Sharp estimates for the Personalized Multiplex PageRank

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Abstract

PageRank can be understood as the stationary distribution of a Markov chain that occurs in a two-layer network with the same set of nodes in both layers: the physical layer and the teleportation layer. In this paper we present some bounds for the extension of this two-layer approach to Multiplex networks, establishing sharp estimates for this Multiplex PageRank and locating the possible values of the personalized PageRank for each node of a network. Several examples are shown to compare the values obtained for both algorithms, the classic and the two-layer PageRank.

Keywords: PageRank, centrality measures, Multiplex networks

1. Introduction

The impressive growth of information available on the web highlights the need for effective web search engines capable to rank pages on the web by assigning authoritative weights to each page. Classic PageRank [18] constitutes a theoretical approach which provides a precise and quantitative measure of the relevance of a webpage giving a direct answer to this need, but it is remarkable that many efforts have been done to improve and modify the performance of web search. New tools, concepts and algorithms have progressively emerged with the advances and new developments of complex networks theory, including the idea of biasing the PageRank vector by using a personalization vector [4, 11, 18]. The development of this science is providing radical new ways of understanding many different mechanisms and processes from physical, social, engineering and

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biological sciences [2, 3, 15]. As a result, a new area is growing up around the concept of Multiplex and multilayer networks [3, 5, 6, 15, 20, 21, 22]. These new paradigms take into account the fact that the interrelations between nodes are heterogeneous [3, 5, 6, 15, 22] and, as a consequence, some structural and dynamical properties and developments emerge from the distinction between different kinds of links. The introduction of the new models of *Multilayer* and *Multiplex* networks requires a revision of all the structural techniques and tools previously developed for (classic) complex network (also called *monoplex* network in this new language), and therefore the centrality measures must be revisited from this new point of view [12, 20, 21, 8]. Roughly speaking, we can say that a Multiplex network is a network formed by several layers (graphs) with the same nodes but different topology inside of each layer. As we have explained, the main idea is that the nodes can interact in different ways and therefore it is needed to consider different layers of interactions. For example, we can imagine the behavior of some people on the social networks WhatsApp, FaceBook and Linkedin to realize that their connections in each one of this networks are radically different. The importance of the study of Multiplex networks is also enhanced by the fact that some authors agree that some key traits of complex systems remain invisible when a multilayer network is considered as if it was a single (monoplex) network [7]. Is thus of big interest to develop analytical tools -similar to those existing in monoplex networks- to analyze the properties of Multiplex networks. Therefore, the centrality measures must be revisited from this new point of view [12, 20, 21] and, in particular, in the new framework represented by Multiplex networks, the PageRank centrality must be rethought. There are some different extensions of the PageRank for Multiplex networks [3, 12, 19, 21]. In [19] a proposal for Multiplex PageRank based on an original approach to the classic PageRank algorithm was introduced together with some theoretical properties of this new centrality measure, and also, an example of application based on the Madrid Metro system in order to illustrate the similarities and differences with the usual concept of PageRank. The key point of this approach is that we can associate two layers to each real layer of the Multiplex and by using this approach it is possible to define a PageRank-like model to the whole Multiplex.

With all this in mind, it is important to highlight that one of the theoretical questions related to Personalized PageRank is to what extend one can use the personalization vector to modify the PageRank vector. In [10] this question was tackled giving some estimates and an analytical characterization of all the possible values of the personalized PageRank for any node. In this paper we present some localization results for the two-layer approach PageRank and the Multiplex PageRank introduced in [19], establishing sharp estimates for the Multiplex PageRank in terms of personalization vector and providing a clear and specific answer to what extent the use of the personalization vector can modify the PageRank vector.

The structure of the paper is as follows. In section 2 the basic definitions and results used in the rest of the paper are presented. In addition, a simplified mathematical formalism is included in order to extend the model for Multiplex networks with any number of layers. Section 3 is devoted to prove the main results of the paper specifically for the PageRank's two-layer approach, by locating the possible values of the personalized PageRank for each node of a network and by illustrating the results with several examples. Finally, section 4 presents a localization theorem for the Multiplex general case. In this final section several examples are presented in order to illustrate the proved results.

2. Notation and some preliminary definitions

We recall some notation from [10] and [19]. Vectors of $\mathbb{R}^{n\times 1}$ will be denoted by column matrices and we will use the superscript T to indicate matrix transposition. The vector of $\mathbb{R}^{n\times 1}$ with all its components equal to 1 will be denoted by $\mathbf{e}$. That is, $\mathbf{e} = (1, \cdots, 1)^T$.

Let $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ be a directed graph where $\mathcal{N} = \{1, 2, \ldots, n\}$ and $n \in \mathbb{N}$. The pair (i, j) belongs to the set $\mathcal{E}$ if and only if there exists a link connecting node i to node j. The *adjacency matrix* of $\mathcal{G}$ is an $n \times n$-matrix

$$A = (a_{ij}) \text{ where } a_{ij} = \begin{cases} 1, & \text{if } (i,j) \text{ is a link of } \mathcal{G} \\ 0, & \text{otherwise.} \end{cases}$$

A link (i, j) is said to be an *outlink* for node i and an *inlink* for node j. We denote $k_{out}(i)$ the *outdegree* of node i, i.e., the number of outlinks of a node i. Notice that $k_{out}(i) = \sum_k a_{ik}$. The graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ may have *dangling nodes*, which are nodes $i \in \mathcal{N}$ with zero outdegree. Dangling nodes are characterized by a vector $\mathbf{d} \in \mathbb{R}^{n\times 1}$ with components d_i defined by

$$d_i = \begin{cases} 1, & \text{if } i \text{ is a dangling node of } \mathcal{G} \\ 0, & \text{otherwise.} \end{cases}$$

Let $P_A = (p_{ij}) \in \mathbb{R}^{n\times n}$ be the *row stochastic matrix* associated to $\mathcal{G}$ defined in the following way:

- if i is a dangling node, $p_{ij} = 0$ for all $j = 1, \ldots, n$,
- otherwise, $p_{ij} = \frac{a_{ij}}{k_{out}(i)} = \frac{a_{ij}}{\sum_k a_{ik}}$.

