Instantaneous and finite time blow-up of solutions to a reaction-diffusion equation with Hardy-type singular potential

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Abstract

We deal with radially symmetric solutions to the reaction-diffusion equation with Hardy-type singular potential

$$u_t = \Delta u^m + \frac{K}{|x|^2} u^m,$$

posed in $\mathbb{R}^N \times (0, T)$, in dimension $N \geq 3$, where $m > 1$ and $0 < K < (N-2)^2/4$. We prove that, in dependence of the initial condition $u_0 \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, its solutions may either *blow up instantaneously* or blow up in finite time at the origin, thus developing a singularity at $x = 0$, but they can be continued globally in weak sense. The instantaneous blow-up occurs for example for any data u_0 such that $u_0(0) > 0$. The proofs are based on a transformation mapping solutions to our equation into solutions to a non-homogeneous porous medium equation.

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1 Introduction

This short note deals with the blow-up behavior of solutions to the following reaction-diffusion equation with Hardy-type potential

$$u_t = \Delta u^m + \frac{K}{|x|^2} u^m, \tag{1.1}$$

posed in $\mathbb{R}^N \times (0, T)$, in dimension $N \geq 3$ and with $m > 1$. As it is well known from the celebrated paper by Baras and Goldstein [3], at least for $m = 1$ there exist no solutions

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when the constant K in (1.1) exceeds the Hardy constant $(N-2)^2/4$. Thus, we will restrict ourselves in all the present work to the range

$$0 < K < K_*(N) := \frac{(N-2)^2}{4}.$$

Equation (1.1) in the semilinear case $m = 1$ has been considered by Baras and Goldstein, and it has been shown [3, Theorem 2.2] that it has no solutions for $K > K_*(N)$ while existence of solutions is proved for $0 \leq K \leq K_*(N)$, under suitable restrictions on the initial condition $u_0(x) = u(x, 0)$. Later Cabré and Martel [7] extended the non-existence results to more general potentials $V(x)$ instead of $|x|^{-2}$ establishing a sharp condition for the non-existence of solutions in terms of the spectrum of the (linear) operator $-\Delta - V$. Starting from these works, Hardy-type potentials (that is, coefficients of the form $K|x|^{-2}$ or $K|x|^{-a}$ with $a \in (0, 2)$, or more general potentials $V(x)$ behaving as a negative power when $x \to 0$, appearing in front of the zero order term) became of interest for many researchers and have been considered quite regularly in papers on both elliptic and parabolic partial differential equations. A seminal paper on the Hardy inequality and the heat equation with a Hardy potential is [21], followed by new results in [22]. Remaining in the linear setting, more recently authors considered Hardy-like potentials either in the boundary condition [17] or related to the Brownian motion and Feynman-Kac formula [13] where in particular the non-existence condition from [3] is improved. Similar equations in the fast-diffusion range $0 < m < 1$ have been studied in various papers by Goldstein and Kombe such as [11, 18, 12], the authors being concerned with the non-existence of solutions in dependence of the singular potential, both in $\mathbb{R}^N$ and in bounded domains. In some of these works extensions to different diffusion operators (such as the p-Laplacian operator or the doubly nonlinear operator) have been also considered. In the last decade, the fractional Laplacian diffusion has been considered together with a Hardy potential depending usually on the fractional power of the Laplacian (see [1] and references therein). More general diffusion operators and potentials were considered in works such as [9].

Recently, the authors of the present note, in the framework of a larger project of understanding the influence of unbounded weights on the reaction term over the behavior of solutions to reaction-diffusion equations, classified in [16] (for the moment only in dimension $N = 1$) the blow-up self-similar profiles to the more general equation

$$u_t = \Delta u^m + |x|^\sigma u^m, \tag{1.2}$$

with $m > 1$ and $\sigma > 0$. Although we require σ to be positive, a closer inspection of the paper shows that in fact a part of the analysis performed there holds true when only the condition $(\sigma + 2)(m - 1) > 0$ is fulfilled, that is, $\sigma > -2$. Thus, we can also understand (1.1) as a limiting case for the weighted reaction-diffusion equations studied in [16].

