1} \neq 0$.

Let us see that $(1-e)a$ is ad-nilpotent of index $n-1$: since $(1-e)a^t = 0$ we get that $\mathrm{ad}_{(1-e)a}^{n-1} K = \mathrm{ad}_{(1-e)a}^{2t-1} K = 0$. In addition, $\mathrm{ad}_{(1-e)a}^{n-2} K = \binom{2t-2}{t-1}(1-e)a^{t-1}K(1-e)a^{t-1} \neq 0$ by Lemma 2.6(3). Thus $(1-e)a$ is nilpotent of index $t$ and ad-nilpotent of index $2t-1 = n-1$.

(2) Suppose that $n \equiv_4 3$. Let us see that in this case $ea^{t-1} = a^{t-1}$: for every $k \in K$, using that $a^t ka^t = 0$, $a^{t-1}ka^t = a^t ka^{t-1}$ and $a^t ba^t = a^t$,

$$\begin{aligned}(ea^{t-1} - a^{t-1})ka^t = (a^{t-1}ba^t + a^t ba^{t-1} - a^{t-1})ka^t = a^t ba^{t-1}ka^t - a^{t-1}ka^t = \\ = a^t ba^t ka^{t-1} - a^{t-1}ka^t = a^t ka^{t-1} - a^{t-1}ka^t = 0.\end{aligned}$$

Hence $(ea^{t-1} - a^{t-1})Ka^t = 0$. Since $ea^{t-1} - a^{t-1} \in K$, $a^t \in H(R,*)$, $a^t Ka^t = 0$ and the ideal generated by $a^t$ is essential in $R$, we have by Lemma 2.6(2) that $ea^{t-1} - a^{t-1} = 0$. In particular we get that $(1-e)a$ is nilpotent of index $\leq t-1$. Moreover, since in this case $n-2 = 2t-2$, $\mathrm{ad}_{(1-e)a}^{2t-3} K = 0$, implying that the index of ad-nilpotence of $(1-e)a$ in $K$ must be $\leq n-2$.

Let us see that $ea$ is ad-nilpotent of index $n$: since $n = 2t-1$, $\mathrm{ad}_{ea}^n K = 0$ follows as above. In addition, $\mathrm{ad}_{ea}^{n-1} K = \binom{2t-2}{t-1} ea^{t-1}Kea^{t-1} \neq 0$ by Lemma 2.6(3). So $ea$ is nilpotent of index $t+1$ and ad-nilpotent of index $\leq n$. □

*Remarks* 3.4. Let $e$ be a $*$-Rus idempotent associated to the ad-nilpotent element $a$ of skew index $n$ with $a^t$ von Neumann regular ($t = [\frac{n+1}{2}]$), and consider the grading of $K$ associated to them by Theorem 3.2.

(1) When $a$ is a Clifford element (i.e., $n = 3$) we have $a = ea = a^2ba + aba^2$ by Theorem 3.3(2) (since $t-1 = 1$), and $a \in K_1$ in the grading.
(2) When $n \equiv_4 3$ and $R$ is free of $\binom{2t-2}{t-1}$-torsion we obtain that $a^{t-1}$ is also von Neumann regular: by Theorem 3.3(2) we have $a^{t-1} = ea^{t-1}$, so $a^{t-1} = e_{t,1} + e_{t+1,2} \in eRe \cong \mathcal{M}_{t+1}(S)$ by Theorem 3.2(2) and we get $a^{t-1} = a^{t-1}ca^{t-1}$, $c = ca^{t-1}c$ for $c := e_{1,t} + e_{2,t+1} \in K$. If $t > 2$ then $c^2 = 0$, while when $a$ is Clifford we have $n = 3$, $t = 2$ and $c = e_{1,2} + e_{2,3}$ satisfies $c^2 = e_{1,3} = e_{1,t+1} = b$, so $c$ is a square root of $b$. In addition $c$ is also a Clifford element and $c \in K_{-1}$ in the grading.
(3) Suppose $R$ centrally closed. While when $t > 2$ we have $K_{-(t-1)}, K_{t-1}$ isomorphic to $C(R)$ as additive groups by Theorem 3.2(4), when $t = 2$ they may be larger: since $t = 2$ we have $n \in \{3, 4\}$; in either case, $a' := ea$ is a Clifford element generating the same grading by Theorem 3.3. We can show that $a'Ka' = C(R)a'$ by using $a' = a^2ba + aba^2$, $a^2Ka^2 = 0$ and $a^2xa^2 = \lambda_x a^2$ with $\lambda_x \in C(R)$ for $x \in R$ to show $a'Ka' \subseteq C(R)a'$, and $a'ca' = a'$ with $c \in K$ to show the equality. Then as an abelian group $K_1 = C(R)a' \oplus X$ with $X := \{a^2k + ka^2 \mid k \in K, a'ka' = 0\}$ and analogously for $K_{-1}$ with $c$ in place of $a'$ (see [3, Proposition 4.4 and related results] for the details, which can be easily adapted to our context). The abelian group $X$ can be 0, for example in the ring of $3 \times 3$ matrices over a field (see [3, Remark 4.6(2)]).

