On the tomographic description of classical fields

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Abstract

After a general description of the tomographic picture for classical systems, a tomographic description of free classical scalar fields is proposed both in a finite cavity and the continuum. The tomographic description is constructed in analogy with the classical tomographic picture of an ensemble of harmonic oscillators. The tomograms of a number of relevant states such as the canonical distribution, the classical counterpart of quantum coherent states and a new family of so called Laguerre states, are discussed. Finally the Liouville equation for field states is described in the tomographic picture offering an alternative description of the dynamics of the system that can be extended naturally to other fields.

Keywords: Tomography, Klein-Gordon equation, WGM, Laguerre states

1. Introduction

Recently it has been shown the equivalence between the tomographic picture of quantum states and the various standard representations of them: Schrödinger [1], Heisenberg [2], Wigner [3], etc. (see for instance [4], [5] and references therein). In this paper we try to extend such description to classical fields. In particular we will discuss the tomographic description of the real scalar Klein–Gordon field inspired by the tomographic description of an ensemble of harmonic oscillators. In fact classical and quantum field states are usually considered as classical and quantum mechanics applied to describing these states for systems with infinite number of degrees of freedom (the field modes). Thus a state of a classical free field when restricted to consider just a finite number of modes can be treated as an statistical ensemble of harmonic oscillators.

This attempt will generalize the description of classical (or quantum) states in two directions. On one side, describing classical field states involves dealing with an infinite number of degrees of freedom and on the other, a covariant treatment of fields implies taking into the description of the state its dynamical evolution. In order to show how to proceed with this task we will analyze the foundations of tomography for a classical system with a finite number of degrees of freedom and we will extend straightforwardly such construction to deal with classical fields.

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The tomographic description of classical systems presented here will be directly inspired by the Radon transform, so that our construction can also be considered as an infinite dimensional extension of the Radon transform. The transition to quantum fields should proceed using similar ideas in the realm of quantum mechanical systems, however we will leave such analysis to a subsequent paper. Classical and quantum standard descriptions of the fields are dramatically different. The states of classical mode are identified with probability densities and the states of quantum modes are identified with Hermitian trace-class nonnegative density operators (or density matrices). The observables in the tomographic picture of quantum mechanics are tomographic symbols of corresponding operators which are constructed by means of a specific star–product scheme [6]. The analogous tomographic representation of classical system states by means of classical Radon transform [7] of the classical probability density $\rho(\omega)$ is also available [8], [9], [4].

Till now the classical and quantum field states have not been considered in the tomographic probability representations, except in early attempts [10], [11] and more recently when applying the quantum Radon transform to study tomographic symbols of creation and annihilation field operators for bosons and fermions [12] [13]. The aim of our work is to extend the tomographic approach to the case of quantum and classical systems with infinite number of degrees of freedom and to introduce for classical and quantum fields the tomographic probability density functionals determining their states. We will also find the tomographic form of the classical field Liouville equation for the tomographic probability density functionals.

The paper is organized as follows. In section 2 a generalized description of the tomography of classical systems inspired on the Radon transform will be described. In this picture the description of states of a physical system as normalized positive functionals on the algebra of observables of the system is paramount. Notice that such framework is common to both classical and quantum systems. The tomographic description of a family of states, similar to coherent states, for an ensemble of independent harmonic oscillators will be done in section 3. Then, in section 4, and using as a guideline the results obtained for the family of oscillators before, we will discuss the tomographic picture of a real scalar Klein–Gordon on a finite cavity. For that we will consider the field as described by a countable ensemble of harmonic oscillators and a family of states similar to coherent states for harmonic oscillators will be analized. It will be shown that the tomographic description an such states is equivalent to the original one. The tomographic description of the field states on the continuous case will be discussed in section 5 following similar lines and finally, the Liouville field equation in tomographic form will be discussed in section 6. Conclusions and further perspectives of this work are given in section 7.

2. The tomographic picture of classical physical systems: an overview

The states of a classical system with a finite numbers of degrees of freedom are described by a probability density $\rho(\boldsymbol{\varpi})$ on its phase space $\boldsymbol{\varpi} \in \Omega$. The phase space carries a canonical measure, the Liouville measure $\mu_{\text{Liouville}}$ that in canonical coordinates $(\boldsymbol{q},\boldsymbol{p}), \boldsymbol{q}=(q_1,\ldots,q_n), \boldsymbol{p}=(p_1,\ldots p_n)$, has the form $\mathrm{d}\mu_{\text{Liouville}}(\boldsymbol{q},\boldsymbol{p}) = \mathrm{d}^n q \mathrm{d}^n p = \mathrm{d}q_1 \cdots \mathrm{d}q_n \mathrm{d}p_1 \cdots \mathrm{d}p_n$. In the case that Ω is a domain in $\mathbb{R}^{2n}$, the classical center–of–mass tomogram $\mathcal{W}_{\text{cm}}$ of the state ρ is defined as the Radon transform of the density ρ and consists on the average of ρ along affine hyperplanes on phase space, i.e.,

$$\mathcal{W}_{\text{cm}}(X,\boldsymbol{\mu},\boldsymbol{\nu}) = \int_{\Omega} \rho\,(\boldsymbol{q},\boldsymbol{p})\,\delta(X-\boldsymbol{\mu}\cdot\boldsymbol{q}-\boldsymbol{\nu}\cdot\boldsymbol{p})\mathrm{d}^n q \mathrm{d}^n p, \tag{1}$$

where $\boldsymbol{\mu}=(\mu_1,\ldots,\mu_n)$, $\boldsymbol{\nu}=(\nu_1,\ldots\nu_n)$ and the equation $X-\boldsymbol{\mu}\cdot\boldsymbol{q}-\boldsymbol{\nu}\cdot\boldsymbol{p}=0$ determines an hyperplane Π in Ω. The classical center–of–mass tomogram $\mathcal{W}_{\text{cm}}(X,\boldsymbol{\mu},\boldsymbol{\nu})$ defines a probability density, depending on the random variable X, on the space of hyperplanes in Ω. The state ρ can be reconstructed by using the inverse Radon transform:

$$\rho\,(\boldsymbol{q},\boldsymbol{p}) = \int_{\mathbb{R}^{2n+1}} \mathcal{W}_{\text{cm}}(X,\boldsymbol{\mu},\boldsymbol{\nu}) \exp\left[\mathrm{i}(X-\boldsymbol{\mu}\cdot\boldsymbol{q}-\boldsymbol{\nu}\cdot\boldsymbol{p})\right] \mathrm{d}X \frac{\mathrm{d}^n\mu \mathrm{d}^n\nu}{(2\pi)^{2n}}. \tag{2}$$

The ideas before can be extended by considering with more care the role of the observables of the system in the construction of the tomogram $\mathcal{W}_{\rm cm}$. The description of a physical system involves always the selection of its algebra of observables, call it $\mathcal{O}$, and its corresponding states, denoted by $\mathcal{S}$. The outputs of measuring a given observable $A \in \mathcal{O}$ when the system is in the state ρ are described by a probability measure $\mu_{A,\rho}$ on the real line such that $\mu_{A,\rho}(\Delta)$ is the probability that the output of A belongs to the subset $\Delta \subset \mathbb{R}$. Thus a measure theory for the physical system under consideration is a pairing between observables A and states ρ assigning to pairs of them measures $\mu_{A,\rho}$. In this setting the expected value of the observable A in the state ρ is given simply by:

$$\langle A \rangle_\rho = \int \lambda \mathrm{d}\mu_{A,\rho}(\lambda). \tag{3}$$

Such picture applies equally to both classical and quantum systems. Thus for closed quantum systems the observables are described by self–adjoint operators A on a Hilbert space $\mathcal{H}$ while states are described as density operators ρ acting on such Hilbert space. The pairing above is provided by the assignment of the measure $\mu_{A,\rho} = \mathrm{Tr}(\rho E_A)$ where E_A denotes the projector–valued spectral measure associated to the Hermitian operator A.

