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Noise induced aperiodic rotations of particles trapped by a non-conservative force

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We describe a mechanism whereby random noise can play a constructive role in the manifestation of a pattern, aperiodic rotations, that would otherwise be damped by internal dynamics. The mechanism is described physically in a theoretical model of overdamped particle motion in two dimensions with symmetric damping and a non-conservative force field driven by noise. Cyclic motion only occurs as a result of stochastic noise in this system. However, the persistence of the cyclic motion is quantified by parameters associated with the non-conservative forcing. Unlike stochastic resonance or coherence resonance, where noise can play a constructive role in amplifying a signal that is otherwise below the threshold for detection, in the mechanism considered here, the signal that is detected does not exist without the noise. Moreover, the system described here is a linear system. *Published by AIP Publishing.* https://doi.org/10.1063/1.5018443

In recent years, there have been a handful of model systems that have been proposed where the presence of noise can play a constructive role, amplifying coherent signals or giving rise to motions that can perform work. Here, we have introduced another model system in which noise plays a constructive role. The model system is a linear overdamped system, and in the absence of noise, the system approaches a steady state. However, when noise is applied to this system, over very long times, a winding motion becomes established with a well-defined average number of windings over a given time interval. We have provided the mathematical analysis to understand this behaviour. Similar behaviour has been observed in the motion of particles trapped by highly focussed laser beams.

I. INTRODUCTION

In a very general sense, random noise typically degrades the quality of patterns. An exception is stochastic resonance[1] which occurs when a weak coherent signal is amplified by moderate levels of noise. In this case, noise may play a constructive role. A signal which is below the threshold for detection in the absence of noise may become above the threshold for detection in the presence of moderate levels of noise. The conventional setup for stochastic resonance involves a periodic forcing coupled with noise, and the signal is a periodic response. A different but related phenomenon is the so-called coherence resonance in which the auto-correlations, or coherence, may become more pronounced by increasing the noise to moderate levels.[2] More recently, it has been shown that noise can not only amplify signals but may actually be used to perform work. The canonical model for this has been described as Brownian vortexes.[3,4] In these systems, noise coupled to a non-conservative force field gives rise to steadily circulating currents that do not exist without the noise.[3–5] Noise-induced circulation, manifesting as cyclic probability currents, has been observed in optical tweezers experiments[5] and in simulations of stochastic population dynamics with two species.[6] In each of these examples, the deterministic component of the dynamical system is nonlinear.

Here, we consider a simple two-dimensional linear dynamical system in which noise is required for the pattern to appear. The dynamical system describes overdamped particle motion in a non-conservative restoring force field under the influence of Gaussian noise. In the absence of noise, the particle is simply attracted to an origin. The noise gives rise to two different types of signals: oscillations, which could be observed as a non-zero peak in the power spectrum, and a persistent aperiodic cyclic motion that does not manifest as a peak in power spectrum analysis.

Aperiodic cyclic motion has been observed in simulations of a model equation for a nanowire trapped by optical tweezers in a non-conservative force field.[7] It was found that the nanowire does not exhibit cyclic motion if a conservative force is considered, implying that the response is a consequence of a non-conservative nature of the optical force.[7] The analysis in Ref. 7 suggested that the cyclic motion could be expected for non-symmetric particles with linear restoring forces. The phenomenon that we describe here is qualitatively similar to that proposed in Ref. 7, and we have reinforced the conclusions of this paper by providing mathematical results for the mean velocity of the cyclic motion as a function of the system parameters. Noise induced periodic motions in a theoretical model for a nanowire trapped by optical tweezers were reported in Ref. 8. Noise induced periodic motions and aperiodic motions have been experimentally observed in the motion of optically trapped nanowires.[8,9]

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The remainder of this paper is organized as follows: In Sec. II, we introduce the model of a two-dimensional overdamped dynamical system with noise, described by a Langevin equation. In Sec. III, we carry out a change in variables in order to facilitate the calculations of the power spectrum for the oscillations, as well as the mean phase velocity of the aperiodic motion. In Sec. IV, parameters are chosen to characterise the simplified system. The power spectrum and dominant oscillation frequency are obtained in Sec. V. In Sec. VI, we derive the expression for the mean phase velocity of the aperiodic motion. In Sec. VIII, we carry out numerical simulations of the Langevin equations for a range of parameters in the model and we compare theoretical predictions with the simulations. We conclude with a summary and a discussion of more general results in Sec. IX.