The extra hypothesis of $a^t$ being von Neumann regular required in Theorems 3.2 and 3.3 is not too restrictive. When $R$ is a $*$-prime ring, $a^tKa^t = 0$ implies by Lemma 2.6(1) the von Neumann regularity of $a^t$ in the central closure of $R$ and therefore in $Q_m^s(R)$. In general, if $R$ is semiprime we can move to the symmetric Martindale ring of quotients $Q_m^s(R)$ because, as we will show in the following theorem, any ad-nilpotent element $a$ of skew index $n$ is still ad-nilpotent in $\mathcal{K} = \mathrm{Skew}(Q_m^s(R), *)$ of skew index $n$ with $a^t$ von Neumann regular in $Q_m^s(R)$. Although the liftings of GPIs and $*$-GPIs respectively to the maximal right ring of quotients and the Martindale symmetric ring of quotients are well known (see for example [1, Theorems 6.4.1 and 6.4.7]), we will include all the calculations for the sake of completeness.

**Theorem 3.5.** *Let $R$ be a semiprime ring with involution $*$ with $\frac{1}{2} \in \Gamma(R)$. Let $a \in K$ be an ad-nilpotent element of skew index $n$. Let $t := [\frac{n+1}{2}]$, let $Q_m^s(R)$ be the symmetric Martindale ring of quotients of $R$ and denote $\mathcal{K} := \mathrm{Skew}(Q_m^s(R), *)$. Then $a$ is an ad-nilpotent element of skew index $n$ of $\mathcal{K}$, and $a^t$ is von Neumann regular in $Q_m^s(R)$.*

*Proof.* Let us see that $a^t\mathcal{K}a^t = 0$: let $q \in \mathcal{K}$ and let $I$ be an essential ideal of $R$ such that $Iq + qI \subseteq R$. By Lemma 2.6(1) we know that for any $y \in I$ there exists $\lambda_y \in H(C(R), *)$ with $a^tya^t = \lambda_y a^t$. From $a^tKa^t = 0$ we have $a^txa^t = a^tx^*a^t$ for every $x \in R$. Thus

$$a^t y a^t q a^t = a^t (y a^t q)^* a^t = -a^t q a^t y^* a^t = \\ = -a^t q a^t y a^t = -\lambda_y a^t q a^t = -a^t y a^t q a^t$$

so $2a^t y a^t q a^t = 0$ for every $y$ in the essential ideal $I$ of $R$, so $a^t q a^t = 0$.

Suppose now that $n \equiv_4 3$. In this case we will show that not only $a^t \mathcal{K} a^t = 0$ but also $a^t q a^{t-1} = a^{t-1} q a^t$ for every $q \in \mathcal{K}$. Let $q \in \mathcal{K}$ and let $I$ be an essential ideal of $R$ such that $Iq + qI \subseteq R$. From $a^t k a^{t-1} = a^{t-1} k a^t$ for every $k \in K$ and $a^t \mathcal{K} a^t = 0$ we get $a^t q a^t = a^t q^* a^t$ for every $q \in Q_m^s(R)$, whence

$$\begin{aligned} a^t y a^t q a^{t-1} &= a^t (y a^t q - (y a^t q)^*) a^{t-1} + a^t (y a^t q)^* a^{t-1} = \\ &= a^{t-1} (y a^t q - (y a^t q)^*) a^t - a^t q a^t y^* a^{t-1} = \\ &= a^{t-1} (y a^t q - (y a^t q)^*) a^t = -a^{t-1} (y a^t q)^* a^t = \\ &= a^{t-1} q a^t y^* a^t = a^{t-1} q a^t y a^t. \end{aligned}$$

