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NILPOTENT SUPERDERIVATIONS IN PRIME SUPERALGEBRAS

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ABSTRACT. In this paper we give a full description of nilpotent homogeneous inner superderivations in associative prime superalgebras with and without superinvolution. We also present examples of all the models of the superderivations appearing in our classification.

Key words: Associative superalgebra, Lie superalgebra, inner superderivation, superinvolution, skew-symmetric element

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1. Introduction

An associative superalgebra is a $\mathbb{Z}_2$-graded associative algebra $R = R_0+R_1$. The elements of $R_0 \cup R_1$ are called homogeneous elements and we say that the degree of $a \in R_0 \cup R_1$ is i (denoted $|a| = i$) when $a \in R_i$, $i \in \{0,1\}$. Given an associative superalgebra R, we obtain a Lie superalgebra if the associative product is replaced by the superbracket $[\ ,\ ]$, where $[a,b] := ab-(-1)^{|a|\,|b|}ba$ for homogeneous $a,b \in R$. The Lie structure of prime/simple associative superalgebras was investigated by F. Montaner in [27] and S. Montgomery in [28].

We say that a $\mathbb{Z}_2$-linear map $* : R \to R$ is a superinvolution when $(a^*)^* = a$ and $(ab)^* = (-1)^{|a|\,|b|}b^*a^*$ for homogeneous $a,b \in R$. The set of skew-symmetric elements of an associative superalgebra is a Lie superalgebra and it will be denoted by K along this paper. The study of the Lie structure of K of a simple associative superalgebra with superinvolution was iniciated by C. Gómez-Ambrosi and I. Shestakov in 1997 in [15], and their results were extended to prime superalgebras in [13]. Superinvolutions in associative superalgebras have been a topic of great interest. We highlight the works of J. Laliena [21] on the description of the derived superalgebra $[K,K]$ of a semiprime superalgebra with superinvolution, the papers [22] and [23] of J. Laliena and R. Rizzo on the extension of results of C. Lanski and S. Montgomery to associative superalgebras with superinvolution, and the recent works of T. S. do Nascimento, A. C. Vieira, A. Giambruno, A. Ioppolo, D. La Mattina, F. Martino ([5], [10], [11], [12], [18]) on superinvolutions in superalgebras related to polynomial identities and related to the growth of certain substructures of the superalgebras.

Another interesting and very active topic in superalgebras is the study of superderivations (see for example the works of A. Fošner and M. Fošner [6], H. Ghahramani, M. N. Ghosseiri and S. Safari [9] or Y. Wang [29]). A linear map $d = d_0 + d_1$ in R is called a superderivation if each d_i, $i \in \{0,1\}$, satisfies $d_i(R_j) \subset R_{i+j}$

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and $d_i(ab) = d_i(a)b + (-1)^{i|a|}ad_i(b)$, for homogeneous $a, b \in R$. For instance, if $a \in R_0 \cup R_1$, the map $\mathrm{ad}_a : R \to R$ given by $\mathrm{ad}_a(x) = [a, x]$ is a superderivation (of degree $|a|$). Such superderivation is called inner derivation. In [9] the authors describe the structure of superderivations on some $\mathbb{Z}_2$-graded rings and study when superderivations are inner.

In this paper we are going to study nilpotent inner superderivations in prime associative superalgebras with and without involution. This problem fits into the so called Herstein theory: the study of nonassociative objects in associative prime and semiprime rings perhaps with involution. Indeed, back in 1963 I. N. Hestein showed that any ad-nilpotent element a of index n in a simple ring of characteristic zero or greater than n gives rise to a nilpotent element $a - \lambda$ for some λ in the center of R and that the index of nilpotence of such element is less than or equal to $[\frac{n+1}{2}]$, see [17, Theorem page 84]. This result of Herstein was generalized by W. S. Martindale and C. R. Miers in 1983 [25, Corollary 1] to prime rings of characteristic greater than n, and nilpotent derivations of the skew-symmetric elements of prime rings with involution were described in the 1990's by W. S. Martindale and C. R. Miers in [26, Main Theorem]. The extension of those descriptions to ad-nilpotent elements in semiprime rings was performed by several authors (P. Grzeszczuk [16], T. K. Lee [24] or the authors of this paper together with J. Brox and R. Muñoz Alcázar [3]).

The goal of this paper is to extend the results of Martindale and Miers on the description of ad-nilpotent elements of prime rings with or without involution to the supersetting. We remark that this extension is not just a direct translation of the non-super results because a superinvolution on a superalgebra is not an involution in the underlying non-super structure.

The paper is organized as follows: after a preliminary section where we recall some useful notions and results in the super and non-super setting, in section 3 we will give a detailed description of a homogeneous ad-nilpotent element a of index n in a prime associative superalgebra R free of $\binom{n}{s}$ and s-torsion, where $s = [\frac{n+1}{2}]$, depending on the degree of the element and the equivalence class of n modulo 4. If a belongs to R_0 the description follows, except for some details, from our study of ad-nilpotent elements of semiprime algebras (see [3, Theorem 4.4 and Theorem 5.6]), while if $a \in R_1$ we will work with $a^2 \in R_0$ and we will show that the only possible indexes of ad-nilpotence of a are $n \equiv_4 1, 2$. These two cases correspond to a nilpotent element of index $\frac{n+1}{2}$, when $n \equiv_4 1$, or to an element a for which there exists $\lambda \in C(R)_0$ with $(a^2 - \lambda)^{\frac{n+2}{4}} = 0$, when $n \equiv_4 2$.

In section 4 we will study ad-nilpotent elements of the skew-symmetric elements K of a prime superalgebra with superinvolution and characteristic $p > n$, i.e., elements $a \in K_0 \cup K_1$ such that $\mathrm{ad}_a^n K = 0$ and $\mathrm{ad}_a^{n-1} K \neq 0$. The key point is the fact proven in Proposition 4.2 that any homogeneous ad-nilpotent element a of K of index n is either nilpotent or ad-nilpotent of the whole R with the same index n. When $a \in K$ is an ad-nilpotent homogeneous even element, it will be classified depending on its index of ad-nilpotence modulo 4 (see Theorem 4.3), and when $a \in K_1$ is ad-nilpotent of index n, its description will depend on the congruence class of n modulo 8 (see Theorem 4.4): if $n \equiv_8 1, 2, 5, 6$ then a behaves as an ad-nilpotent element of R and if $n \equiv_8 0, 7$ then a is nilpotent of index $s + 1$ for $s = [\frac{n+1}{2}]$, and $a^s K a^s = 0$, implying that $a^s R a^s$ is a commutative trivial local superalgebra. We will also show that the indexes of ad-nilpotence $n \equiv_8 3, 4$ are not possible.

The last section is devoted to examples. We will work in the matrix superalgebra $\mathcal{M}(r|s)$ over $\mathbb{F}$, where r is an odd natural number, s as an even natural number, and $\mathbb{F}$ is a field. In such superalgebra we will define a superinvolution and we will present examples of elements fitting each of the cases of ad-nilpotent elements appearing in Theorems 3.2, 4.3 and 4.4.

2. Preliminaries

In this section we recall the main definitions and preliminary results. We refer the reader to [7], [14] and [15] for further information on associative superalgebras.