II. DYNAMICAL MODEL

In physical terms, the model that we consider describes the overdamped dynamics of a system with two degrees of freedom in the vicinity of a stable equilibrium position when it is subject to an attractive force, viscous drag, and noise. The model equations are approximated to the first order by the Langevin equation

$$-\Gamma\dot{\mathbf{q}} - K\mathbf{q} + \mathbf{f}(t) = \mathbf{0}. \tag{1}$$

In this equation, $\mathbf{q} = [q_1(t), q_2(t)]$ are the dynamical variables, $-K\mathbf{q}$ is an external restoring force, characterised by the stiffness matrix K, $-\Gamma\dot{\mathbf{q}}$ is the generalized viscous drag force, characterised by the viscous drag coefficient matrix Γ, and $\mathbf{f}(t)$ is the time-uncorrelated white noise.

The stiffness coefficient matrix K is symmetric if the force is conservative and non-symmetric otherwise. In the following, we assume that the viscous drag coefficient matrix Γ is symmetric, positive definite, and, thus, invertible. The noise is assumed to have a spatial correlation function given by the Stokes-Einstein equation[10]

$$\langle \mathbf{f}(t)\rangle = 0, \quad \langle \mathbf{f}(t)\mathbf{f}(t')^T\rangle = 2\kappa_B T\Gamma\delta(t'-t), \tag{2}$$

where κ_B is the Boltzmann constant, T is the temperature, and $\delta(t'-t)$ is the Dirac Delta function. The spatial correlation can be seen more clearly by writing the auto-correlation in Eq. (2) in component form

$$\langle f_j(t) f_k(t')^T\rangle = 2\kappa_B T\Gamma_{jk}\delta(t'-t). \tag{3}$$

The model equations can be written as stochastic differential equations (SDEs)

$$\mathbf{dq} = -M\,\mathbf{q}\,dt + \mathbf{dW} \tag{4}$$

by defining $M = \Gamma^{-1}K$ and the Wiener process $\mathbf{dW}(t) = \Gamma^{-1}\mathbf{f}(t)dt$. In this formulation, the Wiener process $\mathbf{dW}$ is governed by the Ito Calculus conditions

$$\langle \mathbf{dW}\rangle = 0, \quad \mathbf{dW}\,\mathbf{dW}^T = Hdt, \quad dt\,\mathbf{dW} = 0, \tag{5}$$

where $H = 2\kappa_B T\Gamma^{-1}$. It is assumed that the equilibrium point $\mathbf{q} = 0$ is stable; thus, M is invertible and its eigenvalues are either real and positive or complex with a positive real part. The probability density function for the position of a particle at time t obeying the Langevin equation, Eq. (4), is governed by a Fokker-Planck equation whose stationary solution is a standard bivariate distribution.[11]

III. DIMENSIONLESS MODEL EQUATIONS

In order to facilitate the subsequent calculations, transformation to dimensionless variables is useful. We consider a transformation based on the covariance matrix in the stochastic equilibrium

$$C = \mathrm{cov_s}(\mathbf{q}) = \lim_{t\to\infty}\langle[\mathbf{q}(t) - \langle\mathbf{q}(t)\rangle][\mathbf{q}(t) - \langle\mathbf{q}(t)\rangle]^T\rangle, \tag{6}$$

where the angle brackets denote an expected value. The covariance matrix satisfies the following condition:[11]

$$MC + CM^T = H. \tag{7}$$

Furthermore, for the two-dimension system of linear SDEs defined by Eqs. (4) and (5), it can be shown that

$$C = \frac{1}{2\mathrm{tr}(M)}\left\{H + \frac{[\mathrm{tr}(M)I - M]H[\mathrm{tr}(M)I - M]^T}{\det(M)}\right\}, \tag{8}$$

where I is the 2×2 identity matrix. We also note that the covariance matrix, which is symmetric and positive-definite, can be written as

$$C = RDR^T, \tag{9}$$

where R is a rotation matrix ($R^{-1} = R^T$ and $\det(R) = 1$) and D is a diagonal matrix with non-negative entries. For a system that exhibits noise driven fluctuations in its coordinates, the entries are strictly positive. Thus, the diagonal matrix $\sqrt{D}$ can be defined as the diagonal matrix whose entries are the square roots of the entries of D, and its inverse, along with R, is used to define new, dimensionless variables and parameters as follows:

$$\tau = \sqrt{\det(M)}t, \tag{10a}$$

$$\mathbf{x} = \sqrt{D}^{-1}R^T\mathbf{q}, \tag{10b}$$

$$\mathbf{w} = \sqrt{D}^{-1}R^T\mathbf{W}, \tag{10c}$$

$$\Lambda = \frac{\sqrt{D}^{-1}R^T MR\sqrt{D}}{\sqrt{\det(M)}}. \tag{10d}$$

Equation (4) can now be written as a dimensionless equation of motion

$$\mathbf{dx} = -\Lambda\,\mathbf{x}\,d\tau + \mathbf{dw}, \tag{11}$$

and using Eqs. (5), (7), (9), and (10), it can be shown that

$$\langle\mathbf{dw}\rangle = 0, \quad \mathbf{dw}\,\mathbf{dw}^T = (\Lambda + \Lambda^T)d\tau, \quad d\tau\,\mathbf{dw} = \mathbf{0}. \tag{12}$$