As we know, for any $y \in I$ there is $\lambda_y \in H(C(R), *)$ such that $a^t y a^t = \lambda_y a^t$. Therefore, for every $x \in R$, if we multiply $a^t y a^t q a^{t-1} - a^{t-1} q a^t y a^t = 0$ by $a^t x$ on the left we obtain

$$\begin{aligned} 0 &= a^t x a^t y a^t q a^{t-1} - a^t x a^{t-1} q a^t y a^t = \lambda_y a^t x a^t q a^{t-1} - \lambda_y a^t x a^{t-1} q a^t = \\ &= a^t y a^t x a^t q a^{t-1} - a^t y a^t x a^{t-1} q a^t = a^t y a^t x (a^t q a^{t-1} - a^{t-1} q a^t), \end{aligned}$$

so $a^t q a^{t-1} - a^{t-1} q a^t = 0$ because $a^t I a^t$ generates an essential ideal of $R$.

• If $n \equiv_4 0$, for any $q \in \mathcal{K}$,

$$\mathrm{ad}_a^n(q) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} a^i q a^{n-i} = (-1)^t \binom{n}{t} a^t q a^t = 0.$$

• If $n \equiv_4 3$, for any $q \in \mathcal{K}$,

$$\begin{aligned} \mathrm{ad}_a^n(q) &= (-1)^{t-1} \binom{n}{t-1} a^t q a^{t-1} + (-1)^t \binom{n}{t} a^{t-1} q a^t = \\ &= (-1)^{t-1} \binom{n}{t-1} (a^t q a^{t-1} - a^{t-1} q a^t) = 0. \end{aligned}$$

Moreover, since $a^t$ generates an essential ideal of $R$, it also generates an essential ideal of $Q_m^s(R)$.

Let us see that $a^t$ is von Neumann regular in $Q_m^s(R)$. Since $Q_m^s(R) = Q_m^s(\hat{R})$ we will suppose in the rest of this proof that $R$ is centrally closed. As we know, for every $x \in R$ there exists $\lambda_x \in H(C(R), *)$ such that $a^t x a^t = \lambda_x a^t$. Since $C(R)$ is von Neumann regular there exists $\lambda'_x \in C(R)$ such that $\lambda_x \lambda'_x \lambda_x = \lambda_x$ and $\epsilon_x := \lambda_x \lambda'_x$ is an idempotent of $C(R)$, i.e., for every $x \in R$ we have $a^t \lambda'_x x a^t = \epsilon_x a^t$. Let us consider the family $\{\epsilon_x\}_{x \in R}$ of these idempotents and take a maximal subfamily $\{\epsilon_{x_\gamma}\}_{\gamma \in \Delta}$ of nonzero orthogonal idempotents. Note that for every $\gamma \in \Delta$ there exists $c_{x_\gamma} := \lambda'_{x_\gamma} x_\gamma \in R$ such that $a^t c_{x_\gamma} a^t = \epsilon_{x_\gamma} a^t$.

Let us prove that $I := \sum_{\gamma \in \Delta} \epsilon_{x_\gamma} R$ is an essential ideal of $R$: by [5, Proposition 2.10] there exists an idempotent $\epsilon \in C(R)$ such that $\epsilon\, \epsilon_{x_\gamma} = \epsilon_{x_\gamma}$ for every $\gamma \in \Delta$ and $\mathrm{Ann}_R(I) = (1 - \epsilon)R$. We claim that $\epsilon = 1$; otherwise, if $\epsilon \neq 1$, we can produce a new orthogonal idempotent that does not belong to $\Delta$, which contradicts the maximality of $\Delta$: since $R$ is semiprime and the ideal generated by $a^t$ is essential, $a^t R a^t R (1 - \epsilon) \neq 0$ and for every $x \in R$ such that $0 \neq a^t x a^t R (1 - \epsilon)$ we have