2.1. Throughout the article, $R = R_0 + R_1$ will denote a superalgebra over a unital commutative ring Φ with $\frac{1}{2} \in \Phi$. In these conditions the map $\sigma : R \to R$ defined by $\sigma(x_0 + x_1) = x_0 - x_1$, for every $x_0 \in R_0$, $x_1 \in R_1$, is an algebra automorphism with $\sigma^2 = \mathrm{id}$. Conversely, given an associative algebra R, every algebra automorphism $\sigma : R \to R$ with $\sigma^2 = \mathrm{id}$ defines a $\mathbb{Z}_2$-graduation on R given by $R_0 = \{a \in R \mid \sigma(a) = a\}$ and $R_1 = \{a \in R \mid \sigma(a) = -a\}$. Therefore, a $\mathbb{Z}_2$-graduation in R is equivalent to an algebra automorphism σ with $\sigma^2 = \mathrm{id}$.

Notice that a Φ-module S of R is graded if and only if $\sigma(S) \subset S$.

2.2. A semiprime associative superalgebra R is a superalgebra without nonzero nilpotent graded ideals. We remark that a semiprime associative superalgebra is just an associative superalgebra which is semiprime as an algebra (for every nonzero ideal I of R, $I^2 \neq 0$). A prime associative superalgebra R is an associative superalgebra without nonzero orthogonal graded ideals (for every nonzero graded ideals I, J of R, $IJ \neq 0$). Prime superalgebras have the following property: for every nonzero graded ideal I of a prime superalgebra R and any two elements $a, b \in R$ where at least a or b is homogeneous, the condition $aIb = 0$ implies that either a or b is zero (see [7, pag. 693]).

Lemma 2.3. [27, Lemma 1.2] *If $R = R_0 + R_1$ is a semiprime associative superalgebra, then R and R_0 are semiprime algebras.*

Lemma 2.4. [27, Lemma 1.3] *If $R = R_0 + R_1$ is a prime associative superalgebra, then either R or R_0 are prime as algebras.*

2.5. The notion of extended centroid for associative superalgebras is due to M. Fošner, see [7]. Let R be a semiprime associative superalgebra. Since R is semiprime as an algebra, we can consider the extended centroid $C(R)$ of R. Let $\hat{R} = RC(R) + C(R)$ be the central closure of R. Let $\sigma : R \to R$ be the automorphism associated to the $\mathbb{Z}_2$-grading of R ($\sigma^2 = \mathrm{id}$). This automorphism can be extended to $\hat{R}$ and we denote this extension by $\hat{\sigma}$. Since $\hat{\sigma}^2 = \mathrm{id}$, $\hat{R}$ is again a superalgebra and $\hat{\sigma}(C(R)) = C(R)$, i.e., $C(R) = C(R)_0 + C(R)_1$ where $C(R)_0 = \{\lambda + \hat{\sigma}(\lambda) \mid \lambda \in C(R)\}$ and $C(R)_1 = \{\lambda - \hat{\sigma}(\lambda) \mid \lambda \in C(R)\}$. We will say that R is centrally closed if $R = \hat{R}$, i.e., if R is centrally closed as an algebra.

2.6. Let R be a prime associative superalgebra such that R is not prime as an algebra. Let σ denote the automorphism associated to the $\mathbb{Z}_2$-grading of R and consider a nonzero ideal P of R with $P \cap \sigma(P) = 0$. Then $P \oplus \sigma(P)$ is a graded essential ideal of R, where $(P\oplus\sigma(P))_0=\{x+\sigma(x)\mid x\in P\}\cong P$ as an algebra and $(P\oplus\sigma(P))_1=\{x-\sigma(x)\mid x\in P\}$. Since $P\oplus\sigma(P)$ is essential in R,

$$C(R)\cong C(P\oplus\sigma(P))=C(P)\oplus\sigma(C(P)),$$

where the isomorphism is given by the restriction of permissible maps (for any $\lambda=[I,f]\in C(R)$ we define $\hat{\lambda}=[(I\cap(P\oplus\sigma(P)))^2,g]$ where $g:(I\cap(P\oplus\sigma(P))^2\to P\oplus\sigma(P)$ is the restriction of f to the essential ideal $(I\cap(P\oplus\sigma(P)))^2$ of $P\oplus\sigma(P))$. Notice that the $\mathbb{Z}_2$-grading of $C(P)\oplus\sigma(C(P))$ comes from the $\mathbb{Z}_2$-grading of $P\oplus\sigma(P)$: $(C(P)\oplus\sigma(C(P)))_0=\{\lambda+\sigma(\lambda)\mid\lambda\in C(P)\}$ and $(C(P)\oplus\sigma(C(P)))_1=\{\lambda-\sigma(\lambda)\mid\lambda\in C(P)\}$. In particular,

$$C(R)_0\cong\{\lambda+\sigma(\lambda)\mid\lambda\in C(P)\}\cong C(P).$$

On the other hand, by Lemma 2.4, R_0 is prime as an algebra, and therefore its nonzero ideals are essential. By restricting permissible maps from R_0 to $(P\oplus\sigma(P))_0$ we get $C(R_0)\cong C((P\oplus\sigma(P))_0)\cong C(P)$.

We have obtained that $C(R)_0\cong C(R_0)$.

Lemma 2.7. [7, Lemma 3.1] *Let R be a semiprime associative superalgebra. Then the following assertions are equivalent:*

(i) *R is a prime superalgebra.*
(ii) *all nonzero homogeneus elements on $C(R)$ are invertible.*
(iii) *$C(R)_0$ is a field.*

2.8. Let R be an associative superalgebra over Φ and take an element $a\in R_0\cup R_1$. Then $R_a:=aRa$ with $(aRa)_i:=aR_{i+|a|}a$, $i\in\{0,1\}$, is a $\mathbb{Z}_2$-graded Φ-module. Moreover, the product $(axa)(aya):=axaya$ for any $x,y\in R$ induces an associative superalgebra structure in R_a, which is called the local superalgebra of R at a, see [8]. When R is an associative superalgebra with superinvolution $*$, the superinvolution induces a superinvolution $\star$ in R_a given by $(axa)^\star:=(-1)^{|a|}ax^*a$, for every $x\in R$.

2.9. Given an associative superalgebra R with superinvolution $*$, the set of skew-symetric elements $K:=\{a\in R\mid a^*=-a\}$ and the set of symmetric elements $H:=\{a\in R\mid a^*=a\}$ are graded submodules of R. Since $\frac{1}{2}\in\Phi$, $R=H\oplus K$. We will denote $H_i=H\cap R_i$ and $K_i=K\cap R_i$, $i=0,1$. Notice that

$$a\in K_0\implies\begin{cases}a^s\in H_0, & \text{when } s \text{ is even,}\\ a^s\in K_0, & \text{when } s \text{ is odd,}\end{cases}$$

$$a\in K_1\implies\begin{cases}a^s\in H_0, & \text{when } s\equiv_4 0,\\ a^s\in K_1, & \text{when } s\equiv_4 1,\\ a^s\in K_0, & \text{when } s\equiv_4 2,\\ a^s\in H_1, & \text{when } s\equiv_4 3.\end{cases}$$

Moreover, if R is a prime superalgebra and $\mathrm{Skew}(C(R),*)\neq 0$, then $R=K+\mu K$ for any nonzero homogeneous $\mu\in\mathrm{Skew}(C(R),*)$ (indeed, $\mu^2\in C(R)_0$ is invertible because $C(R)_0$ is field, and therefore $R\subseteq K+\mu^2H\subseteq K+\mu K\subseteq R$).