Main results. The goal of this short note is to show that the blow-up behavior of solutions to Eq. (1.1) strongly depends on the behavior of the initial condition u_0 at the origin. We will always consider radially symmetric and compactly supported initial conditions

$$u_0(x) = u_0(|x|) \in C(\mathbb{R}^N), \quad \operatorname{supp} u_0 \text{ compact in } \mathbb{R}^N, \quad u_0 \geq 0, \ u_0 \not\equiv 0. \tag{1.3}$$

The notion of solution to Eq. (1.1) will be understood in the sense of distributions, adapting the theory for the porous medium equation to tackle with the possible singularity induced by the Hardy-like term. To fix the notation, we denote in the sequel by $u(t)$ the mapping $x \mapsto u(x,t)$ for a fixed $t \geq 0$. We state the following

Definition 1.1. *By a* local weak solution *to the Cauchy problem* (1.1)-(1.3) *we understand a function* $u \in C([0,T) : L^1(\mathbb{R}^N))$ *for some* $T > 0$*, which moreover satisfies the following assumptions:*

- $$u^m(t) \in W^{1,1}_{\text{loc}}(\mathbb{R}^N), \qquad \frac{u^m(t)}{|x|^2} \in L^1_{\text{loc}}(\mathbb{R}^N), \qquad \text{for any } t \in (0,T).$$
- u *is a solution in the sense of distributions to Eq.* (1.1)*, that means that for any* $\varphi \in C_0^\infty(\mathbb{R}^N \times [0,T))$ *and for any* t_1*,* $t_2 \in (0,T)$ *with* $t_1 < t_2$ *we have*
$$\int_{\mathbb{R}^N} u(t_2)\varphi(t_2)\,dx - \int_{\mathbb{R}^N} u(t_1)\varphi(t_1)\,dx - \int_{t_1}^{t_2}\int_{\mathbb{R}^N} u(t)\varphi_t(t)\,dx\,dt + \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \nabla u^m(t)\cdot\nabla\varphi(t)\,dx\,dt = \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \frac{u^m(t)}{|x|^2}\varphi(t)\,dx\,dt.$$
- *The initial condition is taken in* L^1 *sense, that is* $u(t) \to u_0$ *as* $t \to 0$ *with convergence in* $L^1(\mathbb{R}^N)$*.*

Since Eq. (1.1) is invariant to rotations, solutions $u(x,t)$ to it with initial data as in (1.3) are radially symmetric at any time $t > 0$. Concerning the existence and uniqueness of radially symmetric solutions, we have the following

Proposition 1.2. *Let* u_0 *be as in* (1.3)*. Then there exists a unique radially symmetric local weak solution to the Cauchy problem for Eq.* (1.1) *with initial condition* u_0 *in the sense of Definition 1.1. Moreover, the solution is global, that is, it exists for any* $t \in (0,\infty)$*.*

The proof of this partial well-posedness result (partial in the sense that it is stated only for the class of radially symmetric solutions and data) is based on a one-to-one correspondence between the radially symmetric solutions to Eq. (1.1) and the radially symmetric solutions of the following partial differential equation in radial variables

$$\psi_s = (\psi^m)_{zz} + \frac{\overline{N}-1}{z}(\psi^m)_z, \qquad \overline{N} > 2, \tag{1.4}$$

which can be seen as a porous medium equation in (possibly non-integer) dimension $\overline{N}$. The transformation is performed in two steps, along Section 2 and the beginning of Section 3, where the explicit definitions of the function ψ, the variables z and s and the dimension $\overline{N}$ will be given. We stress here that the well-established theory for the porous medium equation (see for example [20]) stays valid in radially symmetric variables also for non-integer dimensions (as in Eq. (1.4), the dimension $\overline{N}$ is just a parameter) with completely identical proofs. Let us state here that we have no proof of either existence or uniqueness of solutions to Eq. (1.1) with general (that is, not necessarily radially symmetric) initial conditions, and up to our knowledge, such proof does not exist in literature.

The next theorem is our most interesting result and shows that for most of such data a phenomenon known as *instantaneous blow-up* at the origin takes place.

Theorem 1.3. *Let u_0 be as in* (1.3) *and which satisfies in addition that*

$$\lim_{x\to 0} |x|^{-2/(m-1)} u_0(x) = +\infty. \tag{1.5}$$

Then the solution u to (1.1) *with initial condition u_0 develops an instantaneous singularity at the origin: $\lim_{x\to 0} u(x,t) = +\infty$ for any $t > 0$.*

This fact is due to the combined strength of the singular potential and of the reaction term u^m with $m > 1$, that produces finite time blow-up by itself, and is in *contrast with the semilinear case* $m = 1$, where solutions are bounded and exist globally in time for initial data u_0 as in the statement of Theorem 1.3, according to [3, Theorem 2.2(i)]. Let us remark at this point that initial conditions u_0 such that $u_0(0) = C > 0$ are included in Theorem 1.3.