$0 \neq (1-\epsilon)a^t x a^t = (1-\epsilon)\epsilon_x \lambda_x a^t$, i.e., $(1-\epsilon)\epsilon_x$ is a new orthogonal idempotent, a contradiction. Therefore $\epsilon = 1$ and $I$ is an essential ideal of $R$.
Define $c : I \to R$ by $c(\sum_\gamma \epsilon_{x_\gamma} y_\gamma) := \sum_\gamma c_{x_\gamma} y_\gamma$. It is clear that $c$ is a homomorphism of right $R$-modules; moreover, for every $\delta \in \Delta$,

$$L_{\epsilon_{x_\delta}} c(\sum_\gamma \epsilon_{x_\gamma} y_\gamma) = \epsilon_{x_\delta} c_{x_\delta} y_\delta = L_{c_{x_\delta}}(\sum_\gamma \epsilon_{x_\gamma} y_\gamma) \in R,$$

where $L_{\epsilon_{x_\delta}} : R \to R$ and $L_{c_{x_\delta}} : R \to R$ are the corresponding left multiplication maps, so $[R, L_{\epsilon_{x_\delta}}] \cdot [I, c] = [R, L_{c_{x_\delta}}]$, and by the usual embedding of $R$ into $Q^s_m(R)$ we obtain $Iq \subseteq R$ for $q := [I, c]$. Furthermore, since each $\epsilon_{x_\delta}$ lies in $C(R)$, with the same argument we can prove that $qI \subseteq R$. Thus $q \in Q^s_m(R)$.
Finally, for every $\gamma \in \Delta$ we have $\epsilon_{x_\gamma}(a^t q a^t - a^t) = a^t c_{x_\gamma} a^t - \epsilon_{x_\gamma} a^t = 0$ which implies that $a^t q a^t - a^t \in \mathrm{Ann}_R(I) = 0$, i.e., $a^t q a^t = a^t$. □

## 4. Examples

In this section we construct examples of ad-nilpotent elements of full index and of skew index for any possible index of ad-nilpotence.

**4.1.** Let $m$ be a natural number, let $F$ a field of characteristic zero (or big enough) with involution denoted by $\overline{\alpha}$ for any $\alpha \in F$, and denote the simple ring $\mathcal{M}_m(F)$ by $R$ and its standard matrix units by $e_{ij}$, $1 \leq i, j \leq m$. We endow $R$ with the involution $* : R \to R$ given by

$$X^* := D^{-1}\overline{X}^{\mathrm{tr}} D$$

where $D := \sum_{i=1}^m (-1)^i e_{i,m+1-i} \in R$ and $\overline{X}^{\mathrm{tr}} := (\overline{x}_{ji})_{i,j=1}^m$ for $X = (x_{ij})_{i,j=1}^m \in R$. As before, we denote the set of skew-symmetric elements of $R$ with respect to the involution $*$ by $K$. When $m$ is odd (the only case we actually need) we have $D^{-1} = D$ and

$$e^*_{ij} = (-1)^{i+j} e_{m-j+1, m-i+1},$$

and thus $A = (a_{ij})_{i,j=1}^m \in K$ if and only if

$$\overline{a_{ij}} = (-1)^{i+j+1} a_{m-j+1, m-i+1}$$

for all $1 \leq i, j \leq m$; in particular $\overline{a_{i,m-i+1}} = -a_{i,m-i+1}$, so $a_{i,m-i+1} \in \mathrm{Skew}(F, -)$ for all $1 \leq i \leq m$.

**4.2.** <u>Ad-nilpotent elements of full index</u>: Let $R := \mathcal{M}_m(F)$ with the involution $*$ given in 4.1, and let $m$ be odd. As in 4.1, consider

$$A_1 := \sum_{i=1}^{m-1} e_{i,i+1} \in K,$$

which is a nilpotent element of index $m$ and ad-nilpotent of $R$ of index $2m-1$ (see [5, Lemma 4.2]). If the involution $-$ in the field $F$ is not the identity, for any $0 \neq \alpha \in \mathrm{Skew}(F, -)$, the element $0 \neq \alpha e_{m,1}$ is skew-symmetric in $R$, and

$$\mathrm{ad}^{2m-2}_{A_1}(\alpha e_{m,1}) = \binom{2m-2}{m-1} A_1^{m-1} \alpha e_{m,1} A_1^{m-1} = \binom{2m-2}{m-1} \alpha e_{1,m} \neq 0.$$

Thus $A_1$ is an ad-nilpotent element of $K$ (and of $R$) whose index of ad-nilpotence is $n = 2m-1 \equiv_4 1$.