Lemma 2.10. *Let $R=R_0+R_1$ be an associative superalgebra with superinvolution $*$, and let $a\in R_0\cup R_1$. If there exists $\lambda\in C(R)$ such that $a-\lambda$ is nilpotent of index n then:*

(i) *if R is prime, has no n-torsion and $a\in R_0$, then $\lambda\in C(R)_0$,*
(ii) *if R is semiprime, λ is the unique element of $C(R)$ such that $a-\lambda$ is nilpotent: moreover, if $a\in K$ then $\lambda\in\mathrm{Skew}(C(R),*)$.*

Proof. (i) Let us consider $a \in R_0$ and suppose that there exists $\lambda = \lambda_0 + \lambda_1 \in C(R)$ such that $a - \lambda$ is nilpotent of index n. If $\lambda_1 \neq 0$, it is invertible by Lemma 2.7 and there exists $\mu_1 \in C(R)_1$ such that $\lambda_1 \mu_1 = 1$. From the nilpotence of $a - \lambda_0 - \lambda_1$ we get that $\mu_1 a - \mu_1 \lambda_0 - 1$ is again nilpotent of index n, i.e., the element $b = \mu_1 a - \mu_1 \lambda_0 \in R_1$ satisfies a polynomial of the form $p(X) = (X-1)^n \in C(R)_0[X]$. Since $C(R)_0$ is a field, $p(X) \in C(R)_0[X]$ is the minimal polynomial of b over $C(R)_0$. In particular

$$b^n - \binom{n}{1} b^{n-1} + \binom{n}{2} b^{n-2} + \cdots = 0$$

and by homogeneity

$$\binom{n}{1} b^{n-1} + \binom{n}{3} b^{n-3} + \cdots = 0,$$

i.e., b satisfies the polynomial $q(X) = \sum_{i=1}^{[\frac{n+1}{2}]} \binom{n}{2i-1} X^{n-2i+1}$. But $n-1 = \deg q(X) < \deg p(X) = n$, a contradiction with the minimality of $p(X)$. Therefore $\lambda_1 = 0$ and $\lambda \in C(R)_0$.

(ii) It follows as in [4, Lemma 2.11]. □

The following technical result appears in [4] and is a direct consequence of a theorem of Beidar, Martindale and Mikhalev [1, Theorem 2.3.3].

Lemma 2.11. [4, Corollary 2.14] *Let R be a semiprime ring and let $\hat{R}$ denote its central closure. Let $a_i, b_i \in R$ for $i = 1, 2, \ldots, n$ be such that $\mathrm{Id}_R(a_1) \subset \mathrm{Id}_R(b_1)$ and $\sum_{i=1}^n a_i x b_i = 0$ for every $x \in R$. Then there exist $\lambda_i \in C(R)$ for $i = 2, \ldots, n$ such that $a_1 = \sum_{i=2}^n \lambda_i a_i$ in $\hat{R}$.*

2.12. Given a Lie superalgebra $L = L_0 + L_1$ we say that an element $a \in L$ is ad-nilpotent of index n if $\mathrm{ad}_a^n(L) = 0$ and $\mathrm{ad}_a^{n-1}(L) \neq 0$, where $\mathrm{ad}_a(x) := [a, x]$ for every $x \in L$, equivalently, if the inner superderivation ad_a is nilpotent of index n.

Every associative superalgebra $R = R_0 + R_1$ can seen as a Lie superalgebra for the superbracket $[a, b] := ab - (-1)^{|a||b|} ba$ for every $a, b \in R_0 \cup R_1$. When $a \in R_0$, ad_a behaves as the usual adjoint map in the non-super setting; when $a \in R_1$, $\mathrm{ad}_a^2 = \mathrm{ad}_{a^2}$.

2.13. In this paper we will use some results about the description of ad-nilpotent elements in the non-super setting. In them, the notion of pure ad-nilpotent element of an associative algebra was crucial. We recall here that notion and some of the results of [3] that will be used in this paper:

Let R be an associative algebra with or without involution $*$. Let $\hat{R}$ denote the central closure of R and let K be the set of skew-symmetric elements of R with respect to $*$.

(i) Let us consider R^-, i.e., the Lie algebra R with product $[a, b] := ab - ba$ for every $a, b \in R$. We say that an element a is a pure ad-nilpotent element of R^- of index n if for every $\lambda \in C(R)$ with $\lambda a \neq 0$, λa is ad-nilpotent in $\hat{R}^-$ of index n.

(ii) Let us consider K. We say that an element a is a pure ad-nilpotent element of K of index n if for every $\lambda \in H(C(R)), *)$ with $\lambda a \neq 0$, λa is ad-nilpotent in $\mathrm{Skew}(\hat{R}, *)$ of index n.

Lemma 2.14. [3, Lemma 3.2] *If R is a semiprime ring and a is an ad-nilpotent element of R of index n, the following conditions are equivalent:*

(i) *a is a pure ad-nilpotent element of R^-.*

(ii) $\mathrm{Id}_R(\mathrm{ad}_a^{n-1}(R))$ *is an essential ideal of* $\mathrm{Id}_R(a)$.
(iii) $\mathrm{Ann}_R(\mathrm{Id}_R(\mathrm{ad}_a^{n-1}(R))) = \mathrm{Ann}_R(\mathrm{Id}_R(a))$.

Theorem 2.15. [3, Theorem 4.4] *Let R be a semiprime ring with no 2-torsion, let $\hat{R}$ be its central closure, and let $a \in R$ be a pure ad-nilpotent element of R^- of index n. Put $t := [\frac{n+1}{2}]$, and suppose that R is free of $\binom{n}{t}$-torsion and t-torsion. Then n is odd and there exists $\lambda \in C(R)$ such that $a - \lambda \in \hat{R}$ is nilpotent of index $\frac{n+1}{2}$.*

Proposition 2.16. [3, Proposition 5.3] *Let R be a semiprime ring with involution $*$ and free of 2-torsion, let $\hat{R}$ be its central closure, and let $a \in K$ be a nilpotent element of index of nilpotence s. Then a is ad-nilpotent in R. If the index of ad-nilpotence of a in K is n and R is free of $\binom{n}{t}$-torsion for $t := [\frac{n+1}{2}]$, then:*

(1) *If $n \equiv_4 0$ then $s = t + 1$ and $a^t K a^t = 0$.*
(2) *If $n \equiv_4 1$ then $s = t$ and the index of ad-nilpotence of a in R is also n.*
(3) *The case $n \equiv_4 2$ is not possible.*
(4) *If $n \equiv_4 3$ then there exists an idempotent $\epsilon \in C(R)$ such that $\epsilon a^t = a^t$. Moreover, when we write $a = \epsilon a + (1 - \epsilon)a$, we have:*
  (4.1) If $0 \neq \epsilon a \in \hat{R}$ then ϵa is nilpotent of index $t + 1$, $\epsilon a^t = a^t$ generates an essential ideal in $\epsilon\hat{R}$ and $(\epsilon a)^{t-1}k(\epsilon a)^t = (\epsilon a)^t k(\epsilon a)^{t-1}$ for every $k \in \mathrm{Skew}(\hat{R}, *)$.
  (4.2) If $0 \neq (1 - \epsilon)a \in \hat{R}$, then the index of ad-nilpotence of $(1 - \epsilon)a$ in $\hat{R}$ is not greater than n, and $(1 - \epsilon)a^t = 0$.