On the other hand, when the initial data are supported away from the origin or they are "too tangent" to the x-axis at $x = 0$, solutions blow up in finite time but not instantaneously, as the following result states:

Theorem 1.4. *Let u_0 be as in* (1.3) *and such that*

$$\limsup_{x\to 0} |x|^{-2/(m-1)} u_0(x) < \infty. \tag{1.6}$$

Then there exists a finite positive time $T \in (0,\infty)$ such that the solution u satisfies $u(t) \in L^{\infty}(\mathbb{R}^N)$ for $t \in (0,T)$ and $\lim_{x\to 0} u(x,T) = +\infty$, that is, the solution u blows up at $x = 0$ at the positive time T.

In particular, in Theorem 1.4 initial conditions u_0 supported far from the origin (that means, $u_0(x) = 0$ for any $x \in B(0,\delta)$ for some $\delta > 0$) are considered. As explained before, the proof of the two results is based on a transformation mapping solutions to (1.1) into solutions to a non-homogeneous porous medium equation with power density that will be made precise in Section 2 and a second transformation of the latter into the radially symmetric porous medium equation (1.4). The fine difference between the two cases relies on the existence or non-existence of a *waiting time* (see [8]) for the evolution of the interfaces of solutions to the porous medium equation.

Remarks. 1. A closer inspection of the proofs shows that the blow-up, either instantaneous as in Theorem 1.3 or in positive finite time T as in Theorem 1.4, occurs *only at the origin*, while at any radius $|x| = r > 0$ solutions remain bounded. Indeed, these solution are mapped by the transformation in Section 2 into solutions to a porous medium equation which remain bounded at every point. Thus, the only singularity appears at $r = 0$ when undoing the transformation (2.4) which includes multiplication by a power of r. This fact is strongly contrasting with the reaction-diffusion equation without the Hardy potential, where it is shown in [19, Chapter 4] that blow-up is not instantaneous and occurs on intervals with respect to $r = |x|$ and not only at the origin. We can thus say that the (either instantaneous or finite time) blow-up only at $x = 0$ *is the effect of the singular potential.*

2. We thus discovered an interesting example of *continuation after blow-up*: indeed, according to Theorems 1.3 and 1.4 solutions to Eq. (1.1) blow up at $x = 0$ in the sense that they become unbounded (either instantaneously or in finite time). However, these solutions live forever in weaker spaces ($L^1(\mathbb{R}^N)$ plus a weighted space given by the condition $u^m/|x|^2 \in L^1(\mathbb{R}^N)$) as stated in Proposition 1.2.

2 The new transformation

The main tool in the proof of our results is a new transformation (but based on similar ideas to the one introduced in the recent work [14]), mapping radially symmetric solutions to Eq. (1.1) into solutions to a one-dimensional non-homogeneous porous medium equation. We describe this transformation below. Let u be a radially symmetric solution to Eq. (1.1). Then $u = u(r,t)$ solves the radial form of Eq. (1.1), that is

$$u_t = (u^m)_{rr} + \frac{N-1}{r}(u^m)_r + \frac{K}{r^2}u^m, \tag{2.1}$$

where here and in the sequel we let $r = |x|$. Let us consider the associated stationary equation solved by the function u^m, that is

$$\xi_{rr} + \frac{N-1}{r}\xi_r + \frac{K}{r^2}\xi = 0, \tag{2.2}$$

which is an ordinary differential equation of Euler type having two linearly independent solutions $v(r) = r^{\lambda_1}$ and $w(r) = r^{\lambda_2}$, where

$$\lambda_1 = \frac{2-N+\sqrt{(N-2)^2-4K}}{2} > \lambda_2 = \frac{2-N-\sqrt{(N-2)^2-4K}}{2}. \tag{2.3}$$

We proceed as in [14, Section 2] and introduce the new independent variables and function

$$\theta(y,\tau) = \frac{u(r,t)}{w(r)^{1/m}}, \quad y = \frac{v(r)}{w(r)} = r^{\lambda_1-\lambda_2}, \ \tau = (\lambda_1-\lambda_2)^2 t. \tag{2.4}$$