The description of a classical system whose phase space is Ω can be easily established in these terms by considering that the algebra of observables $\mathcal{O}$ is a class (large enough) of functions on Ω, and that the states of the system are normalized positive functionals on $\mathcal{O}$, thus for instance if $\mathcal{O}$ contains the algebra of continuous functions on Ω, states are probability measures on phase space. If we assume that the phase space is originally equipped with a measure μ, for instance the Liouville measure $\mu_{\rm Liouville}$ in the case of mechanical systems, then we may restrict ourselves to the statistical states considered by Boltzmann corresponding to probability measures which are absolutely continuous with respect to the Liouville measure, thus determined by probability densities $\rho(\boldsymbol{\varpi})$ on Ω. We denote such space of states by $\mathcal{S}$ as before. Given an observable $f(\boldsymbol{\varpi})$ on Ω, the pairing between states and observables will be realized by assigning to the observable f its characteristic distribution ρ_f with respect to the probability measure $\rho(\boldsymbol{\varpi})\mathrm{d}\mu(\boldsymbol{\varpi})$, then the probability of finding the measured value of the observable f in the interval Δ is given by

$$\int_\Delta \rho_f(\lambda)\mathrm{d}\lambda, \tag{4}$$

and the expected value of f on the state ρ will be given by:

$$\langle f \rangle_\rho = \int \lambda \rho_f(\lambda)\mathrm{d}\lambda. \tag{5}$$

The tomographic description provided by the classical center–of–mass tomograms $\mathcal{W}_{\rm cm}$ above (1) does not allow to cope with systems whose phase space is not of the previous form (for instance spin systems) and it is convenient to expand the scope of the formalism to make it more flexible and allow for alternative and more general pictures. Other tomographic pictures have been proposed for both classical and quantum systems (see for instance [14] for a description of quantum tomograms in the realm of C^*–algebras, *include more references here!!!*).

We will construct a tomographic picture of a classical system starting with a family of elements in $\mathcal{O}$ parametrized by an index x that can be discrete or continuous. Often x is a point on a finite dimensional manifold that we will denote by $\mathcal{M}$, thus $x \in \mathcal{M}$. We will denote the observable associated to the element x by $U(x)$ or U_x depending on the context. Given a state ρ of the system, the correspondence $x \mapsto U_x$, allows to pull–back the observables U_x to $\mathcal{M}$ defining the function $F_\rho(x)$ on $\mathcal{M}$ associated to the state $\rho(\boldsymbol{\varpi})$ by:

$$F_\rho(x) = \int_\Omega U_x(\boldsymbol{\varpi})\rho(\boldsymbol{\varpi})\mathrm{d}\mu(\boldsymbol{\varpi}). \tag{6}$$

The functions U_x must be chosen so that the previous integral is defined. For instance we could have chosen $\mathcal{M} = \Omega$ as in the definition of the Radon transform above (1), and then consider $U_{\boldsymbol{\varpi}} = \delta(\boldsymbol{\varpi})$, thus the function F_ρ associated to the state $\rho(\boldsymbol{\varpi})$ will be again $\rho(\boldsymbol{\varpi})$ itself. The original state $\rho(\boldsymbol{\varpi})$

could be reconstructed from F_ρ iff the family of observables $U(x)$ separate states, that is, given $\rho \neq \rho'$ two different states, there exists $x \in \mathcal{M}$ such that $F_\rho(x) = \langle \rho, U(x)\rangle \neq \langle \rho', U(x)\rangle = F_{\rho'}(x)$. Then two states are different if and only if the corresponding representing functions F_ρ are different.

Clearly up to now, our construction does not discriminate the description of classical systems from quantum systems. The difference will appear only at the level of the product structure on the induced functions F_ρ. Another important ingredient for the tomographic description is the Radon transform. To give an abstract presentation of this transform, we shall assume for the time being that $\mathcal{M}$ is a manifold which carries a measure, so that we can consider integrable functions on it and perform the corresponding integrals.

Consider now the dual space of $\mathcal{F}(\mathcal{M})$ and a second auxiliary space $\mathcal{N}$ whose points will be denoted by $y \in \mathcal{N}$. The space $\mathcal{N}$ parametrizes a certain subspace $\mathcal{D}(\mathcal{M}) \subset \mathcal{F}(\mathcal{M})'$. In other words, for each $y \in \mathcal{N}$ there is an assignment $y \mapsto D(y)$ with $D(y) \in \mathcal{D}(\mathcal{M})$ a linear functional on the space of functions on $\mathcal{M}$. We obtain a map from $\mathcal{F}(\mathcal{M})$ to $\mathcal{F}(\mathcal{N})$ by setting for each $f \in \mathcal{F}(\mathcal{M})$:

$$\mathcal{W}_f(y) = \langle D(y), f\rangle. \tag{7}$$

For instance suppose that $\mathcal{N}$ parametrizes a family of submanifolds $S(y)$ of Ω, $y \in \mathcal{N}$. If the submanifold $S(y)$ has the form $\Phi(\boldsymbol{q}, \boldsymbol{p}; X_1, \ldots, X_d) = X_0$, $y = (X_0, X_1, \ldots, X_d)$ denoting a parametrization of $\mathcal{N}$, the corresponding generalized Radon transform would be written as:

$$\mathcal{W}(y) = \int_\Omega \rho(\boldsymbol{q}, \boldsymbol{p})\delta(X_0 - \Phi(\boldsymbol{q}, \boldsymbol{p}; X_1, \ldots, X_n))\mathrm{d}^n q \mathrm{d}^n p, \tag{8}$$

which has the same form as eq. (1).

When the imbedding is properly chosen, it turns out that $\mathcal{W}(y)$ is a fair probability distribution on $\mathcal{N}$ which we have constructed out of the initial state ρ. The aim of tomography is to reconstruct ρ out of the experimental distribution functions that we obtain from the measurement of the selected observables parametrized by $\mathcal{M}$. This is the so called inversion formula for the Radon transform. In the case that $\Omega = \mathbb{R}^{2n}$ and $\mathcal{N}$ denotes as in (1) the space of hyperplanes, then because of the homogeneity properties of the Dirac distribution, we find that $\mathcal{W}_{\mathrm{cm}}$ satisfies the condition:

$$\left[X\frac{\partial}{\partial X} + \boldsymbol{\mu}\cdot\frac{\partial}{\partial \boldsymbol{\mu}} + \boldsymbol{\nu}\cdot\frac{\partial}{\partial \boldsymbol{\nu}} + 1\right]\mathcal{W}_{\mathrm{cm}}(X, \boldsymbol{\mu}, \boldsymbol{\nu}) = 0. \tag{9}$$

Due to the homogeneity condition (9), $\mathcal{W}_{\mathrm{cm}}$ depends effectively only on $2n$ variables instead of $2n+1$ and the inversion formula works, out of the "measurements" performed with the family of operators $\{\boldsymbol{\mu}\cdot\hat{\boldsymbol{q}} + \boldsymbol{\nu}\cdot\hat{\boldsymbol{p}}\}, (\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathbb{R}^{2n}$, we are able to recover ρ by mneans of eq. (2).