Proposition 2.17. [3, Proposition 5.5] *Let R be a semiprime ring with involution $*$ and free of 2-torsion, let $\hat{R}$ be its central closure, and let $a \in K$ be a pure ad-nilpotent element of K of index $n > 1$. Then:*

(1) *There exists an idempotent $\epsilon \in H(C(R), *)$ such that $(1 - \epsilon)a$ is an ad-nilpotent element of $\hat{R}$ of index $\leq n$ and ϵa is nilpotent with $\mathrm{ad}^n_{\mu\epsilon a}(\hat{R}) \neq 0$ for every $\mu \in C(R)$ such that $\mu\epsilon a \neq 0$.*
(2) *Moreover, if a is pure ad-nilpotent in K and R is free of $\binom{n}{t}$-torsion and t-torsion for $t := [\frac{n+1}{2}]$, when we write $a = \epsilon a + (1 - \epsilon)a$ we have:*
  (2.1) *If $\epsilon a \neq 0$ then ϵa is nilpotent of index $t + 1$.*
  (2.2) *If $(1 - \epsilon)a \neq 0$ then $(1 - \epsilon)a$ is pure ad-nilpotent in $\hat{R}$ of index n. In this case n is odd and there exists $\lambda \in \mathrm{Skew}(C(R), *)$ such that $((1 - \epsilon)a - \lambda)^t = 0$.*

3. Ad-Nilpotent elements of R

In the following result we will relate the index of nilpotence of a homogeneous element of R with its index of ad-nilpotence. It will be useful in our study of ad-nilpotent elements of K.

Proposition 3.1. *Let $R = R_0 + R_1$ be a semiprime associative superalgebra. If $a \in R$ is a homogeneous nilpotent element of index s and*

(1) *$a \in R_0$ and R is free of $\binom{2s-2}{s-1}$-torsion, then a is ad-nilpotent of R (and of R_0) of index $n = 2s - 1$,*
(2a) *$a \in R_1$, s is even and R is free of $\binom{s-2}{\frac{s-2}{2}}$-torsion, then a is ad-nilpotent of R of index $n = 2s - 2$ ($n \equiv_4 2$),*

(2b) *$a \in R_1$, s is odd and R is free of $\binom{s-1}{\frac{s-1}{2}}$-torsion, then a is ad-nilpotent of R of index $n = 2s-1$ ($n \equiv_4 1$).*

Proof. (1) Since $a \in R_0$, the operator ad_a behaves as the adjoint map in the non-super setting. From $a^s = 0$ we get that $\mathrm{ad}_a^{2s-1}(R) = 0$. On the other hand, $a^{s-1} \neq 0$, so by semiprimeness of R (and of R_0) (see Lemma 2.3) there exists $x \in R$ (respectively, $x \in R_0$) such that $a^{s-1}xa^{s-1} \neq 0$ and, since R has no $\binom{2s-2}{s-1}$-torsion, $\binom{2s-2}{s-1}a^{s-1}xa^{s-1} \neq 0$. Thus

$$\mathrm{ad}_a^{2s-2}(x) = \binom{2s-2}{s-1}(-1)^{s-1}a^{s-1}xa^{s-1} \neq 0.$$

We have shown that a is ad-nilpotent of R (and of R_0) of index $n = 2s-1$.

(2a) Suppose that $a \in R_1$ is a nilpotent element of even index s. Since $\mathrm{ad}_a^2 = \mathrm{ad}_{a^2}$ and $a^2 \in R_0$ is nilpotent of index $\frac{s}{2}$, we have by (1) that a^2 is ad-nilpotent of R of index $2(\frac{s}{2}) - 1 = s-1$. Hence the index of ad-nilpotence of a is less or equal to $2s-2$. Let x be any element in $R_0 \cup R_1$:

$$\begin{aligned}\mathrm{ad}_a^{2s-3}(x) &= \mathrm{ad}_a^{2s-4}\,\mathrm{ad}_a(x) = \mathrm{ad}_{a^2}^{s-2}\,\mathrm{ad}_a(x) = \\ &= \binom{s-2}{\frac{s-2}{2}}(-1)^{\frac{s-2}{2}}a^{s-2}(ax - (-1)^{|x|}xa)a^{s-2} = \\ &= \binom{s-2}{\frac{s-2}{2}}(-1)^{\frac{s-2}{2}}a^{s-1}xa^{s-2} - \binom{s-2}{\frac{s-2}{2}}(-1)^{\frac{s-2}{2}+|x|}a^{s-2}xa^{s-1}, \text{ hence}\end{aligned}$$

$$\mathrm{ad}_a^{2s-3}(x)a = \binom{s-2}{\frac{s-2}{2}}(-1)^{\frac{s-2}{2}}a^{s-1}xa^{s-1}.$$

Therefore $\mathrm{ad}_a^{2s-3}(R)$ cannot be zero, since otherwise $a^{s-1} = 0$ because R is free of $\binom{s-2}{\frac{s-2}{2}}$-torsion and semiprime, a contradiction. We have shown that a is ad-nilpotent of index $n = 2s-2$.

(2b) Suppose that $a \in R_1$ is a nilpotent element of odd index s. For any homogeneous $x \in R_0 \cup R_1$:

$$\begin{aligned}\mathrm{ad}_a^{2s-1}(x) &= \mathrm{ad}_a\,\mathrm{ad}_a^{2s-2}(x) = \mathrm{ad}_a\,\mathrm{ad}_{a^2}^{s-1}(x) = \mathrm{ad}_a\left(\binom{s-1}{\frac{s-1}{2}}(-1)^{\frac{s-1}{2}}a^{s-1}xa^{s-1}\right) = \\ &= \binom{s-1}{\frac{s-1}{2}}(-1)^{\frac{s-1}{2}}(a^sxa^{s-1} - (-1)^{|x|}a^{s-1}xa^s) = 0\end{aligned}$$

so $\mathrm{ad}_a^{2s-1}(R) = 0$. Let us see that $\mathrm{ad}_a^{2s-2}(R) \neq 0$: $a^{s-1} \neq 0$, so there exists $x \in R$ such that

$$\mathrm{ad}_a^{2s-2}(x) = \mathrm{ad}_{a^2}^{s-1}(x) = \binom{s-1}{\frac{s-1}{2}}(-1)^{\frac{s-1}{2}}a^{s-1}xa^{s-1} \neq 0$$

because R is semiprime and free of $\binom{s-1}{\frac{s-1}{2}}$-torsion. We have shown that a is ad-nilpotent of index $n = 2s-1$. □

In the following theorem we describe the homogeneous ad-nilpotent elements of R, depending on the equivalence class of their indexes of ad-nilpotence modulo 4.