One can see by straightforward calculations (see the details in [14]) that $\theta(y,\tau)$ is a solution to the following non-homogeneous porous medium equation in one dimension

$$f(y)\theta_\tau = (\theta^m)_{yy}, \quad f(y) = (\lambda_1-\lambda_2)^2\frac{w(r)^{(3m+1)/m}}{W(r)^2}, \tag{2.5}$$

where $W(r)$ is the Wronskian of the two linearly independent solutions $v(r)$, $w(r)$, that is

$$W(r) = v'(r)w(r) - v(r)w'(r) = (\lambda_1-\lambda_2)r^{\lambda_1+\lambda_2-1}.$$

Thus in our specific case (2.5) leads to

$$y^{-\gamma}\theta_\tau = (\theta^m)_{yy}, \quad y \geq 0, \tag{2.6}$$

where

$$\gamma = -\frac{2m(N-1)+(3m+1)\lambda_2}{m(\lambda_1-\lambda_2)} = \frac{(3m+1)\sqrt{(N-2)^2-4K}+N-mN-2m-2}{2m\sqrt{(N-2)^2-4K}}. \tag{2.7}$$

Recalling the Hardy constant $K_*(N) = (N-2)^2/4$, we readily find that $\gamma = \gamma(K)$ given in (2.7) is decreasing with respect to K and thus

$$-\infty = \lim_{K\to K_*(N)}\gamma(K) < \gamma(K) < \gamma(0) = \frac{m+1}{m} - \frac{2}{N-2} < \frac{m+1}{m},$$

for any $K \in (0, K_*(N))$, hence we are in the (good) range of γ for the one-dimensional non-homogeneous equation established in [10] (see also [14, Section 3]).

The equality case $K = K_*(N)$. Our transformation introduced above can be adapted also for the equality case $K = K_*(N)$, with some changes that can be borrowed from the similar transformation in [14, Section 2]. In this case $\lambda_1 = \lambda_2 = (2-N)/2 < 0$ and the two linearly independent solutions to (2.2) are given by

$$v(r) = r^{(2-N)/2} \ln r, \quad w(r) = r^{(2-N)/2}.$$

The Wronskian of these two solution writes

$$W(r) = v'(r)w(r) - v(r)w'(r) = r^{1-N},$$

and we can consider again the new variables and function

$$\theta(y,t) = \frac{u(r,t)}{r^{(2-N)/2m}}, \quad y = \ln r, \tag{2.8}$$

and taking into account the general construction of the transformation in [14] we once more deduce that θ is a solution to the non-homogeneous equation of porous medium type

$$f(y)\theta_t = (\theta^m)_{yy}, \quad f(y) = \frac{w(r)^{(3m+1)/m}}{W(r)^2}, \tag{2.9}$$

but this time with a density $f(y)$ of exponential form. More precisely, taking into account the expressions of $w(r)$ and the Wronskian $W(r)$ above and making a further change of variables

$$z = \frac{N(m-1)+2m+2}{2m} y, \quad \tau = \left(\frac{N(m-1)+2m+2}{2m}\right)^2 t$$

we find after easy calculations that $\theta(z,\tau)$ is a solution to

$$e^z\theta_\tau = (\theta^m)_{zz}, \quad z = \frac{N(m-1)+2m+2}{2m} \ln r, \tag{2.10}$$

where the local behavior near $r = 0$ is mapped to the limit behavior as $z \to -\infty$. Eq. (2.10) has been considered in [10, Section 4] (with density e^{-z}, which is equivalent to our Eq. (2.10) through the obvious transformation $z \to -z$). However, the analysis performed there holds only for compactly supported solutions, while in our case, a typical radially symmetric initial condition $u_0(r)$ such that $u_0(0) = K > 0$ is mapped into an initial data $\theta_0(z)$ to Eq. (2.10) which is compactly supported from the right but has a tail decaying exponentially fast as $z \to -\infty$. Such initial data do not enter in the framework of the analysis in [10, Section 4] (and it is likely to have a different asymptotic profile presenting an exponential tail on its own, too), thus we cannot continue the analysis in this case. This is why in the statements of our Theorems we refrain from the case $K = K_*(N)$, although the results of Theorem 1.3 and Theorem 1.4 are expected to hold true. We can only conclude that in this case the right interface of $u(r,t)$ blows up (the support of u ceases to be compact) in finite time.