Note that each coefficient p_{ij} can be considered as the probability of jumping from the node i to the node j.

We recall that one of the features of the personalized PageRank algorithm is that some *extra* probability of jumping is given to any node. This extra or *teleportation* probability is assigned by using a *personalization vector* $\mathbf{v}$, which is a probability distribution vector. If, in addition, the graph has dangling nodes then the algorithm needs to assign an additional probability of jumping to these dangling nodes; this is done by introducing a probability distribution vector $\mathbf{u}$. With these ingredients, plus a teleportation parameter α, we have everything to build a primitive and stochastic matrix, called *Google matrix*, that we denote by G.

Formally, $G = G(\alpha, \mathbf{u}, \mathbf{v})$, with $\alpha \in (0, 1)$, is defined as

$$G = \alpha(P_A + \mathbf{d}\mathbf{u}^T) + (1-\alpha)\mathbf{e}\mathbf{v}^T \in \mathbb{R}^{n\times n} \tag{2.1}$$

Note that G is row-stochastic, i.e., $G\mathbf{e} = \mathbf{e}$. Recall that $\mathbf{v} \in \mathbb{R}^{n\times 1}$, with $\mathbf{v} > 0$ and $\mathbf{v}^T\mathbf{e} = 1$. Analogously, $\mathbf{u} \in \mathbb{R}^{n\times 1}$ such that $\mathbf{u} > 0$ and $\mathbf{u}^T\mathbf{e} = 1$.

The *PageRank vector* $\pi = \pi(\alpha, \mathbf{u}, \mathbf{v})$ is the unique positive eigenvector of G^T associated to eigenvalue 1 such that $\pi^T\mathbf{e} = 1$, i.e., $\pi > 0$, $\pi^T\mathbf{e} = 1$ and $\pi^T G = \pi^T$ (see [18]). Since we focus our interest in $\mathbf{v}$ we also refer to π as the *personalized PageRank vector*.

Note also that from (2.1) we easily have

$$\pi^T = \alpha\pi^T(P_A + \mathbf{d}\mathbf{u}^T) + (1-\alpha)\mathbf{v}^T. \tag{2.2}$$

We will write π_A^T when needed.

If we consider the row stochastic matrix M_A associated to A defined as follows

$$M_A = \begin{pmatrix} \alpha P_A & (1-\alpha)I_n \\ \alpha I_n & (1-\alpha)\mathbf{e}\mathbf{v}^T \end{pmatrix} \in \mathbb{R}^{2n\times 2n} \tag{2.3}$$

we can give the following definition (see [19]):

Definition 2.4. $\hat{\pi}_M \in \mathbb{R}^{2n\times 1}$ is the unique vector that satisfies

(i) $\hat{\pi}_M^T = \hat{\pi}_M^T\, M_A$ with $\hat{\pi}_M^T\mathbf{e} = 1$

(ii) $\hat{\pi}_M^T = [\pi_u^T \quad \pi_d^T]$ with $\pi_u, \pi_d \in \mathbb{R}^{n\times 1}$ and $\pi_u^T\mathbf{e} = \alpha$, $\pi_d^T\mathbf{e} = 1-\alpha$.

Definition 2.5. Given an adjacency matrix A we define the two-layer approach PageRank of A and denote it by $\hat{\pi}_A$ as the vector $\hat{\pi}_A = \pi_u + \pi_d \in \mathbb{R}^{n\times 1}$.

Now, given a Multiplex composed of k layers A_1, A_2, …, A_k with the same number of nodes n, there are a wide range of forms to define a PageRank associated to the Multiplex. Our key idea is based upon the following: we consider each real layer as a two-layer for a random walker: the real layer A_i and a teleportation layer $\mathbf{e}\mathbf{v}_i^T$. We allow teleportation between teleportation layers of different real layers A_i. This is the sense of the following definition.

Definition 2.6. ([19]) Given a Multiplex network $\mathcal{M} = (\mathcal{N}, \mathcal{E}, \mathcal{S})$, with layers $\mathcal{S} = \{\ell_1, \ldots, \ell_k\}$ whose adjacency matrices are $A_1, \ldots, A_k \in \mathbb{R}^{n\times n}$ respectively and we fix some personalized vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^{n\times 1}$, we can define a new block matrix by associating to each layer ℓ_i a two-layer Multiplex as follows

$$\mathbb{M}_{\mathbb{A}} = \frac{1}{k}\begin{pmatrix} \mathbb{M}_{1,1} & \mathbb{M}_{1,2} & \cdots & \mathbb{M}_{1,k} \\ \mathbb{M}_{2,1} & \mathbb{M}_{2,2} & \cdots & \mathbb{M}_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{M}_{k,1} & \mathbb{M}_{k,2} & \cdots & \mathbb{M}_{k,k} \end{pmatrix} \in \mathbb{R}^{2kn\times 2kn}, \tag{2.7}$$

where if $1 \le i \le k$

$$\mathbb{M}_{i,i} = \begin{pmatrix} \alpha P_{A_i} & (1-\alpha)I_n \\ k\alpha I_n & (1-\alpha)\mathbf{e}\mathbf{v}_i^T \end{pmatrix} \in \mathbb{R}^{2n\times 2n} \tag{2.8}$$

corresponds to the connections between the physical layer and the teleportation layer of each ℓ_i, while if $1 \leq i \neq j \leq k$