Theorem 3.2. *Let us consider a prime associative superalgebra $R = R_0 + R_1$, let $\hat{R}$ denote the central closure of R, and let $a \in R_0 \cup R_1$ be a homogeneous ad-nilpotent* *element of index n. If R is free of $\binom{n}{s}$-torsion and free of s-torsion, for $s=[\frac{n+1}{2}]$, then:*

(1) *If $a\in R_0$, n is odd and exists $\lambda\in C(R)_0$ such that $a-\lambda\in\hat{R}$ is nilpotent of index $\frac{n+1}{2}$.*

(2) *If $a\in R_1$, then*

  (a) *if $n\equiv_4 1$ and R is free of $\binom{\frac{n-1}{2}}{\frac{s-1}{2}}$-torsion, then a is nilpotent of index $\frac{n+1}{2}$.*

  (b) *if $n\equiv_4 2$ then there is $\lambda\in C(R)_0$ such that $(a^2-\lambda)\in\hat{R}$ is nilpotent of index $\frac{n+2}{4}$.*

  (c) *the cases $n\equiv_4 0$ and $n\equiv_4 3$ do not occur.*

Proof. We will suppose without loss of generality that R is centrally closed.
(1) Let $a\in R_0$ be an ad-nilpotent element of index n. By Lemma 2.3, R is semiprime as an algebra. Moreover, the element a is a pure ad-nilpotent element of R because every graded ideal of R is essential (see 2.14). Therefore, we can use Theorem 2.15 to obtain that n is odd and there exists $\lambda\in C(R)$ such that $a-\lambda$ is nilpotent of index $\frac{n+1}{2}$. Moreover, $a\in R_0$, R is prime and has no $\frac{n+1}{2}$-torsion, so $\lambda\in C(R)_0$ by Lemma 2.10(i).
(2) Let $a\in R_1$ be an ad-nilpotent element of index n. Let us split our argument in two cases:

(2a) If n is odd, $n=2s-1$ for some s. Then $0=\mathrm{ad}_a^{n+1}(R)=\mathrm{ad}_a^{2s}(R)=\mathrm{ad}_{a^2}^{s}(R)$, and $a^2\in R_0$ is ad-nilpotent of index s (notice that $\mathrm{ad}_{a^2}^{s-1}(R)=\mathrm{ad}_a^{2s-2}(R)=\mathrm{ad}_a^{n-1}(R)\neq 0$). Therefore, by (1), s is odd (equivalently, $n\equiv_4 1$) and there exists $\lambda\in C(R)_0$ such that $a^2-\lambda$ is nilpotent of index $\frac{s+1}{2}$. Let us see prove that $\lambda=0$: Let us denote $b=(a^2-\lambda)^{\frac{s-1}{2}}$. Then, for every $x\in R_0\cup R_1$,

$$\begin{aligned}0&=\mathrm{ad}_a^n(x)=\mathrm{ad}_a(\mathrm{ad}_{a^2}^{\frac{n-1}{2}}(x))=\mathrm{ad}_a(\mathrm{ad}_{a^2-\lambda}^{\frac{n-1}{2}}(x))=\\&=[a,\sum_{i=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{i}(-1)^{\frac{n-1}{2}-i}(a^2-\lambda)^i x(a^2-\lambda)^{\frac{n-1}{2}-i}]=\\&=[a,\binom{\frac{n-1}{2}}{\frac{s-1}{2}}(-1)^{\frac{s-1}{2}}(a^2-\lambda)^{\frac{s-1}{2}}x(a^2-\lambda)^{\frac{s-1}{2}}]=\\&=[a,\binom{\frac{n-1}{2}}{\frac{s-1}{2}}(-1)^{\frac{s-1}{2}}bxb]=\binom{\frac{n-1}{2}}{\frac{s-1}{2}}(-1)^{\frac{s-1}{2}}\left(abxb-(-1)^{|x|}bxba\right).\end{aligned}$$

Since R is free of $\binom{\frac{n-1}{2}}{\frac{s-1}{2}}$-torsion, we get that

$$abxb=(-1)^{|x|}bxba,\qquad\text{for every } x\in R_0\cup R_1.$$

Take any $x\in R_0$. Multiplying this last equality by a on the left and taking into account that $ab=ba$ we have $a^2bxb=a(abxb)=a(bxba)=abxab$; but $a^2bxb=ab(ax)b=-b(ax)ba=-abxab$ because $ax\in R_1$. Then $a^2bR_0b=abR_0ab=0$. Similarly, for any $x\in R_1$ we have that $a^2bxb=a(abxb)=-a(bxab)$, and we also have that $a^2bxb=ab(ax)b=b(ax)ba=abxab$ because $ax\in R_0$. Then $a^2bR_1b=abR_1ab=0$. We have obtained

$$a^2bRb=abRab=0.$$

From the definition of b we have that $(a^2-\lambda)b=0$, i.e., $a^2b=\lambda b$, so $0=a^2bRb=\lambda bRb$. If $\lambda\neq 0$, we would have that $bRb=0$ (notice that $\lambda\in C(R)_0$ and $C(R)_0$ is a field (Lemma 2.7)), leading to a contradiction with the semiprimeness of R and $b\neq 0$.

Thus $\lambda=0$, so $0\neq b=a^{s-1}$, $ab=a^s$ and $0=abRab=a^sRa^s$ implies $a^s=0$ by semiprimeness of R.

(2b) If n is even, then $n=2s$ for some s, so $a^2\in R_0$ is ad-nilpotent of index s ($\mathrm{ad}_{a^2}^s(R)=\mathrm{ad}_a^n(R)=0$ and $\mathrm{ad}_{a^2}^{s-1}(R)=\mathrm{ad}_a^{2s-2}(R)=\mathrm{ad}_a^{n-2}(R)\neq 0$). Then by (1) we obtain that s is odd (equivalently, $n\equiv_4 2$) and there exists $\lambda\in C(R)_0$ such that $(a^2-\lambda)^{\frac{s+1}{2}}=0$.

Notice that the cases $n\equiv_4 0$ and $n\equiv_4 3$ do not occur. $\square$

4. Ad-Nilpotent elements of K

We start with a technical lemma, which is also interesting by itself. For example, it claims that every semiprime superalgebra with superinvolution and no nonzero skew even elements is a trivial superalgebra, i.e., the odd part is zero.

Lemma 4.1. *Let $R=R_0+R_1$ be a semiprime superalgebra with superinvolution $*$.*

(i) *If $K_0=0$ then $R_1=0$ and $R=R_0=H_0$ is commutative.*

(ii) *Let us consider $h_0\in H_0$. If $h_0K_0h_0=0$ then $h_0R_1h_0=0$ and $h_0Rh_0=h_0R_0h_0=h_0H_0h_0$ is commutative as the (trivial) local superalgebra of R at h_0.*

Proof. (i) Take any $k_1,k_1'\in K_1$ and $h_1,h_1'\in H_1$. Then, since $R_0=H_0$, we have that

$$k_1h_1=(k_1h_1)^*=h_1k_1,\qquad k_1k_1'=(k_1k_1')^*=-k_1'k_1,\qquad h_1h_1'=(h_1h_1')^*=-h_1'h_1.$$

In particular, $k_1^2=h_1^2=0$.