As an important example of the previous discussion, we introduce another kind of tomographic representation of the state $\rho(\boldsymbol{q}, \boldsymbol{p})$, the classical symplectic tomogram defined as:

$$\mathcal{W}_\rho(\boldsymbol{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \int_{\mathbb{R}^{2n}} \rho(\boldsymbol{q}, \boldsymbol{p}) \prod_{k=1}^{n} \delta(X_k - \mu_k q_k - \nu_k p_k)\mathrm{d}q_1 \ldots \mathrm{d}q_n \mathrm{d}p_1 \ldots \mathrm{d}p_n. \tag{10}$$

Notice that we have taken $\mathcal{M} = \mathbb{R}^{2n}$, the phase space again, and $\mathcal{N} = \mathcal{N}_1 \times \cdots \times \mathcal{N}_n$ with $\mathcal{N}_k$ the space of lines in $\mathbb{R}^2$, the phase space of each individual degree of freedom of the physical system under consideration. Thus, we have obtained a joint probability distribution of the n random variables $(X_1, \ldots, X_n) = \boldsymbol{X}$. In contrast to the center–of–mass case, because of the presence of n Dirac distributions, we find that the symplectic tomogram $\mathcal{W}_\rho$ satisfies n homogeneity conditions:

$$\left[X_k\frac{\partial}{\partial X_k} + \mu_k\frac{\partial}{\partial \mu_k} + \nu_k\frac{\partial}{\partial \nu_k} + 1\right]\mathcal{W}_\rho(\boldsymbol{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = 0 \quad , \quad k = 1, \ldots, n. \tag{11}$$

In other words, the classical symplectic tomogram $\mathcal{W}_\rho(\boldsymbol{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$ depends effectively only on $2n$ variables instead of $3n$. In fact, one can show that the symplectic tomogram $\mathcal{W}_\rho(\boldsymbol{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$ can be transformed into

the center–of–mass tomogram $\mathcal{W}_{\rm cm}$ of the same state ρ, and *vice versa*. Finally, out of the "measurements" performed with the family of operators $\{\mu_k\hat{q}_k + \nu_k\hat{p}_k\}, (\mu_k, \nu_k) \in \mathbb{R}^2, k = 1, \ldots, n$, we are again able to recover ρ, by means of the symplectic inversion formula:

$$\rho(\boldsymbol{q}, \boldsymbol{p}) = \int_{\mathbb{R}^{3n}} \mathcal{W}_\rho(\boldsymbol{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) \exp\left[\mathrm{i}\sum_{k=1}^{n}(X_k - \mu_k q_k - \nu_k p_k)\right] \mathrm{d}^n X \frac{\mathrm{d}^n\mu \mathrm{d}^n\nu}{(2\pi)^{2n}}, \tag{12}$$

where $\mathrm{d}^n X = \mathrm{d}X_1 \ldots \mathrm{d}X_n$.

3. Tomograms for states of an ensemble of classical oscillators

3.1. The canonical ensemble

If we consider a family of n independent one–dimensional oscillators with frequencies $\omega_k > 0$, its phase space Ω will be $\mathbb{R}^{2n}$ with canonical coordinates (q_k, p_k), $k = 1, \ldots, n$. The Hamiltonian of the system will be $H = \sum_{k=1}^{n} H_k$, $H_k(q_k, p_k) = \frac{1}{2}(p_k^2 + \omega_k^2 q_k^2)$. The dynamics of the system will be given by

$$\dot{q}_k = p_k, \quad \dot{p}_k = -\omega_k^2 q_k, \quad k = 1, \ldots n, \tag{13}$$

and the Liouville measure on phase space takes again the form $\mathrm{d}\mu_{\rm Liouville} = \mathrm{d}q_1 \ldots \mathrm{d}q_n \mathrm{d}p_1 \ldots \mathrm{d}p_n$. Making the change of variables $\xi_k = q_k/\sqrt{\omega_k}$, $\eta_k = \sqrt{\omega_k}p_k$, the dynamics is written in the symmetrical form

$$\dot{\xi}_k = \omega_k \eta_k, \quad \dot{\eta}_k = -\omega_k \xi_k. \quad k = 1, \ldots n. \tag{14}$$

and the Hamiltonian becomes

$$H(\xi, \eta) = \sum_{k=1}^{n} H_k(\xi_k, \eta_k) = \frac{1}{2}\sum_{k=1}^{n} \omega_k(\xi_k^2 + \eta_k^2). \tag{15}$$

The Liouville measure remains unchanged under this change of variables $\mathrm{d}\mu_{\rm Liouville}(\boldsymbol{q}, \boldsymbol{p}) = \mathrm{d}^n q \mathrm{d}^n p = \mathrm{d}^n \xi \mathrm{d}^n \eta = \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_n \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_n = \mathrm{d}\mu_{\rm Liouville}(\boldsymbol{\xi}, \boldsymbol{\eta})$ and statistical states are described by probability densities $\rho(\boldsymbol{q}, \boldsymbol{p}) = \rho(\boldsymbol{\xi}, \boldsymbol{\eta})$. Liouville equation determines the evolution of the state:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = \{\rho, H\} \tag{16}$$

where the Poisson brackets are defined by the canonical commutation relations $\{q_k, p_l\} = \delta_{kl}$, $\{q_k, q_l\} = \{p_k, p_l\} = 0$. Notice that if $\boldsymbol{\varpi} = (\boldsymbol{\xi}, \boldsymbol{\eta}) \in \Omega$ is a point in phase space, then $\rho_t(\boldsymbol{\varpi}) = \rho(\boldsymbol{\xi}(t), \boldsymbol{\eta}(t))$ with $(\boldsymbol{\xi}(t), \boldsymbol{\eta}(t))$ the solution of the equations of motion (14) starting at $\boldsymbol{\varpi}$ at time $t = 0$.

In particular, the Gibbs state or canonical distribution is given by $\rho_{\rm can}(\boldsymbol{q}, \boldsymbol{p}) = \mathrm{e}^{-\beta H}/Z_0$ where the normalization constant Z_0 is easily evaluated

$$\begin{aligned} Z_0 &= \int_\Omega \mathrm{e}^{-\beta H(\boldsymbol{q},\boldsymbol{p})} \mathrm{d}\mu_{\rm Liouville}(\boldsymbol{q}, \boldsymbol{p}) = \int_{\mathbb{R}^{2n}} \mathrm{e}^{-\frac{1}{2}\beta\sum_{k=1}^{n}\omega_k(\xi_k^2+\eta_k^2)} \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_n \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_n \\ &= (2\pi)^n \prod_{k=1}^{n} (\beta\omega_k)^{-1}. \end{aligned} \tag{17}$$

Hence for a given observable f we will have:

$$\langle f \rangle_{\rho_{\rm can}} = \frac{1}{Z_0} \int_{\mathbb{R}^{2n}} f(\boldsymbol{\xi}, \boldsymbol{\eta}) \mathrm{e}^{-\frac{1}{2}\beta\sum_{k=1}^{n}\omega_k(\xi_k^2+\eta_k^2)} \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_n \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_n. \tag{18}$$

More detailed information will be found in [15].

The classical tomographic description of a state $\rho(\boldsymbol{\xi},\boldsymbol{\eta})$ will be performed by means of a symplectic tomogram:

$$\mathcal{W}_\rho(\mathbf{X},\boldsymbol{\mu},\boldsymbol{\nu}) = \int_{\mathbb{R}^{2n}} \rho(\boldsymbol{\xi},\boldsymbol{\eta}) \prod_{k=1}^{n} \delta(X_k - \mu_k\xi_k - \nu_k\eta_k)\mathrm{d}\xi_1\ldots\mathrm{d}\xi_n\mathrm{d}\eta_1\ldots\mathrm{d}\eta_n. \tag{19}$$

We recall that here we have taken $\mathcal{M} = \mathbb{R}^{2n}$, the phase space again, and $\mathcal{N} = \mathcal{N}_1 \times \cdots \times \mathcal{N}_n$ with $\mathcal{N}_k$ the space of lines in $\mathbb{R}^2$, the phase space of each individual one–dimensional oscillator. A simple computation shows that the Gibbs state tomogram reads:

$$\mathcal{W}_{\rho_{\mathrm{can}}}(\boldsymbol{X},\boldsymbol{\mu},\boldsymbol{\nu}) = \prod_{k=1}^{n} \sqrt{\frac{\beta\omega_k}{2\pi\left(\mu_k^2+\nu_k^2\right)}} \exp\left[-\frac{\beta\omega_k X_k^2}{2(\mu_k^2+\nu_k^2)}\right]. \tag{20}$$

A interesting family of states which are the classical counterpart of quantum coherent states can be introduced by means of the holomorphic representation $\zeta_k = \frac{1}{\sqrt{2}}(\xi_k + i\eta_k)$ of phase space, hence the phase space becomes the complex space $\mathbb{C}^n$ with the Hermitian structure $H(\boldsymbol{\zeta},\bar{\boldsymbol{\zeta}}) = \sum_{k=1}^{n}\omega_k|\zeta_k|^2$. Given a point $\boldsymbol{z} = (z_1,\ldots,z_n) \in \mathbb{C}^n$ we can construct the distribution

$$\rho_{\boldsymbol{z}}(\boldsymbol{\zeta},\bar{\boldsymbol{\zeta}}) = N(\boldsymbol{z}) \exp\left[\sum_{k=1}^{n} \omega_k(z_k\bar{\zeta}_k + \bar{z}_k\zeta_k)\right] \rho_{\mathrm{can}}(\boldsymbol{\zeta},\bar{\boldsymbol{\zeta}})\big|_{\beta=1} \tag{21}$$

where

$$N(\boldsymbol{z}) = \prod_{k=1}^{n} \pi\omega_k^{-1}\exp\left[-\omega_k|z_k|^2\right]. \tag{22}$$