It follows from Eqs. (9) and (10b) that $\mathrm{cov_s}(\mathbf{x}) = I$, and hence, the stationary probability density is the standard bivariate normal distribution

$$p_s(\mathbf{x}) = \frac{e^{-\frac{1}{2}|\mathbf{x}|^2}}{2\pi}. \tag{13}$$

IV. SPACE OF PARAMETERS

It is convenient at this stage to set a choice of effectively independent parameters of the systems. Since $\Lambda + \Lambda^T$ is symmetric, a system of orthogonal axes can be chosen in a way that the aforementioned matrix is diagonal. Under this choice, Λ can be written as

$$\Lambda = \begin{bmatrix} \lambda_1 & \lambda_0 \\ -\lambda_0 & \lambda_2 \end{bmatrix}. \tag{14}$$

Also, the axes can be chosen so that $\lambda_1 \leq \lambda_2$. Since $\det(\Lambda) = 1$ [see Eq. (10d)], the dimensionless system has only two effectively independent parameters. Our choice of parameters will be $\mu = \mathrm{tr}(\Lambda)$ and $\nu = \lambda_0$. From equations $\det(\Lambda) = 1$ and $\mu = \mathrm{tr}(\Lambda)$, the coefficients λ_1 and λ_2 are expressed in terms of the parameters μ and ν

$$\lambda_1 = \frac{1}{2}\left[\mu - \sqrt{\mu^2 - 4(\nu^2 - 1)}\right], \tag{15a}$$

$$\lambda_2 = \frac{1}{2}\left[\mu + \sqrt{\mu^2 - 4(\nu^2 - 1)}\right]. \tag{15b}$$

The parameters μ and ν are restricted by a band of conditions. Since $\Lambda + \Lambda^T$ is positive-definite, λ_1 and λ_2 must be positive real numbers, which implies that $(\frac{\mu}{2})^2 + \nu^2 \geq 1$ and $|\nu| < 1$. Also, since the field force is restoring, the eigenvalues of Λ are either positive or complex with positive real numbers, which is possible only if $\mu > 0$. Note that $\nu = 0$ corresponds to the case where the matrix λ is symmetric (regardless of the choice of orthogonal axes); in other words, the restoring force is conservative.

V. POWER SPECTRUM ANALYSIS

The power spectral density (PSD) of a stochastic variable describes the distribution of the power related to a given signal over the range of frequencies (ω) the signal oscillates with. For an n-coordinate stochastic variable, the PSD is formally defined as a matrix function of ω. In the Langevin linear system described by Eqs. (1) and (2), this matrix is given by[11]

$$S(\omega) = \frac{1}{2\pi}\left[(M + i\omega I)^{-1}C + C(M^T - i\omega I)^{-1}\right], \tag{16}$$

where C is the covariance matrix defined in Sec. VI. In the case of the dimensionless system governed by Eqs. (4) and (5), we need to substitute M and C by Λ and I, respectively,

$$S(\omega) = \frac{1}{2\pi}\left[(\Lambda + i\omega I)^{-1} + (\Lambda^T - i\omega I)^{-1}\right]. \tag{17}$$

We are interested in the individual spectral densities of x_1 and x_2 [i.e., $S_{11}(\omega)$ and $S_{22}(\omega)$], which are computed from Eq. (17). This results in

$$S(\omega)_{11} = \frac{\lambda_2 + \lambda_1\omega^2}{\pi\left[(1-\omega)^2 + (\mu\omega)^2\right]}, \tag{18a}$$

$$S(\omega)_{22} = \frac{\lambda_1 + \lambda_2\omega^2}{\pi\left[(1-\omega)^2 + (\mu\omega)^2\right]}. \tag{18b}$$

The critical frequencies of S_{11}, other than $\omega = 0$, are found by solving the equation $\frac{dS_{11}}{d(\omega^2)} = 0$. This equation is equivalent to a quadratic equation for ω^2, with the following solutions written in terms of the system parameters μ and ν

$$\omega_\pm^2 = \frac{-\lambda_2 \pm \mu|\nu|}{\lambda_1} = \frac{\mu(\pm 2|\nu| - 1) - \sqrt{\mu^2 + 4(\nu^2 - 1)}}{\mu - \sqrt{\mu^2 + 4(\nu^2 - 1)}}. \tag{19}$$