We claim that $K_1=0$. Take any $k_1\in K_1$. Then for every $h_0\in H_0$, $k_1h_0k_1=(k_1h_0k_1)^*=-k_1h_0k_1$ implies $k_1h_0k_1=0$, so $k_1H_0k_1=0$; similarly, for every $h_1\in H_1$, $(k_1h_1)k_1=h_1k_1^2=0$, so $h_1H_1h_1=0$, and, for every $k_1'\in K_1$, $(k_1k_1')k_1=-k_1'k_1^2=0$, so $k_1K_1k_1=0$. We have shown that $k_1Rk_1=0$, so by semiprimeness of R, $k_1=0$.

Let us show that $H_1=0$. Take any $h_1\in H_1$. For every $h_0\in H_0$, since $h_1h_0h_1=(h_1h_0h_1)^*=-h_1h_0h_1$, we have that $h_1h_0h_1=0$, so $h_1H_0h_1=0$. Similarly, for every $h_1'\in H_1$, $h_1h_1'h_1=-h_1'h_1^2=0$, so $h_1H_1h_1=0$, and, finally, for every $k_1\in K_1$, $h_1k_1h_1=k_1h_1^2=0$, so $h_1K_1h_1=0$. We have shown that $h_1Rh_1=0$, so by semiprimeness of R, $h_1=0$.

Therefore, $R_1=H_1+K_1=0$.

Finally, H_0 is commutative because for every $h_0,h_0'\in H_0$,

$$h_0h_0'=(h_0h_0')^*=h_0'h_0.$$

(ii) Take $h_0\in H_0$ and let us consider the local algebra $R_{h_0}=h_0Rh_0$ as defined in 2.8, which is an associative superalgebra with induced superinvolution $(h_0xh_0)^\star:=h_0x^*h_0$, for every $x\in R$. Clearly $\mathrm{Skew}(h_0Rh_0,\star)=h_0Kh_0$ and $\mathrm{Sym}(h_0Rh_0,\star)=h_0Hh_0$. If we suppose that $h_0K_0h_0=0$ then $\mathrm{Skew}(h_0Rh_0,\star)_0=0$ and by (i) we have

$$(R_{h_0})_1=h_0R_1h_0=0\quad\text{ and }\quad R_{h_0}=h_0Rh_0=(R_{h_0})_0=h_0R_0h_0=h_0H_0h_0.$$

□

Proposition 4.2. *Let R be a prime associative superalgebra with superinvolution $*$ and let $a \in K$ be a homogeneous ad-nilpotent element of K of index $n > 2$. Suppose that R is free of $\binom{n}{s}$-torsion and free of s-torsion, for $s = [\frac{n+1}{2}]$. If $\mathrm{Skew}(C(R), *) \neq 0$ then a is ad-nilpotent of R of index n. Otherwise, a is nilpotent.*

Proof. If there exists a homogeneous $0 \neq \lambda \in \mathrm{Skew}(C(R), *)$ then λ^2 is invertible in the field $C(R)_0$, and $R = K + \lambda^2 H \subseteq K + \lambda K$ so $\mathrm{ad}_a^n(R) = 0$. Suppose from now on that $\mathrm{Skew}(C(R), *) = 0$. We split our proof in two cases, depending on the parity of a:

(I) Suppose that $a \in K_0$. Let us see that a is nilpotent. Every $x \in R$ can be expressed as $x = x_h + x_k$ for $x_h := \frac{x+x^*}{2} \in H$ and $x_k = \frac{x-x^*}{2} \in K$, so for every $x \in R$

$$\begin{aligned}\mathrm{ad}_a^n(ax + xa) = \mathrm{ad}_a^n(ax_k + x_k a) + \mathrm{ad}_a^n(ax_h + x_h a) = a\,\mathrm{ad}_a^n(x_k) + \mathrm{ad}_a^n(x_k)a \\ + \mathrm{ad}_a^n(ax_h + x_h a) = 0\end{aligned}$$

because $ax_h + x_h a \in K$ and $a\,\mathrm{ad}_a^i(x) = \mathrm{ad}_a^i(ax)$ for every $x \in R$ and any $i \in \mathbb{N}$. Expanding this expression

$$0 = \mathrm{ad}_a^n(ax+xa) = (-1)^n xa^{n+1} + \sum_{i=1}^{n}\left(\binom{n}{i} - \binom{n}{i-1}\right)(-1)^{n-i}a^i x a^{n+1-i} + a^{n+1}x.$$

Since R is semiprime as an algebra, by Lemma 2.11, a is an algebraic element of R over $C(R)$.

(I.a) Let us suppose that R is prime as an algebra. The calculations of (1.b) in the proof of [3, Proposition 5.5] show that a is nilpotent.

(I.b) If R is prime as a superalgebra but not prime as an algebra, R_0 is prime by 2.4, $C(R)_0 \cong C(R_0)$ by 2.6, the superinvolution $*$ restricted to R_0 is an involution and $\mathrm{Skew}(C(R_0), *) = 0$ because we are assuming that $\mathrm{Skew}(C(R), *) = 0$. The element a is a pure ad-nilpotent element of K_0 because $C(R_0)$ is a field, so we can apply Proposition 2.17(2) to the prime associative algebra R_0 to obtain that a is nilpotent.

(II) If $a \in K_1$, consider $a^2 \in K_0$ and by (I), a^2 is nilpotent, i.e., a is nilpotent. □

In the following two theorems we will describe the homogeneous ad-nilpotent elements of K. Our goal is to relate the index of ad-nilpotence of a homogeneous element of K with its index of ad-nilpotence in R (and in R_0 and in K_0 when the element is even). Moreover, when these indexes in K and in R do not coincide, we will show that the element is nilpotent of an explicit index.

We begin with the description of even ad-nilpotent elements of K.

Theorem 4.3. *Let R be a prime associative superalgebra of characteristic $p > n$ with superinvolution $*$, let $\hat{R}$ be its central closure, let $a \in K_0 := \mathrm{Skew}(R, *)_0$ be an ad-nilpotent element of K of index $n > 1$ and let $s = [\frac{n+1}{2}]$. Then*

1. *If $n \equiv_4 0$ then a is nilpotent of index $s+1$, ad-nilpotent of R and of R_0 of index $n+1$ and satisfies $a^s K a^s = 0$. Moreover, the index of ad-nilpotence of a in K_0 can be $n-1$ or n.*
2. *If $n \equiv_4 1$ then there exists $\lambda \in \mathrm{Skew}(C(R), *)_0$ such that $a - \lambda \in \hat{R}$ is nilpotent of index s and a is ad-nilpotent of R, of R_0 and of K_0 of index n.*
3. *The case $n \equiv_4 2$ is not possible.*

(4) *If $n \equiv_4 3$ then either:*

(4.1) *there exists $\lambda \in \mathrm{Skew}(C(R),*)_0$ such that $a-\lambda \in \hat{R}$ is nilpotent of index s and a is ad-nilpotent of R, of R_0 and of K_0 of index n, or*