Notice that integrating the Liouville equation for such state yields:

$$\rho_{\boldsymbol{z}}(t) = \rho_{\boldsymbol{z}(t)}, \tag{23}$$

with

$$z_k(t) = e^{-i\omega_k t} z_k(0). \tag{24}$$

The symplectic tomographic distribution corresponding to $\rho_{\boldsymbol{z}}(\boldsymbol{\zeta},\bar{\boldsymbol{\zeta}})$ is a product

$$\mathcal{W}_{\rho_{\boldsymbol{z}}}(\boldsymbol{X},\boldsymbol{\mu},\boldsymbol{\nu},\boldsymbol{z}) = \prod_{k=1}^{n} \mathcal{W}_{\rho_{\boldsymbol{z}}}^{(k)}(X_k,\mu_k,\nu_k,z_k), \tag{25}$$

where the tomogram $\mathcal{W}_{\rho_{\boldsymbol{z}}}^{(k)}$ of a single degree of freedom is a Gaussian distribution

$$\mathcal{W}_{\rho_{\boldsymbol{z}}}^{(k)}(X_k,\mu_k,\nu_k,z_k) = \sqrt{\frac{\omega_k}{2\pi\left(\mu_k^2+\nu_k^2\right)}} \exp\left[-\frac{\omega_k\left(X_k - \langle X_k(\mu_k,\nu_k,z_k)\rangle\right)^2}{2(\mu_k^2+\nu_k^2)}\right] \tag{26}$$

of the random variable X_k, with mean value

$$\langle X_k(\mu_k,\nu_k,z_k)\rangle = \mu_k\Re(z_k) + \nu_k\Im(z_k) \tag{27}$$

and variance given by

$$\sigma_{X_kX_k} = \frac{2(\mu_k^2+\nu_k^2)}{\omega_k}. \tag{28}$$

3.2. A new class of states: Laguerre states

We will consider now the family of classical states, inspired on the distribution probabilities of the bounded states of an electron moving on a plane in a perpendicular magnetic field, given by:

$$\rho_{\mathrm{Laguerre},\{m\}}(\boldsymbol{\xi},\boldsymbol{\eta})=\prod_{k=1}^{n}\rho^{(k)}_{\mathrm{Laguerre},m_k}(\xi_k,\eta_k), \tag{29}$$

where $\{m\}=\{m_1,m_2,\ldots,m_n\}$ is a multi–index and

$$\rho^{(k)}_{\mathrm{Laguerre},m_k}(\xi_k,\eta_k)=\frac{\omega_k}{2\pi}\left[L_{m_k}\left(\frac{\omega_k}{2}(\xi_k^2+\eta_k^2)\right)\right]^2\mathrm{e}^{-\frac{1}{2}\omega_k(\xi_k^2+\eta_k^2)}. \tag{30}$$

Here L_{m_k} is the Laguerre polynomial of degree m_k and the exponential is the not normalized Gibbs state $\varrho_{\mathrm{can}}|_{\beta=1}$. Notice that $\rho^{(k)}_{\mathrm{Laguerre},m_k}(\xi_k,\eta_k)$ is a classical state on a bidimensional phase space.

The symplectic Radon transform of the state factorizes:

$$\mathcal{W}_{\mathrm{Laguerre},\{m\}}(\boldsymbol{X},\boldsymbol{\mu},\boldsymbol{\nu})=\prod_{k=1}^{n}\mathcal{W}^{(k)}_{\mathrm{L},m_k}(X_k,\mu_k,\nu_k). \tag{31}$$

and we will obtain

$$\mathcal{W}^{(k)}_{\mathrm{L},m_k}(X_k,\mu_k,\nu_k)=\frac{\exp\left[-\frac{X_k^2}{\sigma_k^2}\right]}{\pi^{\frac{1}{2}}\sigma_k}\sum_{s=0}^{m_k}\frac{1}{2^{2m_k}}\binom{2\,(m_k-s)}{m_k-s}\binom{2s}{s}\frac{\left[H_{2s}\left(\frac{X_k}{\sigma_k}\right)\right]^2}{2^{2s}\,(2s)!} \tag{32}$$

with

$$\sigma_k=\sqrt{\frac{2\,(\mu_k^2+\nu_k^2)}{\omega_k}}, \tag{33}$$

while H_{2s} is the Hermite polynomial of degree $2s$. The above result can be obtained as follows.

First, we drop the label k and write $\mathcal{W}_m(X,\mu,\nu)$ in place of $\mathcal{W}^{(k)}_{\mathrm{L},m_k}(X_k,\mu_k,\nu_k)$. Thus

$$\begin{aligned}\mathcal{W}_m(X,\mu,\nu) &= \frac{\omega}{2\pi}\int L_m^2\left(\frac{\omega}{2}(\xi^2+\eta^2)\right)\mathrm{e}^{-\frac{1}{2}\omega(\xi^2+\eta^2)}\delta\left(X-\mu\xi-\nu\eta\right)\mathrm{d}\xi\mathrm{d}\eta \\ &= \frac{\omega}{(2\pi)^2}\int \mathrm{d}K\mathrm{e}^{\mathrm{i}KX}\int L_m^2\left(\frac{\omega}{2}(\xi^2+\eta^2)\right)\mathrm{e}^{-\frac{1}{2}\omega(\xi^2+\eta^2)}\mathrm{e}^{-\mathrm{i}K(\mu\xi+\nu\eta)}\mathrm{d}\xi\mathrm{d}\eta.\end{aligned} \tag{34}$$

Now we put $\sqrt{\mu^2+\nu^2}=r_{\mu\nu},\mu=r_{\mu\nu}\cos\alpha_{\mu\nu},\nu=r_{\mu\nu}\sin\alpha_{\mu\nu}$, and $\xi=r\sin\theta,\eta=r\cos\theta$. Then, we recast the previous formula as

$$\mathcal{W}_m(X,\mu,\nu)=\frac{1}{2\pi}\int \mathrm{d}K\mathrm{e}^{\mathrm{i}KX}\tilde{\mathcal{W}}_m(K,\mu,\nu) \tag{35}$$

where the characteristic function of $\mathcal{W}_m$, i.e. its Fourier transform $\tilde{\mathcal{W}}_m$, is given by

$$\tilde{\mathcal{W}}_m(K,\mu,\nu)=\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\int_0^{\infty}\left[L_m\left(\frac{\omega r^2}{2}\right)\right]^2\mathrm{e}^{-\frac{\omega r^2}{2}}\mathrm{e}^{-\mathrm{i}(Kr_{\mu\nu})r\sin(\theta+\alpha_{\mu\nu})}\mathrm{d}\left(\frac{\omega r^2}{2}\right). \tag{36}$$

The integral over the angular variable $\theta_{\mu\nu}=\theta+\alpha_{\mu\nu}$ yields the Bessel function J_0, so:

$$\tilde{\mathcal{W}}_m(K,\mu,\nu)=\int_0^{\infty}\left[L_m\left(\frac{x^2}{2}\right)\right]^2\mathrm{e}^{-\frac{x^2}{2}}J_0\left(\frac{Kr_{\mu\nu}}{\sqrt{\omega}}x\right)\mathrm{d}\left(\frac{x^2}{2}\right). \tag{37}$$

The above integral can be evaluated and gives ([16], n. 7.422 2)

$$\begin{aligned}\tilde{\mathcal{W}}_m(K,\mu,\nu) &= \mathrm{e}^{-\frac{1}{2}\left(\frac{Kr_{\mu\nu}}{\sqrt{\omega}}\right)^2}\left[L_m\left(\frac{1}{2}\left(\frac{Kr_{\mu\nu}}{\sqrt{\omega}}\right)^2\right)\right]^2 \\ &= \mathrm{e}^{-\frac{1}{2}\left(\frac{Kr_{\mu\nu}}{\sqrt{\omega}}\right)^2}\frac{1}{2^{2m}}\sum_{s=0}^{m}\binom{2\,(m-s)}{m-s}\binom{2s}{s}L_{2s}\left(\left(\frac{Kr_{\mu\nu}}{\sqrt{\omega}}\right)^2\right),\end{aligned} \tag{38}$$

where the last line has been obtained by a well known addition formula of Laguerre polynomials ([16], n. 8.976 3).