$$\mathbb{M}_{i,j} = \begin{pmatrix} I_n & 0 \\ 0 & (1-\alpha)\mathbf{e}\mathbf{v}_j^T \end{pmatrix} \in \mathbb{R}^{2n\times 2n} \tag{2.9}$$

corresponds to the cross connections between the physical layer and the teleportation layer of ℓ_i and ℓ_j. From $\mathbb{M}_{\mathbb{A}}$ another block matrix can be defined by reordering the blocks in such a way that all the *physical* layers appear first and later all the *teleportation* layers come together as follows

$$M_k = \frac{1}{k} \begin{pmatrix} \mathbb{B}_{1,1} & \mathbb{B}_{1,2} \\ \mathbb{B}_{2,1} & \mathbb{B}_{2,2} \end{pmatrix} \in \mathbb{R}^{2kn\times 2kn}, \tag{2.10}$$

where

$$\mathbb{B}_{1,1} = \mathbb{P}_A = \begin{pmatrix} \alpha P_{A_1} & I_n & \cdots & I_n \\ I_n & \alpha P_{A_2} & \cdots & I_n \\ \vdots & \vdots & \ddots & \vdots \\ I_n & I_n & \cdots & \alpha P_{A_k} \end{pmatrix}, \tag{2.11}$$

$$\mathbb{B}_{2,2} = (1-\alpha) \begin{pmatrix} \mathbf{e}\mathbf{v}_1^T & \cdots & \mathbf{e}\mathbf{v}_k^T \\ \vdots & \ddots & \vdots \\ \mathbf{e}\mathbf{v}_1^T & \cdots & \mathbf{e}\mathbf{v}_k^T \end{pmatrix}, \tag{2.12}$$

$$\mathbb{B}_{1,2} = (1-\alpha)I_{kn} \in \mathbb{R}^{kn\times kn}, \qquad \mathbb{B}_{2,1} = k\alpha I_{kn} \in \mathbb{R}^{kn\times kn}. \tag{2.13}$$

Note that all the spectral properties of M_k are essentially the same as the corresponding $\mathbb{M}_{\mathbb{A}}$ since they are the same matrices after some block-permutation.

Definition 2.14. ([19]) Let M_k be given by Definition 2.6. $\hat{\pi}_M \in \mathbb{R}^{2kn\times 1}$ is the unique vector that satisfies

(i) $\hat{\pi}_M^T = \hat{\pi}_M^T M_k$ with $\hat{\pi}_M^T \mathbf{e} = k$,

(ii) $\hat{\pi}_M^T = [\pi_{u1}^T \quad \pi_{u2}^T \quad \ldots \quad \pi_{uk}^T \quad \pi_{d1}^T \quad \pi_{d2}^T \quad \ldots \quad \pi_{dk}^T]$ with $\pi_{ui}, \pi_{di} \in \mathbb{R}^{n\times 1}$ $\forall i = 1, 2, \ldots k$, and:

$$\begin{aligned} \pi_{ui}^T\mathbf{e} &= \gamma \\ \pi_{di}^T\mathbf{e} &= 1-\gamma \end{aligned}$$

for all $i = 1, 2, \ldots, k$,

with $\gamma = \frac{k\alpha}{1+\alpha(k-1)}$.

The above result allows us to consider the PageRank associated to a Multiplex in the following form:

Definition 2.15. Given a Multiplex composed of k layers A_1, A_2, …, A_k with the same number of nodes n, we define the PageRank of the Multiplex and denote it by $\hat{\pi}_k$ as the unique vector

$$\hat{\pi}_k = \frac{1}{k}\left(\pi_{u1} + \pi_{u2} + \cdots + \pi_{uk} + \pi_{d1} + \pi_{d2} + \cdots + \pi_{dk}\right) \in \mathbb{R}^{n\times 1}.$$

Note that $\hat{\pi}_k^T\mathbf{e} = 1$.

3. Localization of the two-layer approach PageRank

It is shown in [10] that the i-th component of the classic PageRank, denoted as $\mathcal{PR}(i)$, is located in an open interval that depends on the matrix

$$X = (1-\alpha)Y^{-1} \text{ where } Y = I_n - \alpha(P_A + \mathbf{d}\mathbf{u}^T). \tag{3.1}$$

In more detail, it holds the following result:

Theorem 3.2 ([10]). *Given a graph $\mathcal{G}$ with dangling nodes indicated by some vector $\mathbf{d}$, a fixed damping factor $\alpha \in (0,1)$ and fixed dangling nodes distribution $\mathbf{u}$, for each node $i \in \mathcal{N}$*

$$\mathcal{PR}(i) \in (\min_j x_{ji},\ x_{ii}).$$

Therefore, it is natural to ask if there is a similar relationship applied to the two-layer approach PageRank, $\hat{\pi}_A^T$ given by Definition 2.5. Note that, roughly speaking, the two basic ingredients used in the proof of Theorem 3.2 were *(i)* an analytical well known formula that relates the PageRank with the personalization vector [4] and some localization results based on properties of M-matrices [10].

From here throughout the paper we assume that $\mathbf{d} = \mathbf{0}$, but similar results can be obtained easily when $\mathbf{d} \neq \mathbf{0}$, by replacing P_A by $P_A + \mathbf{d}\mathbf{u}^T$.

The first goal is to give an analytical formula that relates the Multiplex PageRank with the personalization vector, but let us give some tools about Neumann series that will be useful later.