In the same ring $R$, take any $1 < t \leq \frac{m-1}{2}$ and consider the matrix

$$A_2 := \sum_{i=1}^{t-1}(e_{i,i+1} + e_{m-i,m-i+1}) \in K,$$

which is nilpotent of index $t$. The element $A_2$ is ad-nilpotent of $R$ of index $2t-1$ (see [5, Lemma 4.2]). Moreover, $0 \neq B := e_{t,1} + (-1)^t e_{m,m-t+1} \in K$ and $\mathrm{ad}_{A_2}^{2t-2} B \neq 0$. Therefore $A_2$ is ad-nilpotent of $K$ (and of $R$) of index $n = 2t-1$. If $t$ is even then $n \equiv_4 3$, while if $t$ is odd then $n \equiv_4 1$.

**4.3.** Ad-nilpotent elements of skew index: Inspired by Theorem 3.2 we will construct the examples of ad-nilpotent elements of skew index in matrix algebras over fields with identity involution.

• $n \equiv_4 3$: Let $m > 1$ be some odd number. Let us consider $R = \mathcal{M}_m(F)$ where $F$ is a field with identity involution and $R$ is a ring with the involution $*$ given in 4.1. Take any $k$ such that $2k \leq m$. Let us consider the element

$$A_1 := \sum_{i=k}^{m-k} e_{i,i+1} \in K$$

which is nilpotent of index $l = m - 2k + 2$ and ad-nilpotent of $R$ of index $2l-1$ (see [5, Lemma 4.2]). Nevertheless, its index of ad-nilpotence in $K$ is lower: Indeed, any $B = \sum_{i,j=1}^m b_{ij}e_{ij} \in K$ has $b_{k+l-1,k} = 0$ and $b_{k+l-2,k} = b_{k+l-1,k+1}$ by 4.1 since $\mathrm{Skew}(F,-) = 0$, so

$$\begin{aligned}
\mathrm{ad}_{A_1}^{2l-3} B &= \binom{2l-3}{l-2}(A_1^{l-2}BA_1^{l-1} - A_1^{l-1}BA_1^{l-2}) = \\
&= \binom{2l-3}{l-2}(e_{k,k+l-2} + e_{k+1,k+l-1})Be_{k,k+l-1} - \\
&- \binom{2l-3}{l-2}e_{k,k+l-1}B(e_{k,k+l-2} + e_{k+1,k+l-1}) = \\
&= \binom{2l-3}{l-2}(b_{k+l-2,k} - b_{k+l-1,k+1})e_{k,k+l-1} = 0.
\end{aligned}$$

Furthermore, for $C := e_{k+l-2,k} - e^*_{k+l-2,k} = e_{k+l-2,k} + e_{k+l-1,k+1} \in K$ we have $\mathrm{ad}_{A_1}^{2l-4} C \neq 0$, so the index of ad-nilpotence of $A_1$ in $K$ is $2l-3 \equiv_4 3$. For any odd $l$ we have built an ad-nilpotent matrix $A_1$ of index $n := 2l-3 \equiv_4 3$.

• $n \equiv_4 0$: Take any $n \equiv_4 0$. Then $n = 2t$ for some even number $t$. Let $m := 3t+3$. In the ring $R = \mathcal{M}_m(F)$ where $F$ is a field with identity involution and $R$ has the involution $*$ given in 4.1, let us define $A := A_1 + A_2$ where

$$A_1 := \sum_{i=t+2}^{2t+1} e_{i,i+1} \quad \text{and} \quad A_2 := \sum_{i=1}^{t-1}(e_{i,i+1} + e_{m-i,m-i+1}).$$

By construction, $A_1 \in K$ is nilpotent of index $t+1$ and ad-nilpotent of $R$ of index $2t+1$. Moreover, by taking $k = t+2$ this matrix corresponds to the matrix $A_1$ defined in case $n \equiv_4 3$, so it is ad-nilpotent of $K$ of index $2t-1$. Similarly, $A_2 \in K$ is nilpotent of index $t$, and it is ad-nilpotent of $K$ (and of $R$) of index $2t-1$.