We remark that the above equation yields, by multiplication over the restored label k, the characteristic function $\tilde{\mathcal{W}}_{\mathrm{Laguerre},\{m\}}(\boldsymbol{K},\boldsymbol{\mu},\boldsymbol{\nu})$, with $\boldsymbol{K}=(K_1,\ldots,K_k,\ldots K_n)$, of the tomogram $\mathcal{W}_{\mathrm{Laguerre},\{m\}}(\boldsymbol{X},\boldsymbol{\mu},\boldsymbol{\nu})$.

Besides, as $\tilde{\mathcal{W}}_m(K=0,\mu,\nu)=1$, we get at once the normalization property of the tomogram $\mathcal{W}_m(X,\mu,\nu)$.

Finally, we are able to perform the last integration. The Fourier anti–transform of $\tilde{\mathcal{W}}_m(K,\mu,\nu)$ is obtained by means of the integral over $y=Kr_{\mu\nu}/\sqrt{\omega}$ ([16], n. 7.418 2):

$$\frac{1}{\pi}\frac{\sqrt{\omega}}{r_{\mu\nu}}\int_0^\infty L_{2s}\left(y^2\right)\mathrm{e}^{-\frac{1}{2}y^2}\cos\left(\frac{\sqrt{\omega}}{r_{\mu\nu}}Xy\right)\mathrm{d}y=\frac{\sqrt{\omega}}{\sqrt{2\pi}r_{\mu\nu}}\mathrm{e}^{-\frac{\omega}{2(r_{\mu\nu})^2}X^2}\frac{1}{2^{2s}\,(2s)!}\left[H_{2s}\left(\frac{\sqrt{\omega}}{\sqrt{2}r_{\mu\nu}}X\right)\right]^2. \tag{39}$$

So, we get the predicted expression of $\mathcal{W}_m(X,\mu,\nu)$.

4. The tomographic picture of Liouville's equation

Finally, let us discuss the tomographic form of the evolution equation for states, Liouville equation (16). The evolution equation in the tomographic description was recently obtained in [17] in relation with a relativistic wave function description of harmonic oscillators. We will describe it here in the realm of our previous discussion. Notice that because of the symplectic reconstruction formula for a classical state (12) we can compute:

$$\frac{\partial}{\partial t}\rho(\boldsymbol{\xi},\boldsymbol{\eta},t)=\int_{\mathbb{R}^{3n}}\exp\left[\mathrm{i}\sum_{k=1}^{n}(X_k-\mu_k\xi_k-\nu_k\eta_k)\right]\frac{\partial}{\partial t}\mathcal{W}_\rho(\boldsymbol{X},\boldsymbol{\mu},\boldsymbol{\nu},t)\mathrm{d}^nX\frac{\mathrm{d}^n\mu\mathrm{d}^n\nu}{(2\pi)^{2n}}, \tag{40}$$

(notice that the symplectic tomogram is computed at a given fixed time) and, on the other hand:

$$\begin{aligned}\{\rho,H\} &= \sum_{k=1}^{n}\left[\frac{\partial H}{\partial\eta_k}\frac{\partial}{\partial\xi_k}-\frac{\partial H}{\partial\xi_k}\frac{\partial}{\partial\eta_k}\right]\rho \\ &= \sum_{k=1}^{n}\int_{\mathbb{R}^{3n}}\mathrm{d}^nX\frac{\mathrm{d}^n\mu\mathrm{d}^n\nu}{(2\pi)^{2n}}\mathcal{W}_\rho(\boldsymbol{X},\boldsymbol{\mu},\boldsymbol{\nu},t)\left[\frac{\partial H}{\partial\eta_k}\frac{\partial}{\partial\xi_k}-\frac{\partial H}{\partial\xi_k}\frac{\partial}{\partial\eta_k}\right]\exp\left[\mathrm{i}\sum_{j=1}^{n}(X_j-\mu_j\xi_j-\nu_j\eta_j)\right] \\ &= \sum_{k=1}^{n}\int_{\mathbb{R}^{3n}}\mathrm{d}^nX\frac{\mathrm{d}^n\mu\mathrm{d}^n\nu}{(2\pi)^{2n}}\mathcal{W}_\rho(\boldsymbol{X},\boldsymbol{\mu},\boldsymbol{\nu},t)\left[\frac{\partial H}{\partial\xi_k}\nu_k\frac{\partial}{\partial X_k}-\frac{\partial H}{\partial\eta_k}\mu_k\frac{\partial}{\partial X_k}\right]\exp\left[\mathrm{i}\sum_{j=1}^{n}(X_j-\mu_j\xi_j-\nu_j\eta_j)\right].\end{aligned} \tag{41}$$

Eventually, we obtain the evolution equation for the classical tomogram $\mathcal{W}_\rho$:

$$\begin{aligned}&\frac{\partial\mathcal{W}_\rho(\boldsymbol{X},\boldsymbol{\mu},\boldsymbol{\nu},t)}{\partial t}= \\ &\sum_{k=1}^{n}\left[\frac{\partial H}{\partial\eta_k}\left(\left\{\xi_j\to-\left[\frac{\partial}{\partial X_j}\right]^{-1}\frac{\partial}{\partial\mu_j}\right\},\left\{\eta_j\to\left[\frac{\partial}{\partial X_j}\right]^{-1}\frac{\partial}{\partial\nu_j}\right\}\right)\mu_k\frac{\partial}{\partial X_k}\mathcal{W}_\rho(\boldsymbol{X},\boldsymbol{\mu},\boldsymbol{\nu},t)\right. \\ &\left.-\frac{\partial H}{\partial\xi_k}\left(\left\{\xi_j\to-\left[\frac{\partial}{\partial X_j}\right]^{-1}\frac{\partial}{\partial\mu_j}\right\},\left\{\eta_j\to\left[\frac{\partial}{\partial X_j}\right]^{-1}\frac{\partial}{\partial\nu_j}\right\}\right)\nu_k\frac{\partial}{\partial X_k}\mathcal{W}_\rho(\boldsymbol{X},\boldsymbol{\mu},\boldsymbol{\nu},t)\right].\end{aligned} \tag{42}$$

Notice that the arguments $\{\xi_j\}, \{\eta_j\}$ of the derivatives of H, for any j, are replaced by the operators $\left\{-\left[\frac{\partial}{\partial X_j}\right]^{-1}\frac{\partial}{\partial \mu_j}\right\}, \left\{\left[\frac{\partial}{\partial X_j}\right]^{-1}\frac{\partial}{\partial \nu_j}\right\}$, respectively. Explicitly, the operator $\left[\frac{\partial}{\partial X}\right]^{-1}$ is defined in terms of a Fourier transform as

$$\left[\frac{\partial}{\partial X}\right]^{-1}\int_{\mathbb{R}} f(K)\exp(\mathrm{i}KX)\,dK = \int_{\mathbb{R}}\frac{f(K)}{\mathrm{i}K}\exp(\mathrm{i}KX)\,\mathrm{d}K. \tag{43}$$

Due to the presence of such terms, for a generic Hamiltonian H the evolution tomographic equation is integro-differential. In the particular instance of H given by (15), because of the general correspondence rule:

$$\frac{\partial}{\partial \xi_k}\rho \leftrightarrow \mu_k\frac{\partial}{\partial X_k}\mathcal{W}_\rho \quad , \quad \frac{\partial}{\partial \eta_k}\rho \leftrightarrow \mu_k\frac{\partial}{\partial X_k}\mathcal{W}_\rho, \tag{44}$$

the tomographic evolution equation takes the form of a differential equation:

$$\begin{aligned}\frac{\partial \mathcal{W}_\rho(\boldsymbol{X},\boldsymbol{\mu},\boldsymbol{\nu},t)}{\partial t} &= \sum_{k=1}^{n}\omega_k\left[\mu_k\frac{\partial}{\partial \nu_k}-\nu_k\frac{\partial}{\partial \mu_k}\right]\mathcal{W}_\rho(\boldsymbol{X},\boldsymbol{\mu},\boldsymbol{\nu},t) \qquad (45)\\ = \quad \sigma\left(\{\mu_k,\nu_k\}_k, \left\{\xi_k\to\omega_k\frac{\partial}{\partial \mu_k}, \eta_k\to\omega_k\frac{\partial}{\partial \nu_k}\right\}_k\right)&\mathcal{W}_\rho(\boldsymbol{X},\boldsymbol{\mu},\boldsymbol{\nu},t),\end{aligned}$$

where σ is the canonical symplectic form on the linear space $E=\mathbb{R}^{2n}$.