Theorem 3.3 ([16]). *Given a complex $n \times n$ matrix X, then the following statements are equivalent:*

(i) *The Neumann Series $I_n + X + X^2 + \cdots$ converges,*

(ii) *$\rho(X) < 1$, where $\rho(X)$ is the spectral radius of the matrix X,*

(iii) *$\lim_{k\to\infty} X^k = 0$,*

(iv) *$I_n - X$ is nonsingular and*

$$(I_n - X)^{-1} = \sum_{k=0}^{\infty} X^k.$$

Remark 3.4. All the results presented later can be also derived from the original result of C. Neumann [17] about Neumann Series that shows that if $(X, \|\cdot\|)$ is a Banach space and $T : X \longrightarrow X$ is a bounded linear operator with $\|T\| < 1$ ($\|T\|$ is the norm induced by the norm in X), then $I - T$ is nonsingular and

$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k,$$

where I is the identity operator in X, since if we take an $n \times n$ matrix $X = (x_{ij})$ and we consider $(X, \|\cdot\|) = (\mathbb{R}^{n\times 1}, \|\cdot\|_\infty)$, then it is easy to check that the induced norm of X is

$$\|X\| = \max_{1\le i\le n} \left(\sum_{j=1}^{n} |x_{ij}| \right),$$

and therefore if X is row-stochastic then $I_n - sX$ is nonsingular for every $0 \le s < 1$ and

$$(I_n - sX)^{-1} = \sum_{k=0}^{\infty} s^k X^k.$$

By using this tool, we can obtain the following analytical formula that relates the Multiplex PageRank with the personalization vector, which is similar to the well known expression obtained in [4]:

Theorem 3.5. *Given a graph $G = G(\alpha, \mathbf{v})$, $\alpha \in (0,1)$, with personalization vector $\mathbf{v}$, the unique 1-eigenvector $\hat{\pi}_M^T = [\pi_u^T, \pi_d^T]$ with $\hat{\pi}_M^T \mathbf{e} = 1$ associated to the matrix M_A of 2.3 depends on the personalization vector $\mathbf{v}^T$ in the following way*

$$\begin{aligned} \pi_u^T &= \alpha(1-\alpha)^2 \mathbf{v}^T Z^{-1}, \\ \pi_d^T &= (1-\alpha)^2 \mathbf{v}^T Y Z^{-1}, \end{aligned}$$

where $Y = I_n - \alpha P_A$ and $Z = \beta(I_n - \frac{\alpha}{\beta} P_A)$ for $\beta = 1 - \alpha(1-\alpha)$.

Proof. From Definition of $\hat{\pi}_M$ (see Definition 2.4) we have that

$$\hat{\pi}_M^T = [\pi_u^T \quad \pi_d^T] = [\pi_u^T \quad \pi_d^T] \begin{pmatrix} \alpha P_A & (1-\alpha) I_n \\ \alpha I_n & (1-\alpha)\mathbf{e}\mathbf{v}^T \end{pmatrix}.$$

From the first n equations in the previous expression we get that

$$\pi_u^T (I_n - \alpha P_A) = \alpha \pi_d^T.$$

Note that, since P_A is row-stochastic, then $\rho(P_A) = 1$ (see, for example [16]) and therefore $\rho(\alpha P_A) = \alpha < 1$, which makes that $Y = I_n - \alpha P_A$ is nonsingular, by using Theorem 3.3. Hence, π_u^T can be obtained from the last expression as

$$\pi_u^T = \alpha \pi_d^T Y^{-1}. \tag{3.6}$$

By substituting in the last n equations concerning π_d^T, and recalling Definition 2.4, it is easy to check that

$$\pi_d^T = \alpha(1-\alpha)\pi_d^T Y^{-1} + (1-\alpha)^2 \mathbf{v}^T,$$

and by multiplying by Y on both sides

$$\pi_d^T Y = \alpha(1-\alpha)\pi_d^T + (1-\alpha)^2 \mathbf{v}^T Y,$$

which leads us get that

$$\pi_d^T(Y - \alpha(1-\alpha)I_n) = (1-\alpha)^2\mathbf{v}^T Y.$$

If we denote by $Z = Y - \alpha(1-\alpha)I_n = \beta(I_n - \frac{\alpha}{\beta}P_A)$ for $\beta = 1 - \alpha(1-\alpha)$, then $3/4 < \beta < 1$ and $\rho(\frac{\alpha}{\beta}P_A) = \frac{\alpha}{\beta} < 1$ since $0 < \alpha < 1$. Therefore, by using Theorem 3.3, we get that $I_n - \frac{\alpha}{\beta}P_A$ is nonsingular and hence also Z, so

$$\pi_d^T = (1-\alpha)^2\mathbf{v}^T Y Z^{-1}.$$

Finally, by substituting the last formula in (3.6) we find

$$\pi_u^T = \alpha(1-\alpha)^2\mathbf{v}^T Y Z^{-1} Y^{-1} = \alpha(1-\alpha)^2\mathbf{v}^T Z^{-1}, \tag{3.7}$$

since $YZ^{-1} = Z^{-1}Y$. Note that the commutability between Y and Z^{-1} comes from the fact $YZ = ZY$ jointly with the nonsingularity of Z, that leads to $Y = ZYZ^{-1}$ and thus $Z^{-1}Y = YZ^{-1}$.

□

We focus now on the problem of finding the interval in which each component of $\hat{\pi}_A^T$ moves. In order to do so, let us recall that $\hat{\pi}_A^T = \pi_u^T + \pi_d^T$, and take the matrix

$$B = (1-\alpha)^2(YZ^{-1} + \alpha Z^{-1}) = (1-\alpha)^2(Y + \alpha I_n)Z^{-1}, \tag{3.8}$$

where Y and Z were defined in Theorem 3.5.

Theorem 3.9. *Given a graph $\mathcal{G}$ with a fixed damping factor $\alpha \in (0,1)$, for each node $i \in \mathcal{N}$, the i-th component of the PageRank vector $\hat{\pi}_A$ belongs to the interval*

$$(\min_j b_{ji},\ b_{ii}),$$

for the matrix $B = (1-\alpha)^2(YZ^{-1} + \alpha Z^{-1})$ defined above.