Remark. While it is clear that the dimension $N = 2$ cannot be considered since $K_*(2) = 0$, the previous transformations also hold true in dimension $N = 1$. The main change is that λ_1, λ_2 in (2.3) become positive, whence the new function θ defined in (2.4) solution to (2.6) is singular at the origin for any $\tau > 0$ and up to our knowledge a study of such singular solutions to Eq. (2.6) has not been done.

3 Proofs of the main results

With the transformation in Section 2 we have mapped radially symmetric solutions u to Eq. (1.1) onto solutions θ to Eq. (2.6). Since the general theory for Eq. (2.6) is not yet well-established, we need to perform one further change of variables in order to transform θ into a (radially symmetric) solution to the standard porous medium equation via the transformation introduced in [15, Section 2.1, Case 2], by letting

$$\theta(y,\tau)=y^{1/m}\psi(y^{\mu},\mu^2\tau), \quad \mu=\frac{(1-\gamma)m+1}{2m}>0, \tag{3.1}$$

(the latter holds true since $\gamma<(m+1)/m$), where ψ is a solution to the following equation

$$\psi_s=(\psi^m)_{zz}+\frac{\overline{N}-1}{z}(\psi^m)_z, \quad z=y^{\mu},\ s=\mu^2\tau,\ \overline{N}=2+\frac{1}{\mu}>2, \tag{3.2}$$

which can be seen (at a formal level) as the radially symmetric form of a porous medium equation in (fractional) dimension $\overline{N}$. With this latter change of variables, we are now in a position to prove Proposition 1.2 and Theorems 1.3 and 1.4. Assume for the moment that the well-posedness result Proposition 1.2 is true (its proof will be postponed at the end of the paper) and let us prove the two Theorems. We begin with the first one, which is also the most unexpected.

Proof of Theorem 1.3. For the reader's convenience, we divide the proof into two steps.

Step 1. Let us in a first step consider an initial condition u_0 as in (1.3) such that $u_0(0)=C>0$ and assume by contradiction that there exists a time $T>0$ and a radially symmetric solution $u(r,t)$ to Eq. (1.1) such that $u(t)$ is bounded for any $t\in(0,T)$ and $u(r,0)=u_0(r)$ for any $r\geq 0$. We map this solution u via the transformation (2.4) into a solution θ to Eq. (2.6) with initial condition satisfying

$$\theta_0(y)\sim Cr^{-\lambda_2/m}=Cy^{-\lambda_2/m(\lambda_1-\lambda_2)}, \quad \text{as } y\to 0,$$

which is a solution to Eq. (2.6) for $\tau\in(0,T_1)$, where $T_1=(\lambda_1-\lambda_2)^2T>0$. We next map θ via the transformation (3.1) into a solution ψ to Eq. (3.2) having an interface at the origin $z=0$ with the local behavior

$$\psi(z,s)\sim y^{-1/m}\theta(y,\tau)=z^{-\frac{\lambda_1}{m\mu(\lambda_1-\lambda_2)}}. \tag{3.3}$$

Since $N\geq 3$ we find that $\lambda_1<0$, thus the exponent of z in (3.3) is positive. Moreover, it is smaller than $2/(m-1)$. Indeed, we have

$$-\frac{\lambda_1}{m\mu(\lambda_1-\lambda_2)}-\frac{2}{m-1}=\frac{8m}{(m-1)[(m-1)\sqrt{(N-2)^2-4K}-mN-2m+N-2]}<0,$$

since

$$(m-1)\sqrt{(N-2)^2-4K}-mN-2m+N-2<(m-1)(N-2)-(m-1)N-2(m+1)<0.$$

By the results in [8, p. 376] (see also [20, Corollary 14.10] applied for $x_0=0$) it follows that there is no waiting time for the interface of the solution ψ to Eq. (3.2) and thus ψ

(and respectively also θ, since $y = z^{1/\mu}$ and $\mu > 0$) becomes instantaneously positive in a neighborhood of the origin $z = 0$ (respectively $y = 0$). In particular, there exist $\tau_0 \in (0, T_1)$ and $\delta > 0$ such that $\theta(y, \tau_0) > 0$ for $|y| < \delta$. Undoing the transformation, letting

$$t_0 = \frac{\tau_0}{(\lambda_1 - \lambda_2)^2} \in (0, T),$$

and taking into account that $\lambda_2 < 0$ for $N \geq 3$, it follows that $u(r, t_0) = r^{\lambda_2/m}\theta(y, \tau_0) \to \infty$ as $r \to 0$ (which is equivalent to $y \to 0$) and a contradiction with the fact that the solution u was continuous and bounded for times $t \in (0, T)$. The contradiction proves the instantaneous blow-up of u.