5. The tomogram of the real Klein-Gordon field on a cavity

Having shown that an interesting family of states for a finite ensemble of harmonic oscillators is amenable to be described tomographically, we will discuss now the Klein–Gordon equation for a real scalar field $\varphi(\mathrm{x})$ in a finite cavity on $1+d$ Minkowski space–time. Thus we consider Minkowski space–time $\mathbb{M}=\mathbb{R}^{1+d}$ with metric of signature $(+,-,\cdots,-)$. Points in space–time will be written as $\mathrm{x}=(t,x)$ The dynamics of the real scalar field $\varphi(\mathrm{x})=\varphi(t,x)$ is defined by the Lagrangian density:

$$\mathcal{L}[\varphi] = \frac{1}{2}\left(\partial_\mu\varphi\partial^\mu\varphi - V[\varphi]\right), \tag{46}$$

with Euler–Lagrange equations:

$$\partial_\mu\partial^\mu\varphi = -V'[\varphi]. \tag{47}$$

Considering $V[\varphi]=m^2\varphi^2$ we get the Klein–Gordon equation:

$$\varphi_{tt}-\Delta\varphi+m^2\varphi=0, \tag{48}$$

with Δ the d–dimensional Laplacian in $\mathbb{R}^d$. As we have extensively seen, tomographic methods are described on phase space where conjugated variables and Poisson brackets are available. On this carrier space dynamical equations are described by a vector field, first order differential equations in time. Thus, for our Klein–Gordon equations we have to introduce a larger carrier space where the equations will be first order in time. The transition from second order equations in time to first order differential equations in time may be done in many ways [18], here we shall consider one in which the new variables will make the equations of motion more symmetric. We would stress that by using a specific splitting of spacetime into a space part and a time part we break the explicit Poincarè invariant form but of course our description is still relativistic invariant. To proceed, we will consider the Cauchy hypersurface $\mathcal{C}=\{0\}\times\mathbb{R}^d$ and the finite cavity will be defined as $\mathcal{V}\subset\mathcal{C}$. We consider the restriction of the field to the cavity $\mathcal{V}$ using the same notation $\varphi(x):=\varphi(0,x), x\in\mathcal{V}$ and the Klein–Gordon equation becomes the evolution equation in the space of fields $\varphi(x)$:

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = -(-\Delta+m^2)\varphi. \tag{49}$$

Boundary conditions at the boundary of the cavity $\mathcal{V}$ are chosen such that the operator $-\Delta + m^2$ is strictly–positive and self-adjoint on square integrable functions on $\mathcal{V}$ with respect to the Lebesgue measure, thus we can define the invertible positive self–adjoint operator $B = \sqrt{-\Delta + m^2}$. We will also assume for simplicity that boundary conditions are chosen in such a way that the spectrum of B is nondegenerate, so that the eigenvalues of B will be $0 < \omega_1 < \omega_2 < \ldots < \omega_n < \ldots$ with eigenfunctions $\Phi_k(x)$, $B\Phi_k(x) = \omega_k \Phi_k(x)$, $k = 1, 2, \ldots$. Thus equation (49) may be transformed into a first order evolution differential equation by introducing the new fields:

$$\xi = B^{1/2}\varphi \quad ; \quad \eta = B^{-1/2}\varphi_t. \tag{50}$$

(notice that $B^{-1/2}$ is well–defined because B is positive and invertible) and the equations of motion (49) for the field φ take the simple symmetric form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}0 & B\\ -B & 0\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}. \tag{51}$$

Thus the equations of motion for the Klein–Gordon field constitute an infinite dimensional extension of the dynamics of a finite number of independent oscillators (14). Using the Fourier expansion of the fields ξ and η with respect to the eigenfunctions Φ_k of B, $\xi(x) = \sum_{k=1}^{\infty} \xi_k \Phi_k(x), \eta(x) = \sum_{k=1}^{\infty} \eta_k \Phi_k(x)$, then, the mechanical variables $q_k = \sqrt{\omega_k}\xi_k$ and $p_k = \eta_k/\sqrt{\omega_k}$ can be interpreted as position and momentum for a one–dimensional oscillator of frequency ω_k and their evolution in time, given by eq. (13), as a trajectory in phase space $\Omega = \mathbb{R}^{2\infty}$. In the presence of field fluctuations we have to introduce a statistical interpretation to the mechanical degrees of freedom (q_k, p_k) or (ξ_k, η_k) of the field $\varphi(x)$, thus the classical statistical description of the field whose physical meaning corresponds to the probability of a certain fluctuation of the field to take place, will be provided by a probability law ρ on the infinite dimensional phase space $\mathbb{R}^{2\infty}$. Thus in the presence of field fluctuations the state of the field will induce a marginal probability density on each mode $\rho_k(q_k, p_k)$ defined by,

$$\rho_k(q_k, p_k) = \int \rho(q_1, q_2, \ldots, q_k, \ldots; p_1, p_2, \ldots, p_k, \ldots) \prod_{l \neq k} \mathrm{d}q_l \mathrm{d}p_l. \tag{52}$$

Such marginal probability could be understood as a probability density for the k–th mode of the field φ described by the one–dimensional oscillator with Hamiltonian $H_k(\xi_k, \eta_k)$. Similar considerations could be applied to finite dimensional subspaces of modes of the field whose statistical and tomographic description would be made as in the previous section.

The canonical or Gibbs state for the field $\varphi(x)$ is given by the probability distribution on the infinite dimensional phase space of the system as:

$$\rho_{\mathrm{can}}(\xi_1, \xi_2, \ldots; \eta_1, \eta_2, \ldots) = N \exp\left[-\frac{1}{2}\beta \sum_{k \geq 1} \omega_k (\xi_k^2 + \eta_k^2)\right] \tag{53}$$

with the normalization constant N to be determined by regularizing the integral:

$$\int e^{-\frac{1}{2}\beta \sum_{k\geq 1} \omega_k (\xi_k^2 + \eta_k^2)} \prod_{k=1}^{\infty} \mathrm{d}\xi_k \mathrm{d}\eta_k = \left[\det\left(\frac{1}{2}\beta B\right)\right]^{-1}, \tag{54}$$

what amounts to define the determinant of the operator B by using the ζ–function regularization of determinants, i.e.,

$$\det\left(\frac{1}{2}\beta B\right) = \exp\left[\zeta'_{\frac{1}{2}\beta B}(0)\right], \tag{55}$$

with

$$\zeta_{\frac{1}{2}\beta B}(s) = \sum_{k=1}^{\infty} \left(\frac{1}{2}\beta \omega_k\right)^{-s}. \tag{56}$$