Note that the square roots in the above equation are real in the space of parameters and $\omega_-^2 < 0$ is unphysical. Thus, this solution is not accepted. It is possible to demonstrate that $\omega_+^2 > 0$ only if the following conditions apply

$$|\nu| > \frac{1}{2}, \quad \mu^2 < 1 + \frac{1}{|\nu|}. \tag{20}$$

Under these circumstances, $S_{11}(\omega)$ has three critical frequencies, the roots of ω_+^2 and zero. Since $S_{11}(\omega)$ is a positive, symmetric, and decaying function, no second order derivative analysis is necessary to infer that $\pm\omega_+$ maximize $S_{11}(\omega)$, while 0 minimizes it. If the conditions in (20) do not apply, then $\omega = 0$ is the only critical frequency. Thus, it is maximal.

Summarizing, the characteristic oscillation frequency of $x_1(\tau)$ is given by

$$|\omega_1| = \begin{cases} \sqrt{\dfrac{\mu(2|\nu| - 1) - \sqrt{\mu^2 + 4(\nu^2 - 1)}}{\mu - \sqrt{\mu^2 + 4(\nu^2 - 1)}}} & |\nu| \geq \dfrac{1}{2} \text{ and } \mu \leq \sqrt{1 + \dfrac{1}{|\nu|}}. \\ 0 & \sim \end{cases} \tag{21}$$

The analysis to find the characteristic oscillation frequency of $x_2(\tau)$ is quite similar

$$|\omega_2| = \begin{cases} \sqrt{\dfrac{\mu(2|\nu| - 1) + \sqrt{\mu^2 + 4(\nu^2 - 1)}}{\mu + \sqrt{\mu^2 + 4(\nu^2 - 1)}}} & |\nu| \geq \dfrac{1}{2} \text{ or } \mu \leq \sqrt{1 + \dfrac{1}{|\nu|}}. \\ 0 & \sim \end{cases} \tag{22}$$

Figure 1 illustrates the color maps of $|\omega_1|$ and $|\omega_2|$ on the space of parameters. The critical frequency ω_1 is zero almost everywhere except at the left corners of the space of parameters. On the other hand, ω_1 is zero in a relatively small

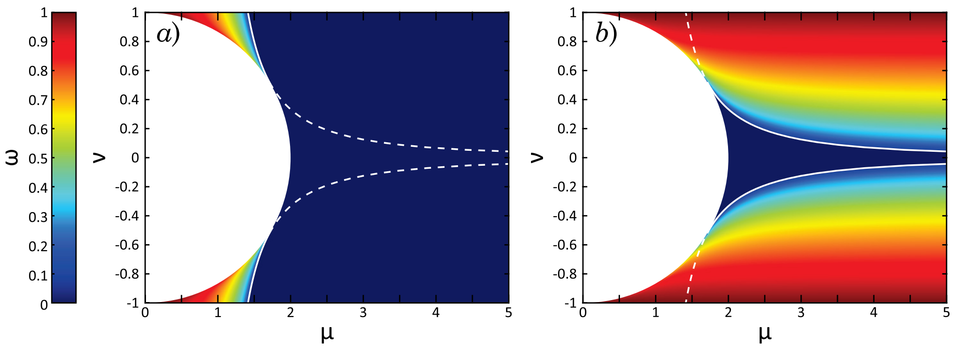

FIG. 1. Color maps of the critical frequencies of (a) $S_{11}(\omega)$ and (b) $S_{22}(\omega)$, over the space of parameters. The curve $\mu = \sqrt{1+\frac{1}{|\nu|}}$ is included in both maps, as it serves as a delimiter for the zero level set of both critical frequencies.

region, given by the inequalities, $\left(\frac{\mu}{2^2}\right) + \nu^2 \geq 1$ and $\mu \leq \sqrt{1+\frac{1}{|\nu|}}$. This region encloses the line $\nu = 0$, where the restoring force is conservative. Note that both oscillation frequencies cannot be bigger than 1 given the time re-scaling in Sec. III [Eq. (10a)].

The procedure described here is valid for any bivariate, first-order Langevin system.