Moreover, every b with $\min_j b_{ji} < b < \ b_{ii}$ can be achieved as the PageRank of node i for a certain personalization vector $\mathbf{v}$.

Proof. Theorem 3.5 shows that $\hat{\pi}_A^T = \mathbf{v}^T(1-\alpha)^2(YZ^1 - \alpha Y) = \mathbf{v}^T B$ and since $\mathbf{v}^T$ is a positive vector such that $\mathbf{v}^T\mathbf{e} = 1$, then each of the i-components of $\hat{\pi}_A$ is a convex combination of the i-column of the matrix B. Note that if we use Lemma 2.3 in [10], then we get that B is diagonal dominant by columns, so the values of $(\hat{\pi}_A)_i$ belong to the interval $(\min_j b_{ji},\ b_{ii})$, $i = 1, \ldots, n$ (see, for example [1, 13]).

Without loss of generality suppose that $i = 1$. The first component of $\hat{\pi}_A$ equals $\sum_j v_j b_{j1}$. In particular, if we admitted the canonical basis vector $\mathbf{e}_1^T = (1, 0, \ldots, 0)$ as a personalization vector, $(\hat{\pi}_A)_1 = b_{11}$. Similarly, if the minimum of the first column of B is in position $b_{j_1 1}$ and we admitted the canonical basis vector $\mathbf{e}_{j_1}^T = (0, \ldots, 1, \ldots, 0)$ the j_1-th component of $\hat{\pi}_A$ would be $b_{j_1 1}$, i.e., the extreme values of the open interval

$$(\min_j b_{j1},\ b_{11})$$

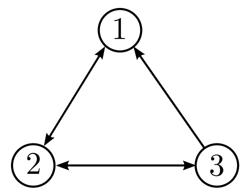

Figure 1: A directed graph with three nodes.

would be achieved.

Now we define

$$\mathbf{v}_{1\varepsilon} = \begin{pmatrix} 1-\varepsilon \\ \frac{\varepsilon}{n-1} \\ \frac{\varepsilon}{n-1} \\ \vdots \\ \frac{\varepsilon}{n-1} \end{pmatrix}, \qquad \mathbf{v}_{j_1\varepsilon} = \begin{pmatrix} \frac{\varepsilon}{n-1} \\ \vdots \\ 1-\varepsilon \\ \vdots \\ \frac{\varepsilon}{n-1} \end{pmatrix} \leftarrow j_1 \text{ coordinate}$$

for every $\varepsilon \in (0,1)$. If we fix $\mathbf{v}_{1\varepsilon}$ as personalization vector, the limit of the first component of the PageRank $\hat{\pi}_A$ when $\varepsilon \to 0^+$ is b_{11}, and if we fix $\mathbf{v}_{j_1\varepsilon}$ as personalization vector, the limit of the j_1-component of the PageRank $\hat{\pi}_A$ when $\varepsilon \to 0^+$ is b_{j_11}. Finally, for every $\lambda \in (0,1)$ if we take $\mathbf{v}_{\lambda\varepsilon} = \lambda\mathbf{v}_{1\varepsilon} + (1-\lambda)\mathbf{v}_{j_1\varepsilon} > 0$ as personalization vector, the first component of $\hat{\pi}_A$ would be $\lambda b_{11} + (1-\lambda)b_{j_11}$, which satisfies that the limit when $\lambda \to 1$ and $\varepsilon \to 0^+$ of the first component of $\hat{\pi}_A$ is b_{11}, and the limit when $\lambda \to 0$ and $\varepsilon \to 0^+$ of the j_1-component of $\hat{\pi}_A$ is b_{j_11}, hence for every b with $b_{j_11} < b < b_{11}$ there exists some $\varepsilon_0, \lambda_0 \in (0,1)$ such that the first component of $\hat{\pi}_A$ is b when $\mathbf{v}_{\lambda_0\varepsilon_0}$ is taken as personalization vector. $\square$

Example 3.10. *Let us consider the graph given in Figure 1 and whose the adjacency matrix is*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Table 1 shows the values of each component of the classic PageRank (denoted by $\mathcal{PR}(i)$) and the two-layer approach PageRank (denoted by $\widehat{PR}(i) \equiv \hat{\pi}_A^T\mathbf{e}_i$) for different personalization vectors.

Notice that the ranking of the nodes given by both methods are the same, for each $\mathbf{v}$ *used. This example was studied in [10] where it is shown that the bounds for the classic PageRank for each node are the following*

$$\mathcal{PR}(1) \in (0.2982, 0.4035),$$
$$\mathcal{PR}(2) \in (0.3872, 0.4925),$$
$$\mathcal{PR}(3) \in (0.1779, 0.3146).$$

	$\mathbf{v} = \mathbf{e}/n$		$\mathbf{v}^T = (1,0,0)$	
Node	$\mathcal{PR}$	$\widehat{PR}$	$\mathcal{PR}$	$\widehat{PR}$
1	0.3333	0.3333	0.4035	0.3596
2	0.4327	0.4401	0.4186	0.4306
3	0.2339	0.2266	0.1779	0.2098

Table 1: Comparison for example 3.10

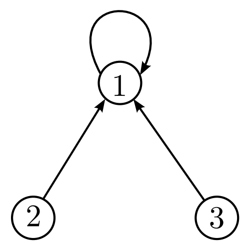

Figure 2: A directed graph with three nodes and a sink.