In other words, the canonical ensemble for the real scalar Klein–Gordon field $\varphi(x)$ is defined as the Gaussian measure with variance $C = (\frac{1}{2}\beta B)^{-1}$ on $\mathbb{R}^{2\infty}$. Notice that

$$\begin{aligned} H[\boldsymbol{\xi},\boldsymbol{\eta}] &= \frac{1}{2}\sum_{k=1}^{\infty}\omega_k(\xi_k^2+\eta_k^2) = \frac{1}{2}\left|\left|B\varphi\right|\right|^2 + \frac{1}{2}\left|\left|\varphi_t\right|\right|^2 = H[\varphi] \\ &= \frac{1}{2}\int_{\mathcal{V}}(\partial_\mu\varphi\partial^\mu\varphi + m^2\varphi^2)\mathrm{d}^d x, \end{aligned} \tag{57}$$

with $\frac{1}{2}||B\varphi||^2$ denoting the potential $U[\varphi]$ of the Klein–Gordon field in the Hamiltonian picture. Observe that $U[\varphi]$ can also be written as:

$$U[\varphi] = \frac{1}{2}||B\varphi||^2 = \frac{1}{2}\langle\varphi, B^2\varphi\rangle = \frac{1}{2}\int_{\mathcal{V}}\varphi(x)(-\Delta+m^2)\varphi(x)\mathrm{d}^d x. \tag{58}$$

Then the canonical ensemble for the Klein–Gordon field at finite temperature will be written in the usual form:

$$\mathrm{d}\mu_{\mathrm{can}}[\varphi] = N\mathrm{e}^{-\frac{\beta}{2}\int_{\mathcal{V}}(\partial_i\varphi\partial^i\varphi+m^2\varphi^2)\mathrm{d}^d x}\mathcal{D}\varphi \tag{59}$$

with $\mathcal{D}\varphi = \prod_{k=1}^{\infty}\mathrm{d}q_k\mathrm{d}p_k$. Moreover, if $F[\varphi]$ denotes an observable on the field φ (like the energy, momentum, etc.), then the expected value of F on the canonical distribution at temperature β will be given by:

$$\langle F\rangle_{\mathrm{can}} = \frac{\int F[\varphi]\,\mathrm{e}^{-\beta H[\varphi]}\mathcal{D}\varphi}{\int \mathrm{e}^{-\beta H[\varphi]}\mathcal{D}\varphi}. \tag{60}$$

The tomographic description of the states of the Klein–Gordon field will be performed as in the case of an ensemble of harmonic oscillators in section 4 by choosing the space $\mathcal{M}$ the phase space $\mathbb{R}^{2\infty}$ itself and $\mathcal{N} = \prod_{k=1}^{\infty}\mathcal{N}_k$ with $\mathcal{N}_k$ the space of straight lines on the phase space of the one–dimensional oscillator (ξ_k, η_k). Then, as in (19), we will define:

$$\begin{aligned} \mathcal{W}_{\rho_{\mathrm{can}}}[\boldsymbol{X},\boldsymbol{\mu},\boldsymbol{\nu}] &= \int \rho_{\mathrm{can}}[\boldsymbol{\xi},\boldsymbol{\eta}]\prod_{k=1}^{\infty}\delta(X_k-\mu_k\xi_k-\nu_k\eta_k)d\xi_k d\eta_k \\ &= \int e^{-\beta H[\boldsymbol{\xi},\boldsymbol{\eta}]}\delta\left[X(x)-\mu(x)\xi(x)-\nu(x)\eta(x)\right]\mathcal{D}\xi\,\mathcal{D}\eta \end{aligned} \tag{61}$$

Here the Dirac functional distribution must be understood as an infinite continuous product:

$$\begin{aligned} \delta\left[X(x)-\mu(x)\xi(x)-\nu(x)\eta(x)\right] &= \prod_k \delta\left(X_k-\mu_k\xi_k-\nu_k\eta_k\right) \\ = \int \exp\left[i\int K(x)\left(X(x)-\mu(x)\xi(x)-\nu(x)\eta(x)\right)\mathrm{d}^d x\right]\mathcal{D}K, \end{aligned} \tag{62}$$

where $X(x)$, $\mu(x)$ and $\nu(x)$ are fields whose expansion on the modes ω_k of the field $\varphi(x)$ are given respectively by:

$$X(x) = \sum_{k=1}^{\infty}X_k\Phi_k(x); \quad \mu(x) = \sum_{k=1}^{\infty}\mu_k\Phi_k(x); \quad \nu(x) = \sum_{k=1}^{\infty}\nu_k\Phi_k(x). \tag{63}$$

Notice that the time dependence of the various fields is encoded in the coefficients of the corresponding expansions. Taking advantage again of the scaling property of the delta function we may use the natural parametrization of optical tomograms defined by the reparametrization $\tilde{\mu}_k = \mu_k/\sqrt{\mu_k^2+\nu_k^2} = \cos\theta_k$, $\tilde{\eta}_k = \nu_k/\sqrt{\mu_k^2+\nu_k^2} = \sin\theta_k$, $\tilde{X}_k = X_k/\sqrt{\mu_k^2+\nu_k^2}$ and after standard computations we get:

$$\mathcal{W}^{\mathrm{opt}}_{\rho_{\mathrm{can}}}(\tilde{\boldsymbol{X}},\boldsymbol{\theta}) = N\mathrm{e}^{-\sum_{k=1}^{\infty}\tilde{X}_k^2} = N\mathrm{e}^{-\int_{\mathcal{V}}\tilde{X}(x)^2\mathrm{d}^d x} = N\mathrm{e}^{-||\tilde{\boldsymbol{X}}||^2}. \tag{64}$$

with $\theta(x) = \tan^{-1}[\eta(x)/\xi(x)]$ and the normalization constant N defined by choosing a proper regularization of the trace of the operator B.

6. Tomographic picture of continuous modes

If we consider the scalar field in a infinite volume cavity or in the full Minkowski space–time for instance, many or all of the modes of the system will become continuous. For simplicity we will assume that we are discussing the field in the $d+1$ Minkowski space–time $\mathbb{R}^{d+1}$ and the continuous modes of the fields $\varphi(x)$, $\xi(x)$, $\eta(x)$ are described by the wave vector k, this is:

$$\xi(x) = \frac{1}{(2\pi)^{d/2}} \int \left(\xi_k \mathrm{e}^{-\mathrm{i}k\cdot x} + \xi_{-k} \mathrm{e}^{\mathrm{i}k\cdot x}\right) \mathrm{d}^d k, \tag{65}$$

etc. Now a state of the field $\varphi(x)$ will be represented by a probability measure $\rho\,[\xi, \eta]$, again nonnegative and normalized. An example of such state will be given by the canonical ensemble, this is the Gaussian measure whose covariance is the operator B as in (59):

$$d\mu_{\mathrm{can}}\,[\varphi] = \mathrm{e}^{-\beta H[\varphi]} \mathcal{D}\varphi = \mathrm{e}^{-\beta H[\xi,\eta]} \mathcal{D}\xi \mathcal{D}\eta \tag{66}$$

with the normalization constant absorbed in the definition of the measure.

We will consider as analogue of Gibbs states, states that are absolutely continuous with respect to the canonical state, i.e., states of the form:

$$\rho\,[\varphi] = f\,[\xi(x), \eta(x)]\,\mu_{\mathrm{can}} \tag{67}$$

with

$$f\,[\xi(x), \eta(x)] \geq 0 \tag{68}$$

$$\int f\,[\xi(x), \eta(x)]\, e^{-\beta H[\xi,\eta]} \mathcal{D}\xi \mathcal{D}\eta = 1. \tag{69}$$

Even though at a formal level, we may introduce as in (61) a tomographic probability density for a state of the field of the form (67) as a functional of three auxiliary tomographic fields $X(x), \xi(x), \eta(x)$ and apply, at the functional level, the usual Radon transform. The expansions (63) will be replaced by the Fourier transform:

$$X(x) = \frac{1}{(2\pi)^{d/2}} \int \left(X_k \mathrm{e}^{-\mathrm{i}k\cdot x} + X_{-k} \mathrm{e}^{\mathrm{i}k\cdot x}\right) \mathrm{d}^d k, \quad \text{etc.} \tag{70}$$

Then,

$$\mathcal{W}_f\,[X(x), \mu(x), \nu(x)] = \int f\,[\xi(x), \eta(x)]\, \delta\,[X(x) - \mu(x)\xi(x) - \nu(x)\eta(x)]\, e^{-\beta H[\xi,\eta]} \mathcal{D}\xi \mathcal{D}\eta. \tag{71}$$

The inverse Radon transform maps the tomographic probability density given by (71) onto the probability density functional

$$\rho_f\,[\xi, \eta] = \int \mathcal{W}_f\,[X, \mu, \nu] \exp\left[\mathrm{i}(X(x) - \mu(x)\xi(x) - \nu(x)\eta(x))\right] \mathcal{D}X(x) \mathcal{D}\mu(x) \mathcal{D}\nu(x) \tag{72}$$

The tomographic probability functional (71) has the properties of nonnegativity and normalization, i.e.

$$\mathcal{W}_f\,[X(x), \mu(x), \nu(x)] \geq 0 \tag{73}$$

$$\int \mathcal{W}_f\,[X(x), \mu(x), \nu(x)]\, \mathcal{D}X(x) = 1. \tag{74}$$

These formulas hold true for any value of the auxiliary fields $X(x), \mu(x), \nu(x)$.