VI. PHASE VELOCITY FOR APERIODIC MOTION

In this section, we show that the stochastic system defined by Eqs. (11) and (12) gives rise to a persistent aperiodic cycling motion that can be characterised by a phase velocity $\langle \frac{d\phi}{dt} \rangle$, where ϕ is the angular displacement in the phase space co-ordinate shown schematically in Fig. 2. Referring to Fig. 2, the infinitesimal angular displacement is given by

$$d\phi = \frac{\mathbf{x}(\tau) \times \mathbf{x}(\tau + d\tau)}{|\mathbf{x}(\tau)||\mathbf{x}(\tau + d\tau)|} = \frac{\mathbf{x}}{|\mathbf{x}|} \times \frac{\mathbf{dx}}{|\mathbf{x} + \mathbf{dx}|}, \tag{23}$$

where "$\times$" denotes the scalar cross-product between two two-dimensional vectors, i.e., $\mathbf{a} \times \mathbf{b} = a_1 b_2 - a_2 b_1$. The right hand of Eq. (23) comes from noting $\mathbf{x}(\tau) = \mathbf{x}$ and $\mathbf{x}(\tau + d\tau) = \mathbf{x} + \mathbf{dx}$. The phase velocity can be found by expanding the right hand side to first order in dt. It follows from Eqs. (11) and (12) that $d\phi$ has to be expanded to second order in $\mathbf{dw}$. This is achieved by expanding $\frac{1}{|\mathbf{x}+\mathbf{dx}|}$ to first order in $\mathbf{dx}$. Explicitly, we have

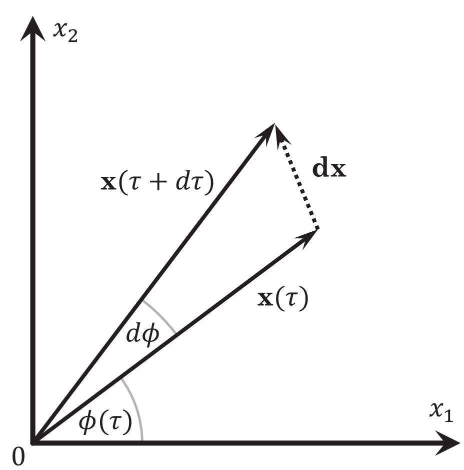

FIG. 2. Schematic illustration of an infinitesimal angular variation $d\phi$ for an infinitesimal time $d\tau$ in the space of dimensionless coordinates.

$$\begin{aligned} d\phi &= \frac{\mathbf{x}}{|\mathbf{x}|} \times \left[\frac{1}{|\mathbf{x}|} - \frac{\mathbf{x}}{|\mathbf{x}|^3} \cdot \mathbf{dx} + \mathcal{O}(\mathbf{dx}^2) \right] \mathbf{dx} \\ &= \frac{\mathbf{x}}{|\mathbf{x}|} \times \left[\frac{\mathbf{dx}}{|\mathbf{x}|} - \frac{(\mathbf{x} \cdot \mathbf{dx})\mathbf{dx}}{|\mathbf{x}|^3} \right] + \mathcal{O}(\mathbf{dx}^3), \end{aligned} \tag{24}$$

where "$\cdot$" denotes the inner scalar product $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$. Equations (11) and (12) can be used to further expand $d\phi$. Note that $\mathcal{O}(\mathbf{dx}^3)$ becomes $\mathcal{O}(d\tau^2)$ with no terms dw_n involved, and we have

$$\begin{aligned} d\phi &= \frac{\mathbf{x}}{|\mathbf{x}|} \times \left[\frac{-\Lambda\,\mathbf{x}\,d\tau + \mathbf{dw}}{|\mathbf{x}|} - \frac{(\Lambda + \Lambda^T)\,\mathbf{x}\,d\tau}{|\mathbf{x}|^3} \right] + \mathcal{O}(d\tau^2) \\ &= \left[\frac{\Lambda\,\mathbf{x} \times \mathbf{x}}{|\mathbf{x}|^2} + \frac{(\Lambda + \Lambda^T)\,\mathbf{x} \times \mathbf{x}}{|\mathbf{x}|^4} \right] d\tau + \frac{\mathbf{x} \times \mathbf{dw}}{|\mathbf{x}|^2} + \mathcal{O}(d\tau^2). \end{aligned} \tag{25}$$

It is useful to express this in terms of the components x_n, dw_n λ_1, λ_2, and ν in which case

$$\begin{aligned} d\phi = &\left[\nu + (\lambda_1 - \lambda_2) \frac{x_1 x_2}{x_1^2 + x_2^2} + 2(\lambda_1 - \lambda_2) \frac{x_1 x_2}{(x_1^2 + x_2^2)^2} \right] d\tau \\ &+ \frac{x_1 dw_2 - x_2 dw_1}{x_1^2 + x_2^2} + \mathcal{O}(d\tau^2). \end{aligned} \tag{26}$$