Now, by using Theorem 3.9 the bounds for the two-layer approach PageRank are

$$\widehat{PR}(1) \in (0.3202, 0.3596),$$
$$\widehat{PR}(2) \in (0.4251, 0.4645),$$
$$\widehat{PR}(3) \in (0.2098, 0.2548),$$

since the matrix $B = (1-\alpha)^2 \left(YZ^{-1} + \alpha Z^{-1}\right)$, *with* $\alpha = 0.85$, *is*

$$B = \begin{pmatrix} 0.3596 & 0.4306 & 0.2098 \\ 0.3202 & 0.4645 & 0.2153 \\ 0.3202 & 0.4251 & 0.2548 \end{pmatrix}.$$

Notice that these intervals are sharp. For example, for $\mathbf{v} = (1,0,0)$ *node* 1 *gets its maximum value and node* 3 *gets its minimum value (see Table 1). Notice also that the interval for the two-layer approach results to be included in the classic interval for each node.*

Example 3.11. *Let us consider now the graph given in Figure 2 and whose the adjacency matrix is*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

In [10] it is shown that for this example the bounds for the classic PageRank are

$$\mathcal{PR}(1) \in (0.8500, 1.0000),$$
$$\mathcal{PR}(2) \in (0.0000, 0.1500),$$
$$\mathcal{PR}(3) \in (0.0000, 0.1500).$$

Now, by using Theorem 3.9 the bounds for the two-layer approach PageRank are

$$\widehat{PR}(1) \in (0.9523, 1.0000),$$
$$\widehat{PR}(2) \in (0.0000, 0.0477),$$
$$\widehat{PR}(3) \in (0.0000, 0.0477),$$

since the matrix $B = (1-\alpha)^2 \left(YZ^{-1} + \alpha Z^{-1}\right)$*, with* $\alpha = 0.85$*, is*

$$B = \begin{pmatrix} 1.0000 & 0.0000 & 0.0000 \\ 0.9523 & 0.0477 & 0.0000 \\ 0.9523 & 0.0000 & 0.0477 \end{pmatrix}.$$

By computing both the classic and the two-layer approach PageRank, by using $\mathbf{v} = (1,0,0)^T$*, we obtain* $\mathcal{PR} = \widehat{PR} = (1,0,0)^T$*. Therefore in this limit case the bounds for the two-layer approach PageRank (and also for the classic PageRank) are reached. Notice that in this example the upper (lower) limit of the interval for node* 1 *(nodes* 2*,*3*) is the same in both methods.*

4. Bounds for the Multiplex PageRank, general case

In a similar manner as we have done in the previous section, we begin by recalling that by Definition 2.14 it is hold that there exists a unique vector

$$\hat{\pi}_M^T = [\pi_{u1}^T, \ldots, \pi_{uk}^T, \pi_{d1}^T, \ldots, \pi_{dk}^T] \in \mathbb{R}^{2kn}$$

such that

$$\hat{\pi}_M^T = \hat{\pi}_M^T \, M_k \tag{4.1}$$

with $\hat{\pi}_M^T \mathbf{e} = k$ and matrix M_k is given by Definition 2.6.

Let us recall, from equation (2.11), the matrix

$$\mathbb{P}_A = \begin{pmatrix} \alpha P_{A_1} & I_n & \cdots & I_n \\ I_n & \alpha P_{A_2} & \cdots & I_n \\ \vdots & \vdots & \ddots & \vdots \\ I_n & I_n & \cdots & \alpha P_{A_k} \end{pmatrix}$$

and define the matrices

$$\tilde{Y} = I_{kn} - \frac{1}{k}\mathbb{P}_A, \qquad \tilde{Z} = \tilde{Y} - \frac{\alpha(1-\alpha)}{k} I_{kn} = \frac{\beta_k}{k}(I_{kn} - \frac{1}{\beta_k}\mathbb{P}_A)$$

for $\beta_k = k - \alpha(1-\alpha)$.

Theorem 4.2. *Given a Multiplex composed of k layers defined by adjacency matrices A_i, $i = 1, 2, \ldots, k$, with the same number of nodes n, fixed damping factor $\alpha \in (0, 1)$ and personalization vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, the unique 1-eigenvector $\hat{\pi}_M^T = [\pi_{u1}^T, \ldots, \pi_{uk}^T, \pi_{d1}^T, \ldots, \pi_{dk}^T]$ with $\hat{\pi}_M^T \mathbf{e} = k$ associated to the matrix M_k in Definition 2.6 depends on the personalization vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in the following way*

$$[\pi_{u1}^T, \ldots, \pi_{uk}^T] = \frac{\alpha(1-\alpha)^2}{1+\alpha(k-1)}[\mathbf{v}_1^T, \ldots, \mathbf{v}_k^T]\tilde{Z}^{-1}$$
$$[\pi_{d1}^T, \ldots, \pi_{dk}^T] = \frac{\alpha(1-\alpha)}{1+\alpha(k-1)}[\mathbf{v}_1^T, \ldots, \mathbf{v}_k^T]\tilde{Y}\tilde{Z}^{-1}.$$

Proof. From the first kn equations of (4.1) we have that

$$[\pi_{u1}^T, \ldots, \pi_{uk}^T]\tilde{Y} = \alpha[\pi_{d1}^T, \ldots, \pi_{dk}^T]. \tag{4.3}$$

Note that every row of the non-negative matrix $\frac{1}{k}\mathbb{P}_A$ sums $\frac{\alpha+k-1}{k}$ and therefore

$$\rho(\frac{1}{k}\mathbb{P}_A) = \frac{\alpha+k-1}{k} < 1,$$

which makes that $\tilde{Y}$ is nonsingular, simply by using Theorem 3.3. Hence, from (4.3) we get that

$$[\pi_{u1}^T, \ldots, \pi_{uk}^T] = \alpha[\pi_{d1}^T, \ldots, \pi_{dk}^T]\tilde{Y}^{-1}.$$