(4.2) *a is nilpotent of index $s+1$, ad-nilpotent of K_0 of index n, ad-nilpotent of R and of R_0 of index $n+2$ and satisfies $a^ska^{s-1}-a^{s-1}ka^s=0$ for every $k \in K$. In particular R satisfies $a^sKa^s=0$.*

Proof. Suppose without loss of generality that R is centrally closed. Let $a \in K_0$ be an ad-nilpotent element of K of index n.
– If $\mathrm{Skew}(C(R),*) \neq 0$, by Proposition 4.2, a is ad-nilpotent of index n of R and by Theorem 3.2 n has to be odd ($n \equiv_4 1$ or $n \equiv_4 3$) and there exists $\lambda \in C(R)_0$ such that $a-\lambda$ is nilpotent of index s, so a is ad-nilpotent of R and of R_0 of the same index $n = 2s-1$, see Proposition 3.1(1). Moreover, $\lambda \in \mathrm{Skew}(C(R),*)_0$ by Lemma 2.10 and since $\mathrm{Skew}(C(R),*)_0 \subset \mathrm{Skew}(C(R_0),*)$, the index of ad-nilpotence of $a-\lambda$ in K_0 is again $n = 2s-1$ (notice that, by Lemma 2.10(ii), λ is the unique element of $C(R_0)$ such that $a-\lambda$ is nilpotent). These are the cases (2) and (4.1).
– If $\mathrm{Skew}(C(R),*) = 0$, by Proposition 4.2, a is nilpotent. We are going to approach this case considering the index of ad-nilpotence of a in K_0 and comparing it with its index of ad-nilpotence in K and in R. Let us suppose that a is ad-nilpotent of K_0 of index $m \leq n$ and let $r = [\frac{m+1}{2}]$. Since R_0 is a semiprime algebra and the superinvolution $*$ restricted to R_0 is an involution, by Proposition 2.16 we have four possibilities:

• $m \equiv_4 0$ then a is nilpotent of index $r+1$ and $a^rK_0a^r = 0$, which, by Lemma 4.1(ii), implies that $a^rR_1a^r = 0$, so a is also ad-nilpotent of index m of K, i.e., $m = n$ and a is nilpotent of index $s+1$ with $s = \frac{n}{2} = r$. Now, since $s+1$ is the index of nilpotence of a, by Proposition 3.1(1) a is ad-nilpotent of index $n+1$ of R and of R_0. This is the case (1) ($n \equiv_4 0$) with the index of ad-nilpotence of a in K_0 equal to the index of ad-nilpotence of a in K.

• $m \equiv_4 1$ then a is nilpotent of index r. This implies, by Proposition 3.1(1), that a is ad-nilpotent of R and of R_0 of index m. So n has to be equal to m and therefore the index of nilpotence of a is $s = \frac{n+1}{2} = r$. This is the case (2), i.e., $n \equiv_4 1$.

• $m \equiv_4 2$ does not occur.

• $m \equiv_4 3$ then there exists an idempotent $\epsilon \in C(R_0)$ such that $\epsilon a^r = a^r$ and a decomposes as $a = \epsilon a + (1-\epsilon)a$ (although the elements ϵa and $(1-\epsilon)a$ do not belong to R but in central closure of R_0, this decomposition will be useful for our purposes):

◇ If $\epsilon a = 0$ then $a = (1-\epsilon)a$ is nilpotent of index r. By Proposition 3.1(1), this implies that a is ad-nilpotent of R and of R_0 of index m, so $n = m$ and the index of nilpotence of a is $s = \frac{n+1}{2} = r$. This is the case (4.1), i.e., $n \equiv_4 3$.

◇ If $\epsilon a \neq 0$ then a is nilpotent of index $r+1$ and $a^rk_0a^{r-1} - a^{r-1}k_0a^r = (\epsilon a)^rk_0(\epsilon a)^{r-1} - (\epsilon a)^{r-1}k_0(\epsilon a)^r = 0$ for every $k_0 \in K_0$. Since $a^{r+1} = 0$, $a^rK_0a^r = 0$ and, by Lemma 4.1(ii), $a^rR_1a^r = 0$, so $a^rKa^r = 0$ and therefore $\mathrm{ad}_a^{m+1} K = 0$. There are two possibilities:

– Either $a^rka^{r-1} - a^{r-1}ka^r = 0$ for every homogeneous $k \in K$ and therefore a is ad-nilpotent of index m of K. Then $n = m$, $r = \frac{n+1}{2} = s$, so $a^ska^{s-1} - a^{s-1}ka^s = 0$ and a is nilpotent of index $s+1$ which, by

Proposition 3.1(1), implies that a is ad-nilpotent of R and of R_0 of index $n+2$ and fits with the case (4.2), i.e., $n \equiv_4 3$,

– or there exists $k \in K$ such that $a^r k a^{r-1} - a^{r-1} k a^r \neq 0$, so a is ad-nilpotent of K of index $m+1$. Hence $n = m+1$, $r = \frac{n}{2} = s$, and a is nilpotent of index $s+1$. Therefore, by Proposition 3.1(1), a is ad-nilpotent of R and of R_0 of index $n+1$. This is again case (1) with the index of ad-nilpotence of a in K_0 equal to $n-1$ and $n \equiv_4 0$.

□

In the following theorem we describe the odd ad-nilpotent elements of K. We will first distinguish whether $C(R)$ has skew-symmetric elements, in which case a is ad-nilpotent of R of the same index, or $\mathrm{Skew}(C(R), *) = 0$, which implies by Proposition 4.2 that a is nilpotent. In this second case, we will consider $a^2 \in K_0$ and use Theorem 4.3 applied to a^2 to obtain the description of a.

Theorem 4.4. *Let R be a prime associative superalgebra of characteristic $p > n$ with superinvolution $*$, let $\hat{R}$ be its central closure, let $a \in K_1 := \mathrm{Skew}(R, *)_1$ be an ad-nilpotent element of K of index $n > 1$ and let $s = [\frac{n+1}{2}]$.*

(1) *If $n \equiv_8 0$ then a is nilpotent of index $s+1$, ad-nilpotent of R of index $n+1$ and $a^s K a^s = 0$ (so $a^s R a^s$ is a commutative trivial local superalgebra).*
(2) *If $n \equiv_8 1$ then $a^{s-1} \in H_0$, and a is nilpotent of index s and ad-nilpotent of R of index n.*
(3) *If $n \equiv_8 2$ then there exists $\lambda \in \mathrm{Skew}(C(R), *)_0$ such that $a^2 - \lambda \in \hat{R}$ is nilpotent of index $\frac{s+1}{2}$ and a is ad-nilpotent of R of index n.*
(4) *If $n \equiv_8 5$ then $a^{s-1} \in K_0$, and a is nilpotent of index s and ad-nilpotent of R of index n.*
(5) *If $n \equiv_8 6$ then there exists $\lambda \in \mathrm{Skew}(C(R), *)_0$ such that $a^2 - \lambda \in \hat{R}$ is nilpotent of index $\frac{s+1}{2}$ and a is ad-nilpotent of R of index n.*
(6) *If $n \equiv_8 7$ then a is nilpotent of index $s+1$, ad-nilpotent of R of index $n+2$ and $a^s k a^{s-1} + (-1)^{|k|} a^{s-1} k a^s = 0$ for every homogeneous $k \in K$ (so $a^s R a^s$ is a commutative trivial local superalgebra).*
(7) *The cases $n \equiv_8 3$ and $n \equiv_8 4$ do not occur.*

Proof. Suppose without loss of generality that R is centrally closed.