We now consider the expected value of $d\phi$ in the stochastic equilibrium. Using the result that under Ito Stochastic Calculus, $\mathbf{x}(\tau)$ and any analytical function of it are stochastically independent of $\mathbf{dw}(\tau)$, we can write

$$\begin{aligned} \langle d\phi \rangle_s = &\left[\nu + (\lambda_1 - \lambda_2) \left\langle \frac{x_1 x_2}{x_1^2 + x_2^2} \right\rangle_s \right. \\ &\left. + 2(\lambda_1 - \lambda_2) \left\langle \frac{x_1 x_2}{(x_1^2 + x_2^2)^2} \right\rangle_s \right] d\tau \\ &+ \left\langle \frac{x_1}{x_1^2 + x_2^2} \right\rangle_s \langle dw_2 \rangle - \left\langle \frac{x_2}{x_1^2 + x_2^2} \right\rangle_s \langle dw_1 \rangle + \langle \mathcal{O}(d\tau^2) \rangle_s, \end{aligned} \tag{27}$$

where given an analytical function $f(\mathbf{x})$

$$\langle f(\mathbf{x}) \rangle_s = \iint_{\mathbb{R}^2} f(\mathbf{x}) p_s(\mathbf{x})\, dx_1\, dx_2. \tag{28}$$

Using the stationary distribution p_s given by Eq. (13), together with symmetry arguments, it is possible to infer that

$$\left\langle \frac{x_1}{x_1^2+x_2^2} \right\rangle_s = \left\langle \frac{x_2}{x_1^2+x_2^2} \right\rangle_s = \left\langle \frac{x_1 x_2}{x_1^2+x_2^2} \right\rangle_s = \left\langle \frac{x_1 x_2}{(x_1^2+x_2^2)^2} \right\rangle_s = 0. \tag{29}$$

From the Ito rules, Eq. (12), we also have $\langle dw_1 \rangle = \langle dw_2 \rangle = 0$. Finally, using the above results in Eq. (27), dividing by $d\tau$, and taking the limit $d\tau \to 0$, the stationary expected phase velocity is found to be given by

$$\left\langle \frac{d\phi}{d\tau} \right\rangle_s = \nu = \frac{\lambda_{12} - \lambda_{21}}{2}. \tag{30}$$

The phase velocity is thus zero if the matrix Λ is a symmetric matrix.

VII. PHASE VELOCITY IN THE PHYSICAL SYSTEM

In this section, we find an expression for the phase velocity in terms of the physical parameters of the system. First, we note that the phase velocities of both $\mathbf{q}(t)$ and $\mathbf{x}(\tau)$ should be the same. This follows from the transformation Eq. (10b) which simply involves a rotation of the axes q_1 and q_2 (given by R^T), followed by each axis being elongated or contracted separately (given by $\sqrt{D}^{-1}$). In both frames, the number of cycles over a very long period of time will be the same. Moreover, since the system under study is ergodic, the expected phase velocity is equal to the average angular velocity in both the physical and dimensionless frames. Thus

$$\langle \omega \rangle_s = \left\langle \frac{d\phi}{dt} \right\rangle_s = \sqrt{\det(M)} \left\langle \frac{d\phi}{d\tau} \right\rangle_s = \frac{\sqrt{\det(M)}\operatorname{atr}(\Lambda)}{2}, \tag{31}$$

where we have defined $\operatorname{atr}(A) = a_{12} - a_{21}$ for a 2×2 matrix. Given two 2×2 matrices A and B, it can be shown that

$$\operatorname{atr}(ABA^T) = \det(A)\operatorname{atr}(B). \tag{32}$$

Using Eqs. (9) and (10d), we can write

$$\Lambda = \frac{\left(\sqrt{D}^{-1} R^T\right) MC \left(\sqrt{D}^{-1} R^T\right)^T}{\sqrt{\det(M)}}, \tag{33}$$

and then using Eqs. (32) and (33), we can write

$$\begin{aligned} \langle \omega \rangle_s &= \frac{1}{2}\det\left(\sqrt{D}^{-1} R^T\right)\operatorname{atr}(MC) = \frac{\det(R)\operatorname{atr}(MC)}{2\sqrt{\det(D)}} \\ &= \frac{\operatorname{atr}(MC)}{2\sqrt{\det(C)}}. \end{aligned} \tag{34}$$

In obtaining the above result, we used the intermediate results that $C = RDR^T$, $R^{-1} = R^T$, and $\det R = 1$.