Now, by substituting in the last kn equations of (4.1) concerning $[\pi_{d1}^T, \ldots, \pi_{dk}^T]$ we obtain that

$$[\pi_{d1}^T, \ldots, \pi_{dk}^T] = \frac{\alpha(1-\alpha)}{k}[\pi_{d1}^T, \ldots, \pi_{dk}^T]\tilde{Y}^{-1} + \frac{(1-\alpha)^2}{1+\alpha(k-1)}[\mathbf{v}_1^T, \ldots, \mathbf{v}_k^T],$$

and multiplying by $\tilde{Y}$ on both sides

$$[\pi_{d1}^T, \ldots, \pi_{dk}^T]\tilde{Y} = \frac{\alpha(1-\alpha)}{k}[\pi_{d1}^T, \ldots, \pi_{dk}^T] + \frac{(1-\alpha)^2}{1+\alpha(k-1)}[\mathbf{v}_1^T, \ldots, \mathbf{v}_k^T]\tilde{Y},$$

hence

$$[\pi_{d1}^T, \ldots, \pi_{dk}^T]\left(\tilde{Y} - \frac{\alpha(1-\alpha)}{k}I_{kn}\right) = \frac{(1-\alpha)^2}{1+\alpha(k-1)}[\mathbf{v}_1^T, \ldots, \mathbf{v}_k^T]\tilde{Y}.$$

If we now denote

$$\tilde{Z} = \tilde{Y} - \frac{\alpha(1-\alpha)}{k}I_{kn} = \left(1 - \frac{\alpha(1-\alpha)}{k}\right)\left(I_{kn} - \frac{1}{\beta_k}\mathbb{P}_A\right),$$

then it is easy to check that every row of the non-negative matrix $\frac{1}{\beta_k}\mathbb{P}_A$ sums $\frac{\alpha+k-1}{\beta_k}$ and therefore

$$\rho(\frac{1}{\beta_k}\mathbb{P}_A) = \frac{\alpha+k-1}{\beta_k} < 1,$$

since $0 < \alpha < 1$. Therefore, Theorem 3.3 shows that $\tilde{Z}$ is nonsingular and then

$$[\pi_{d1}^T, \ldots, \pi_{dk}^T] = \frac{(1-\alpha)^2}{1+\alpha(k-1)}[\mathbf{v}_1^T, \ldots, \mathbf{v}_k^T]\tilde{Y}\tilde{Z}^{-1}.$$

Finally, by substituting on $[\pi_{u1}^T, \ldots, \pi_{uk}^T]$ we obtain that

$$\begin{aligned}
[\pi_{u1}^T, \ldots, \pi_{uk}^T] &= \frac{\alpha(1-\alpha)^2}{1+\alpha(k-1)}[\mathbf{v}_1^T, \ldots, \mathbf{v}_k^T]\tilde{Y}\tilde{Z}^{-1}\tilde{Y}^{-1} \\
&= \frac{\alpha(1-\alpha)^2}{1+\alpha(k-1)}[\mathbf{v}_1^T, \ldots, \mathbf{v}_k^T]\tilde{Z}^{-1}
\end{aligned}$$

since $\tilde{Y}\tilde{Z}^{-1} = \tilde{Z}^{-1}\tilde{Y}$. Note that, similarly as in the proof of Theorem 3.5, the commutability between $\tilde{Y}$ and $\tilde{Z}^{-1}$ comes from the fact $\tilde{Y}\tilde{Z} = \tilde{Z}\tilde{Y}$ jointly with the nonsingularity of $\tilde{Z}$.

□

Now we turn to the problem of finding the interval in which each component of the Multiplex PageRank $\hat{\pi}_k$ moves. To that end, let us recall that the PageRank vector of the Multiplex is $\hat{\pi}_k = \frac{1}{k}(\pi_{u1} + \cdots + \pi_{uk} + \pi_{d1} + \cdots + \pi_{dk})$ and take the matrices

$$\tilde{B} = \frac{(1-\alpha)^2}{k(1+\alpha(k-1))}(\tilde{Y}\tilde{Z}^{-1} + \alpha\tilde{Z}^{-1}),$$

and

$$\begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_k \end{pmatrix} = \tilde{B}\begin{pmatrix} I_n \\ I_n \\ \vdots \\ I_n \end{pmatrix}.$$

Theorem 4.4. *Given a Multiplex composed of k layers defined by adjacency matrices A_i, $i = 1, 2, \ldots, k$, with the same number of nodes n, fixed damping factor $\alpha \in (0,1)$ and personalization vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, each i-th component of the PageRank vector $\hat{\pi}_k$ belongs to the interval $(c(i), d(i))$ where*

$$\begin{aligned}
c(i) &= \min_j (C_1)_{ji} + \min_j (C_2)_{ji} + \cdots + \min_j (C_k)_{ji} \\
d(i) &= (C_1)_{ii} + (C_2)_{ii} + \cdots + (C_k)_{ii}
\end{aligned}$$

for the matrices $C_1, \ldots, C_k$ defined above.

Moreover, every b with $c(i) < b < d(i)$ can be achieved as the PageRank of node i for certain personalization vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

Proof. By using the matrices $C_1, \ldots, C_k$ defined above, Theorem 4.2 implies that $\hat{\pi}_k = \mathbf{v}_1^T C_1 + \mathbf{v}_2^T C_2 + \cdots + \mathbf{v}_k^T C_k$. Recall that the personalization vectors are positive vectors such that the sum of their components are one. This implies that the n components of each $\mathbf{v}_i^T C_i$ are convex combinations of the elements of the columns of the matrix C_i, so each of them lies between the maximum of each column and the minumum of each column. Therefore, each i-th component of the PageRank vector $\hat{\pi}_k$ is greater than $\min_j(C_1)_{ji} + \cdots + \min_j(C_k)_{ji}$ and less than $\max_j(C_1)_{ji} + \cdots + \max_j(C_k)_{ji}$.