In the current case the manifold $\mathcal{N}$ used to construct the generalized Radon transform is described by the tomographic fields $X(x), \nu(x), \mu(x)$, which would be a continuum version of the finite–mode version of the straight lines:

$$X_k - \mu_k \xi_k - \nu_k \eta_k = 0. \tag{75}$$

We will end this discussion by emphasizing again the homogeneity property of the tomographic description of the scalar field we just presented, homogeneity that is described by the condition:

$$\left[X(x)\frac{\delta}{\delta X(x)}+\mu(x)\frac{\delta}{\delta\mu(x)}+\nu(x)\frac{\delta}{\delta\nu(x)}+1\right]\mathcal{W}_f\left[X(x),\mu(x),\nu(x)\right]=0 \tag{76}$$

7. The tomographic picture of evolution equation for classical fields

Thus the state of the classical scalar field $\varphi(x)$ can be described either by a probability density functional $f\left[\xi(x),\eta(x)\right]$ on the field phase–space or by the tomographic probability density functional $\mathcal{W}_f\left[X(x),\mu(x),\nu(x)\right]$. Both probability density functionals are connected by the invertible functional Radon transform (71), (72) and in view of this, they both contain equivalent information on the random field states. The dynamical evolution of states of the field $\varphi(t,x)$ will be determined by the Klein–Gordon equation (51)

If the Hamiltonian providing the evolution of the field is given by the sum of kinetic and potential energy

$$\mathcal{H}\left[\varphi\right]=\frac{1}{2}\int\left(\dot{\varphi}(x)^2+V\left[\varphi(x)\right]\right)\mathrm{d}^dx=\frac{1}{2}||\dot{\varphi}||^2+U\left[\varphi\right], \tag{77}$$

the evolution of the probability density functional on the classical phase–space of the field obeys a Liouville functional differential equation:

$$\frac{\mathrm{d}f}{\mathrm{d}t}=\{f,\mathcal{H}\}. \tag{78}$$

The functional Poisson brackets above are given by:

$$\left\{F\left[\xi,\eta\right],G\left[\xi,\eta\right]\right\}=\int\left(\frac{\delta F}{\delta\xi(x)}\left\{\xi(x),\eta(y)\right\}\frac{\delta G}{\delta\eta(y)}+\frac{\delta F}{\delta\eta(x)}\left\{\eta(x),\xi(y)\right\}\frac{\delta G}{\delta\xi(y)}\right)\mathrm{d}^dx\mathrm{d}^dy, \tag{79}$$

where the fields $\xi(x)$, $\eta(y)$ satisfy the canonical commutation relations:

$$\left\{\xi(x),\eta(y)\right\}=\delta^d(x-y). \tag{80}$$

Then we obtain for the Hamiltonian $\mathcal{H}$ above (77) the expression:

$$\frac{\mathrm{d}}{\mathrm{d}t}f\left[\xi,\eta\right]+\int\eta(x)\frac{\delta f\left[\xi,\eta\right]}{\delta\xi(x)}\mathrm{d}^dx-\int\frac{\delta V\left[\xi,\eta\right]}{\delta\xi(x)}\frac{\delta f\left[\xi,\eta\right]}{\delta\eta(x)}\mathrm{d}^dx=0 \tag{81}$$

which is just the $n\to\infty$ limit of the Liouville equation for finite number of field modes discussed in section 4.

In the case that $V\left[\varphi\right]=0$ we have the functional Liouville equation

$$\frac{\mathrm{d}}{\mathrm{d}t}f\left[\xi,\eta\right]+\int\eta(x)\frac{\delta f\left[\xi,\eta\right]}{\delta\xi(x)}\mathrm{d}^dx=0. \tag{82}$$

and the corresponding tomographic form of this equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_f\left[X,\mu,\nu\right]-\int\mu(x)\frac{\delta\mathcal{W}_f\left[X,\mu,\nu\right]}{\delta\nu(x)}\mathrm{d}^dx=0. \tag{83}$$

To get this equation starting from (78) we used the correspondences:

$$\begin{aligned}&\frac{\delta}{\delta\xi(x)}\leftrightarrow\mu(x)\frac{\delta}{\delta X(x)};\frac{\delta}{\delta\eta(x)}\leftrightarrow\nu(x)\frac{\delta}{\delta X(x)}\\&\xi(x)\leftrightarrow-\frac{\delta}{\delta\mu(x)}\left[\frac{\delta}{\delta X(x)}\right]^{-1};\eta(x)\leftrightarrow-\frac{\delta}{\delta\nu(x)}\left[\frac{\delta}{\delta X(x)}\right]^{-1}.\end{aligned} \tag{84}$$

These relations correspond to a realization of the infinite Heisenberg–Weyl algebra generators (and enveloping algebra) on the field phase–space and the map of the representation in terms of the generator action onto the tomograms.

The rule (84) provide a possibility to construct the tomographic form of the Liouville equation (81). Using the substitution $f \to \mathcal{W}_f$ in (81), and the substitutions (84), we get the field evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_f\left[X,\mu,\nu\right] = \int \mathrm{d}^d x' \mu(x') \frac{\delta \mathcal{W}_f\left[X(x),\mu(x),\nu(x)\right]}{\delta \nu(x')}$$
$$+ \int \mathrm{d}^d x' \left[\frac{\delta V}{\delta \xi(x')}\left(\xi\left(x\right) \to -\frac{\delta}{\delta \mu(x)}\left[\frac{\delta}{\delta X(x)}\right]^{-1}\right)\nu(x')\frac{\delta}{\delta X(x')}\right]\mathcal{W}_f\left[X,\mu,\nu\right]. \tag{85}$$

For the case of field which is a collection of noninteracting oscillators described by the potential energy

$$V\left[\varphi\right] = \frac{1}{2}m^2\varphi^2 \tag{86}$$

then (85) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_f = \int \mathrm{d}^d x \left(\mu(x)\frac{\delta \mathcal{W}_f}{\delta \nu(x)} - \nu(x)\frac{\delta \mathcal{W}_f}{\delta \mu(x)}\right). \tag{87}$$

which is the equivalent of eq. (45) to the continuous scalar field φ.

8. Conclusion and perspectives

A proposal for the tomographic description of a family of statistical states for a classical real scalar Klein-Gordon field has been presented inspired by the tomographic description of statistical states for an ensemble of harmonic oscillators. This tomographic description of classical fields shares most of the tomographic properties of tomograms for classical states: homogeneity, positivity and normalization. Moreover the field equations, represented as the evolution equation for field states, are reproduced in tomographic terms, paving the way towards a tomographic description of the quantum scalar field. Notice that the tomographic description presented in this work, a natural extension of Radon transform, breaks the Lorentz covariance of the field theory, thus the Lorentz covariance of the tomographic description should be restored at the end. Lorentz covariance, as well as gauge invariance (when interactions are introduced), should be incorporated as a natural ingredient in the tomographic picture. The tomographic picture of other fields like Maxwell, Dirac, Proca, Einstein could be addressed following similar arguments. Such issues will be discussed in subsequent works.

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