The matrix $A$, which is an orthogonal sum of $A_1$ and $A_2$, is nilpotent of index $t+1$ and ad-nilpotent of $R$ of index $2t+1$. Let us see that $\mathrm{ad}_A^{2t} K = 0$: for any $B = \sum_{ij} b_{ij} e_{ij} \in K$ we have

$$\begin{aligned}\mathrm{ad}_A^{2t} B &= \binom{2t}{t} A^t B A^t = \binom{2t}{t} e_{t+2,2t+2} B e_{t+2,2t+2} = \\ &= \binom{2t}{t} b_{2t+2,t+2} e_{t+2,2t+2} = 0\end{aligned}$$

because $b_{2t+2,t+2} \in \mathrm{Skew}(F,-) = 0$. Furthermore, for $C := e_{t,t+2} - e^*_{t,t+2} = e_{t,t+2} - e_{2t+2,2t+4} \in K$ we have $\mathrm{ad}_A^{2t-1} C \neq 0$, so $A$ is ad-nilpotent of $K$ of index $n = 2t \equiv_4 0$.

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## References

[1] K. I. Beidar, W. S. Martindale, III, and A. V. Mikhalev. *Rings with generalized identities*, volume 196 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1996.

[2] Jose Brox, Antonio Fernández López, and Miguel Gómez Lozano. Inner ideals of Lie algebras of skew elements of prime rings with involution. *Proc. Amer. Math. Soc.*, 144(7):2741–2751, 2016.

[3] Jose Brox, Antonio Fernández López, and Miguel Gómez Lozano. Clifford elements in Lie algebras. *J. Lie Theory*, 27(1):283–294, 2017.

[4] Jose Brox, Esther García, and Miguel Gómez Lozano. Jordan algebras at Jordan elements of semiprime rings with involution. *J. Algebra*, 468:155–181, 2016.

[5] Jose Brox, Esther García, Miguel Gómez Lozano, Rubén Muñoz Alcázar, and Guillermo Vera de Salas. A description of ad-nilpotent elements in semiprime rings with involution. *Bull. Malays. Math. Sci. Soc.*, 44(4):2577–2602, 2021.

[6] V. De Filippis, N. Rehman, and M. A. Raza. Strong commutativity preserving skew derivations in semiprime rings. *Bull. Malays. Math. Sci. Soc.*, 41(4):1819–1834, 2018.

[7] A. Fernández López, E. García Rus, M. Gómez Lozano, and M. Siles Molina. Jordan canonical form for finite rank elements in Jordan algebras. *Linear Algebra Appl.*, 260:151–167, 1997.

[8] Antonio Fernández López. *Jordan structures in Lie algebras*, volume 240 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2019.

[9] Antonio Fernández López, Esther García, and Miguel Gómez Lozano. The Jordan algebras of a Lie algebra. *J. Algebra*, 308(1):164–177, 2007.

[10] Esther García, Miguel Gómez Lozano, Rubén Muñoz Alcázar, and Guillermo Vera de Salas. A Jordan canonical form for nilpotent elements in an arbitrary ring. *Linear Algebra Appl.*, 581:324–335, 2019.

[11] Carlos Gómez-Ambrosi, Jesús Laliena, and Ivan P. Shestakov. On the Lie structure of the skew elements of a prime superalgebra with superinvolution. *Comm. Algebra*, 28(7):3277–3291, 2000.

[12] Tsiu-Kwen Lee. Ad-nilpotent elements of semiprime rings with involution. *Canad. Math. Bull.*, 61(2):318–327, 2018.

[13] Cheng-Kai Liu. On skew derivations in semiprime rings. *Algebr. Represent. Theory*, 16(6):1561–1576, 2013.

[14] W. S. Martindale, III and C. Robert Miers. Nilpotent inner derivations of the skew elements of prime rings with involution. *Canad. J. Math.*, 43(5):1045–1054, 1991.

[15] Edward C. Posner. Derivations in prime rings. *Proc. Amer. Math. Soc.*, 8:1093–1100, 1957.

[16] Oleg N. Smirnov. Finite **Z**-gradings of Lie algebras and symplectic involutions. *J. Algebra*, 218(1):246–275, 1999.

[17] E. I. Zelmanov. Lie algebras with finite gradation. *Mat. Sb. (N.S.)*, 124(166)(3):353–392, 1984.

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