Step 2. Assume now that u_0 is a radially symmetric initial condition as in (1.3) and satisfying $u_0 = 0$ and (1.5). This means that for any $M > 0$, there exists $\delta = \delta(M) > 0$ such that $u_0(x) > M|x|^{2/(m-1)}$, for any x such that $|x| < \delta$. Applying the successive transformations (2.4) and (3.1) as we also did in Step 1, we get first an initial condition θ_0 such that

$$\theta_0(y) = \frac{u_0(r)}{r^{\lambda_2/m}} \geq My^{\eta}, \quad \eta = \left(\frac{2}{m-1} - \frac{\lambda_2}{m}\right)\frac{1}{\lambda_1 - \lambda_2},$$

and then an initial condition to a solution to the standard porous medium equation (3.2) such that

$$\psi_0(z) \geq My^{\eta - 1/m} = Mz^{(\eta - 1/m)/\mu}, \quad \mu = \frac{(1-\gamma)m + 1}{2m}, \tag{3.4}$$

all these two estimates taking place in corresponding neighborhoods of the origin whose amplitudes depend on M. But a simple calculation shows that $(\eta - 1/m)/\mu = 2/(m-1)$, thus (3.4) implies that $\psi_0(z) \geq Mz^{2/(m-1)}$ in a neighborhood of $z = 0$ depending on M and for any $M > 0$, thus one can readily establish that

$$\lim_{z \to 0} z^{-2/(m-1)}\psi_0(z) = +\infty$$

and by [20, Corollary 14.10] we again infer that there is no waiting time to filling the hole in the origin and thus $\psi(0, s) > 0$ for any $s \in (0, s_0)$ for some $s_0 > 0$. Undoing the two transformations and using again the same argument by contradiction as at the end of Step 1, we obtain the instantaneous blow-up at $x = 0$ of u, as desired. □

Proof of Theorem 1.4. For the reader's convenience, we also divide the proof into two steps.

Step 1. Assume now that u_0 is a radially symmetric initial condition as in (1.3) and satisfying (1.6) but such that $u_0 > 0$ in a neighborhood of the origin. This condition implies that there exists $C > 0$ such that

$$u_0(x) \leq C|x|^{2/(m-1)},$$

in some neighborhood of $x = 0$. Using successively the transformations (2.4) and (3.1) and performing the same calculations as in Step 2 of the proof of Theorem 1.3 above, we end up with an initial condition $\psi_0(z)$ to Eq. (3.2) such that

$$\limsup_{z \to 0} z^{-2/(m-1)}\psi_0(z) \leq C.$$

Once more by [20, Corollary 14.10] we deduce that there is a positive waiting time $S > 0$ such that $\psi(0, s) = 0$ for any $s \in (0, S)$ before $\psi(S)$ becoming positive at the origin. Undoing the two transformations and using a similar argument by contradiction as in the proof of Theorem 1.3 but at time $t = T > 0$ instead of $t = 0$, we obtain that the solution u to (1.1) blows up in finite time

$$T = \frac{S}{(\lambda_1 - \lambda_2)^2 \mu^2},$$

but not instantaneously.

Step 2. Assume now that u_0 is a radially symmetric initial condition as in (1.3) but with a compact support away from the origin, that is, $\operatorname{supp} u_0 \subset (0, \infty)$. Applying once more the transformations (2.4) and (3.1) we arrive to an initial condition ψ_0 to the radially symmetric porous medium equation (3.2) such that $\operatorname{supp} \psi_0 \subset (0, \infty)$ is a compact set. We infer from the theory of focusing (filling the hole) developed by Aronson and Graveleau in [2] (see also [20, Section 19.2]) that the solution to (3.2) fills the hole (its backward interface arriving at the origin) in a finite time (usually called "focusing time") $S_0 > 0$. Moreover, it is shown in [20, Section 19.2] that at the focusing time $s = S_0$ the behavior of $\psi(z, S_0)$ as $z \to 0$ is of the asymptotic form

$$\psi(z, S_0) \sim c z^{\varepsilon/(m-1)}, \ \varepsilon = \frac{2\beta_* - 1}{\beta_*}, \quad \text{as } z \to 0,$$

where $\beta_* \in (1/2, 1)$ is the self-similar (anomalous) focusing exponent (see [2]). It is known that $0 \leq \varepsilon \leq 1$, hence $\varepsilon/(m-1) \leq 1/(m-1) < 2/(m-1)$ and thus

$$\lim_{z \to 0} z^{-2/(m-1)} \psi(z, S_0) = \lim_{z \to 0} z^{(\varepsilon - 2)/(m-1)} = +\infty.$$