Notice that the matrices $C_1, \ldots, C_k$ are diagonal dominant by columns as a direct consequence of Lemma 2.3 in [10], so the maximum of each column is achieved in the diagonal position, i.e., the maximum of the first column of C_i lies in $(C_i)_{11}$, the maximum of the second column of C_i is the element of $(C_i)_{22}$, etc.

Without loss of generalization suppose that $i = 1$. We can argue as in Theorem 3.9 in order to show that for every b_1 with $\min_j(C_1)_{j1} < b_1 < (C_1)_{11}$ there exists an appropriate $\tilde{\mathbf{v}}_1$ such that the first component of $\tilde{\mathbf{v}}_1^T C_1$ equals b_1, i.e., $(\tilde{\mathbf{v}}_1^T C_1)_1 = b_1$. Similarly, for every b_2 with $\min_j(C_2)_{j1} < b_2 < (C_2)_{11}$ there exists an appropriate $\tilde{\mathbf{v}}_2$ such that the first component of $\tilde{\mathbf{v}}_2^T C_2$ equals b_2, i.e., $(\tilde{\mathbf{v}}_1^T C_1)_1 = b_2 \ldots$, and for every b_k with $\min_j(C_k)_{j1} < b_k < (C_k)_{11}$ there exists an appropriate $\tilde{\mathbf{v}}_k$ with $(\tilde{\mathbf{v}}_k^T C_k)_1 = b_k$. Since every b with $c(1) < b < d(1)$ is a sum of certain $b_1, b_2, \ldots, b_k$ for $\min_j(C_1)_{j1} < b_1 < (C_1)_{11}$, $\min_j(C_2)_{j1} < b_2 < (C_2)_{11}, \ldots$, and $\min_j(C_k)_{j1} < b_k < (C_k)_{11}$, there exist appropriate personalization vectors $\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \ldots, \tilde{\mathbf{v}}_k$ such that the first component of the PageRank of node 1 equals b. □

Example 4.5. *Let us consider the Multiplex network given by a directed cycle graph of $n = 5$ nodes in $k = 4$ layers, with adjacency matrices*

$$A_1 = A_2 = A_3 = A_4 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then it can be shown that, by using $\mathbf{v} = \mathbf{e}/n$, it results $\hat{\pi}_4 = 0.2\mathbf{e}$, while the interval given by Theorem 4.4 is

$$(0.1885, 0.2156)$$

for any component of the Multiplex PageRank. When using the personalization vectors $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}_3 = \mathbf{v}_4 = [1, 0, 0, 0, 0]^T$, the Multiplex PageRank results to be:

$$[0.2156, 0.2039, 0.1986, 0.1935, 0.1885]$$

and therefore the limits of the interval are reached for node 1 (upper limit) and node 5 (lower limit).

Example 4.6. *Let us consider the case given by a Multiplex network, with $k = 4$, and*

$$A_1 = A_2 = A_3 = A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

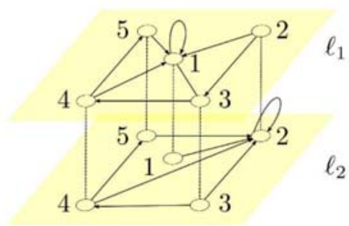

Figure 3: A directed biplex graph with five nodes.

Then it can be shown that, by using $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}_3 = \mathbf{v}_4 = [1,0,0,0,0]^T$, *it results* $\hat{\pi}_4 = [1,0,0,0,0]^T$ *while the bounds given by Theorem 4.4 are*

$$\hat{\pi}_4^T \mathbf{e}_1 \in (0.9680, 1.0000)$$

and

$$\hat{\pi}_4^T \mathbf{e}_i \in (0.0000, 0.0320) \quad \textit{for} \quad i = 2, \dots 5.$$

Therefore, once again, the first component of the Multiplex PageRank achieves the maximum value indicated by the sharp upper bound and the rest of the nodes achieve its minimum value.

Example 4.7. *Let us consider the biplex network given in Figure 3 and whose the adjacency matrices are*

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

By the structure of the layers it is clear that the most central node (as seen by PageRank) in layer 1 *is node* 1 *and the most central node in layer* 2 *is node* 2*. Actually nodes* 1 *and* 2 *interchange their roles. Therefore it is expected that the bounds for the whole biplex be equal for nodes* 1 *and* 2*. In fact, by using the bounds in Theorem 4.4 we confirm this in obtaining the following bounds for the Multiplex PageRank*

$$\begin{aligned} \hat{\pi}_2^T \mathbf{e}_1 &\in (0.3636, 0.4103), \\ \hat{\pi}_2^T \mathbf{e}_2 &\in (0.3636, 0.4103), \\ \hat{\pi}_2^T \mathbf{e}_3 &\in (0.1246, 0.1615), \\ \hat{\pi}_2^T \mathbf{e}_4 &\in (0.0615, 0.0984), \\ \hat{\pi}_2^T \mathbf{e}_5 &\in (0.0308, 0.0676), \end{aligned}$$

and, as a final illustration of the applicability of our result, we show the value of the Multiplex PageRank for some personalization vectors in Table 2.

Node	$\mathbf{v} = \mathbf{e}/n$	$\mathbf{v} = \mathbf{e}_1$	$\mathbf{v} = \mathbf{e}_2$
1	0.3758	0.4103	0.3636
2	0.3758	0.3636	0.4103
3	0.1349	0.1311	0.1311
4	0.0721	0.0638	0.0638
5	0.0414	0.0311	0.0311

Table 2: Multiplex PageRank for some $\mathbf{v}$. Example 4.7

Acknowledgements

This work has been partially supported by the projects MTM2014-59906-P, MTM2014-52470-P (Spanish Ministry) and the grant *URJC-Grupo de Excelencia Investigadora* GARECOM (2014-2016). The authors would like to thank an anonymous referee for his/her valuable comments and remarks that have improved the readability of the manuscript.

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