We can multiply Eq. (8) by M from the left and use the property $M^{-1} = \frac{1}{\det(M)}(\operatorname{tr}(M)I - M^T)$, valid for any invertible 2×2 matrix, to write

$$\begin{aligned} MC &= \frac{1}{2}\left[H + \frac{MH - HM^T}{\operatorname{tr}(M)}\right] \\ &= \frac{1}{2}\left\{H + \frac{1}{\operatorname{tr}(M)}\begin{bmatrix} 0 & \operatorname{atr}(MH) \\ -\operatorname{atr}(MH) & 0 \end{bmatrix}\right\}. \end{aligned} \tag{35}$$

We have also used the property that H is symmetric and $MH - HM^T$ is anti-symmetric. We can now write

$$\operatorname{atr}(MC) = \frac{\operatorname{atr}(MH)}{\operatorname{tr}(M)} \tag{36}$$

and

$$\det(C) = \frac{\det(MC)}{\det(M)} = \frac{1}{4\det(M)}\left\{\det(H) + \left[\frac{\operatorname{atr}(MH)}{\operatorname{tr}(M)}\right]^2\right\}. \tag{37}$$

Substituting these results into Eq. (34), the stationary mean phase velocity is found in terms of M and H, viz.,

$$\langle \omega \rangle_s = \sqrt{\frac{\det(M)}{\operatorname{tr}(M)^2\det(H) + \operatorname{atr}(MH)^2}}\operatorname{atr}(MH). \tag{38}$$

An expression for $\langle \omega \rangle_s$ in terms of the physical parameters K, Γ, and κ_{BT} still remains to be found. As stated in Sec. II, $M = \Gamma^{-1}K$ and $H = 2\kappa_B T\Gamma^{-1}$. Provided that all the matrices are 2×2, it follows that

$$\operatorname{tr}(M) = \frac{\operatorname{tr}(\Gamma)\operatorname{tr}(K) - \operatorname{tr}(\Gamma K)}{\det(\Gamma)}. \tag{39}$$

Substituting this final result into Eq. (38) leads to

$$\langle \omega \rangle_s = \sqrt{\frac{\det(K)}{[\operatorname{tr}(\Gamma)\operatorname{tr}(K) - \operatorname{tr}(\Gamma K)]^2 + \det(\Gamma)\operatorname{atr}(K)^2}}\operatorname{atr}(K). \tag{40}$$

This expression shows that if the viscous drag is symmetric, then noise in this system produces an aperiodic cyclic motion if and only if the stiffness coefficient matrix K is non-symmetric, i.e., if and only if the external force is non-conservative.

VIII. NUMERICAL SIMULATIONS

In this section, we have carried out numerical simulations of the Langevin equation, Eq. (4). The phase angle was measured from the simulations, and measurements of the mean phase velocity of aperiodic motion were compared with the theoretical result in Eq. (40).

In the numerical simulations, with the Wiener Process $\mathbf{dW}$ governed by Eq. (5), we considered $M = \Gamma^{-1}K$ and $H = 2\kappa_B T\Gamma^{-1}$, with

$$K = \begin{bmatrix} 2 & 1 + \frac{1}{2}\Delta k_c \\ 1 - \frac{1}{2}\Delta k_c & 3 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 31 & 11 \\ 11 & 4 \end{bmatrix}. \tag{41}$$

Here, $\Delta k_c = \operatorname{atr}(K)$ is a parameter that accounts for the non-symmetry of the stiffness matrix K. Figure 3(a) illustrates the evolution of the phase angle over time for each integer value of Δk_c between –10 and 10. Each simulation has a total of 10^6 steps separated by $dt = 10^{-2}$. The figure shows that the