According to [20, Corollary 14.10], there is no waiting time and the solution ψ becomes immediately positive at the origin $z = 0$ after the focusing time S_0. Undoing the two transformations, we infer that the solution $u(x, t)$ to Eq. (1.1) blows up in finite time

$$T = \frac{S_0}{(\lambda_1 - \lambda_2)^2 \mu^2},$$

concluding the proof. □

We end the paper with the proof of the postponed well-posedness result.

Proof of Proposition 1.2. Let u_0 be a radially symmetric and compactly supported function as in (1.3). By performing the two changes of variables (2.4) and (3.1) we are left with a continuous, compactly supported initial data ψ_0 for (3.2). But the well-posedness for the porous medium equation is well established, thus there exists a unique solution ψ to (3.2) which stays bounded and compactly supported for any $t > 0$. By undoing the transformations we arrive to a solution $u(r, t)$ which is bounded and continuous for any $r > 0$, $t > 0$ and develops a singularity at $r = 0$ (either instantaneously or in a finite positive time T) having the form

$$u(r, t) \sim C r^{\lambda_2/m}, \quad C > 0, \tag{3.5}$$

Recalling that we are working in dimension $N \geq 3$ we show that despite the singularity, we have

$$u(t) \in L^1(\mathbb{R}^N), \qquad u^m \in W^{1,1}_{\rm loc}(\mathbb{R}^N), \qquad \frac{u^m(t)}{|x|^2} \in L^1(\mathbb{R}^N),$$

thus $u(t)$ belongs to the spaces required in Definition 1.1 and this holds true for any $t > 0$, hence the solution is global in time (in the sense of Definition 1.1). Indeed, taking into account the value of λ_2 obtained in (2.3), we have on the one hand

$$\frac{\lambda_2}{m} = \frac{2 - N - \sqrt{(N-2)^2 - 4K}}{2m} > \frac{2 - N - (N-2)}{2m} = \frac{2-N}{m} > -N,$$

since $mN - N + 2 > 0$ for $m > 1$. Thus, the singularity is integrable near $r = 0$. On the other hand,

$$\frac{u^m(t)}{|x|^2} \sim Cr^{\lambda_2 - 2}, \qquad \nabla u^m \sim Cr^{\lambda_2 - 1}, \text{ near } r = 0,$$

and also

$$\lambda_2 - 1 > \lambda_2 - 2 = \frac{-N - 2 - \sqrt{(N-2)^2 - 4K}}{2} > \frac{-N - 2 - N + 2}{2} = -N,$$

whence also the singularities of $u^m/|x|^2$ and ∇u^m are integrable near $x = 0$. The fact that the equation is verified in the sense of distributions and that it takes the initial value follows from the similar results for Eq. (3.2) in a standard way, similarly as in, for example, [15, Section 4]. □

Final remark. There are further interesting transformations related to our equation and the transformations used here. For example, by letting $z(r,t) = r^{-\lambda_1/m} u(r,t)$ we transform Eq. (1.1) into the following equation

$$\begin{aligned} |x|^{\alpha} \partial_t z = |x|^{-\beta} {\rm div}(|x|^{\beta} \nabla z^m), \quad & \beta = N - 1 + 2\lambda_1 = 1 + \sqrt{(N-2)^2 - 4K}, \\ & \alpha = -\frac{m-1}{m} \lambda_1 < N. \end{aligned} \tag{3.6}$$

The same transformation is a particular case of the transformation at the beginning of [15, Subsection 2.1] by letting there $\theta = 1$, $C = 1$, $\gamma = -(m-1)\delta$ and $n = N - 2m\delta$, where $\delta = -\lambda_1/m > 0$. Equation (3.6) became recently very fashionable after some works showing some very interesting properties of it related to the Caffarelli-Kohn-Nirenberg inequalities and the property of *symmetry breaking* in some cases, see for example [4, 5, 6].

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