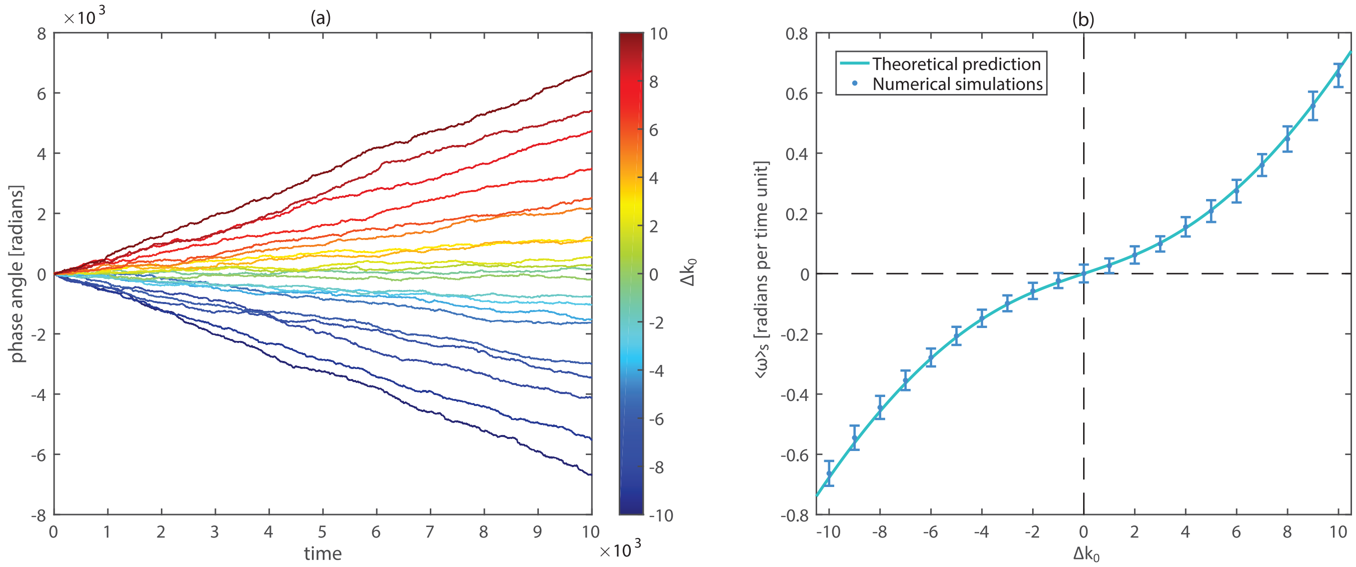

FIG. 3. (a) Computations of the phase angle over time from simulations of the physical system described in Sec. II, where the matrices K and Γ are given by Eq. (41). (b) Numerical estimations of the mean phase velocity in terms of the non-symmetry parameter, Δk_c, together with the theoretical prediction, Eq. (42).

phase angle has a fluctuating behaviour with rises and drops. However, it can be seen that it changes at a mean constant rate when observed over a sufficiently long period of time. The mean phase velocity has the same sign as Δk_c.

A more precise estimation of the mean phase velocity, $\langle\omega\rangle_s$, was obtained by running 100 simulations of Eq. (4) for every integer value of Δk_c between –10 and 10. The phase angle was computed from each simulation, and a linear regression was carried out. The mean phase velocity was estimated as the average between the slopes of the regressions. The results are summarized in Fig. 3, with error bars from the standard deviation of the slopes.

A theoretical prediction for $\langle\omega\rangle_s$ can be inferred from Eqs. (40) and (41)

$$\langle\omega\rangle_s = \sqrt{\frac{5+\frac{1}{4}\Delta k_c^2}{79^2+3\Delta k_c^2}}\Delta k_c. \tag{42}$$

The plot of $\langle\omega\rangle_s$ versus Δk_c is included in Fig. 3(b), thus showing that the numerical estimations of $\langle\omega\rangle_s$ are in excellent agreement with the theoretical prediction.

IX. SUMMARY AND DISCUSSION

In this work, we derived expressions for the frequency of oscillations of periodic motions and for the mean phase velocity of cyclic aperiodic motion in a two-dimensional overdamped system with noise and linear restoring forces. The periodic motions and the cyclic aperiodic motion do not arise in the corresponding deterministic system without noise. Our mathematical result for the mean phase velocity of cyclic aperiodic motion, Eq. (39), shows the dependence on the linear restoring forces and the viscous drag coefficients. We carried out numerical simulations of the Langevin equation for the model system, and we obtained excellent agreement with the theoretical result. In particular, it can readily be seen from Eq. (39) that a non-conservative force field is required for the cyclic aperiodic motion to occur in this system. In this sense, the model shares features with Brownian vortexes.[4] However, unlike Brownian vortexes, the cyclic aperiodic motion that is reported here occurs in a linear system.

Our results for the cyclic aperiodic motion are also consistent with the observations reported in Ref. 7 on simulations of a model nanowire trapped by optical tweezers in a non-conservative force field. A key observation in this work was that the mean phase velocity of the cyclic motion increases approximately linearly with the optical trapping power. This is consistent with Eq. (39) because the optical forces and torques, and hence the components of the stiffness matrix, are linearly proportional to the beam trapping power.

Finally, it should be remarked that the result in Eq. (39) was derived considering an overdamped system with a symmetric viscous drag, and correlated noise can be applied to an overdamped system with time- and space-uncorrelated white noise by considering the special case, $\Gamma = I$. In this case, the stochastic system, Eq. (3), simplifies to

$$\mathbf{dq} = -K\mathbf{q}dt + \mathbf{f}(t)dt \tag{43}$$

and the mean phase velocity simplifies to

$$\langle\omega\rangle_s = \sqrt{\frac{\det(K)}{\mathrm{tr}(K)^2+\mathrm{atr}(K)^2}}\mathrm{atr}(K). \tag{44}$$

An interesting aspect of this result is that the aperiodic cyclic motion does not simply arise by uncorrelated noise perturbing a stable node, or stable spiral, in a planar dynamical system, i.e., $\mathrm{tr}(K) > 0$ and $\det(K) > 0$. The non-conservative nature of the restoring force is also fundamental, i.e., $\mathrm{atr}(K) \neq 0$.

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