Let $a \in K_1$ be an ad-nilpotent element of K of index n. If $\mathrm{Skew}(C(R), *) \neq 0$, by Proposition 4.2, a is ad-nilpotent of R of index n. By Theorem 3.2 n can be:

- $n \equiv_4 1$ and therefore a is nilpotent of index s (cases (2) and (4)), or
- $n \equiv_4 2$ and therefore there exists $\lambda \in \mathrm{Skew}(C(R)_0, *)$ such that $a^2 - \lambda$ is nilpotent of index $\frac{s+1}{2}$ (cases (3) and (5)).

Let us suppose that $\mathrm{Skew}(C(R), *) = 0$. By Proposition 4.2, a is nilpotent. Then, since $a^2 \in K_0$ and $\mathrm{ad}_a^2(x) = \mathrm{ad}_{a^2}(x)$, a^2 is an ad-nilpotent element of K. Let us denote by m the index of ad-nilpotence of a^2 in K and let $r = \left[\frac{m+1}{2}\right]$. By Theorem 4.3 applied to the element a^2 we have:

• If $m \equiv_4 0$ and $r = \frac{m}{2}$, $(a^2)^r \neq 0$, $(a^2)^{r+1} = 0$ and $a^{2r} K a^{2r} = 0$. We are going to show that $a^{2r+1} = 0$: let x be any homogeneous element in R, so

$ax + (-1)^{|x|}x^*a \in K_{1+|x|}$,

$$
\begin{aligned}
0 = \mathrm{ad}_{a^2}^m(ax + (-1)^{|x|}x^*a)a &= \binom{m}{\frac{m}{2}}(-1)^{\frac{m}{2}}(a^m(ax + (-1)^{|x|}x^*a)a^m)a = \\
&= \binom{m}{r}(-1)^r a^{2r}(ax + (-1)^{|x|}x^*a)a^{2r+1} = \binom{m}{r}(-1)^r a^{2r+1}xa^{2r+1} \\
&+ \binom{m}{r}(-1)^r(-1)^{|x|}a^{2r}x^*a^{2r+2} = \binom{m}{r}(-1)^r a^{2r+1}xa^{2r+1}.
\end{aligned}
$$

Since R is semiprime and free of $\binom{m}{r}$-torsion, $a^{2r+1} = 0$. Moreover, since $ad_{a^2}^{m-1}(K) \neq 0$, we have two possibilities:

- If $\mathrm{ad}_a^{2m-1}(K) \neq 0$, then a is an ad-nilpotent element of K of index $n = 2m$. In this case $n \equiv_8 0$ and for $s = \frac{n}{2}$ we have that $a^{s+1} = 0$, $a^s \neq 0$ and $a^sKa^s = 0$. Moreover, by Proposition 3.1, a is ad-nilpotent of R of index $n+1$, case (1).
- If $\mathrm{ad}_a^{2m-1}(K) = 0$, then a is an ad-nilpotent element of K of index $n = 2m-1$. So in this case we have got $n \equiv_8 7$ and for $s = \frac{n+1}{2}$ we have that $a^{s+1} = 0$, $a^s \neq 0$. Moreover, for every homogeneous $k \in K$,

$$
\begin{aligned}
0 = \mathrm{ad}_a^{2m-1}(k) &= \binom{m-1}{\frac{m}{2}}(-1)^{\frac{m}{2}}(a^mka^{m-1} + (-1)^{|k|}a^{m-1}ka^m) = \\
&= \binom{m-1}{\frac{s}{2}}(-1)^{\frac{s}{2}}(a^ska^{s-1} + (-1)^{|k|}a^{s-1}ka^s)
\end{aligned}
$$

and since R is free of $\binom{m-1}{\frac{s}{2}}$-torsion we have that $a^ska^{s-1} + (-1)^{|k|}a^{s-1}ka^s = 0$. In addition, by Proposition 3.1, a is ad-nilpotent element of R of index $n+2$, case (6).

- If $m \equiv_4 1$ and $r = \frac{m+1}{2}$ we have that $(a^2)^r = 0$, $(a^2)^{r-1} \neq 0$ and $\mathrm{ad}_{a^2}^m(R) = 0$. Since $ad_{a^2}^{m-1}(K) \neq 0$, we have two possibilities:
  - If $\mathrm{ad}_a^{2m-1}(K) \neq 0$, then a is an ad-nilpotent element of K of index $n = 2m$ and there exists a homogeneous k in K such that:

$$
\begin{aligned}
0 \neq \mathrm{ad}_a^{2m-1}(k) &= \mathrm{ad}_{a^2}^{m-1}\,\mathrm{ad}_a(k) = \\
&= \binom{m-1}{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}}(a^mka^{m-1} - (-1)^{|k|}a^{m-1}ka^m) = \\
&= \binom{m-1}{r}(-1)^r(a^{2r-1}ka^{2r-2} - (-1)^{|k|}a^{2r-2}ka^{2r-1}).
\end{aligned}
$$

Therefore, since R is free of $\binom{m-1}{r}$-torsion, $a^{2r-1} \neq 0$. In this case $n \equiv_8 2$ and for $s = \frac{n}{2}$ we have that $a^{s+1} = 0$, $a^s \neq 0$. By Proposition 3.1, a is ad-nilpotent of index n, case (3).

• If $\mathrm{ad}_a^{2m-1}(K)=0$, then a is ad-nilpotent of K of index $n=2m-1$. Let x be any homogeneous element in R and let us consider $ax+(-1)^{|x|}x^*a\in K_{1+|x|}$:

$$\begin{aligned}
0 &= \mathrm{ad}_a^{2m-1}(ax+(-1)^{|x|}x^*a)=\mathrm{ad}_a^{2m-2}\,\mathrm{ad}_a(ax+(-1)^{|x|}x^*a)=\\
&= \mathrm{ad}_{a^2}^{m-1}\,\mathrm{ad}_a(ax+(-1)^{|x|}x^*a)=\\
&= \binom{m-1}{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}}a^{m-1}(a^2x+(-1)^{|x|}ax^*a-(-1)^{1+|x|}(axa+(-1)^{|x|}x^*a^2))a^{m-1}=\\
&= \binom{m-1}{r-1}(-1)^{r-1}a^{2r-2}(a^2x+(-1)^{|x|}ax^*a-(-1)^{1+|x|}(axa+(-1)^{|x|}x^*a^2))a^{2r-2}=\\
&= \binom{m-1}{r-1}(-1)^{\frac{m-1}{2}+|x|}a^{2r-1}(x^*+x)a^{2r-1}
\end{aligned}$$

and

$$\begin{aligned}
0 &= \mathrm{ad}_a^{2m-1}(x-x^*)a=\mathrm{ad}_a^{2m-2}\,\mathrm{ad}_a(x-x^*)a=\mathrm{ad}_{a^2}^{m-1}\,\mathrm{ad}_a(x-x^*)a=\\
&= \binom{m-1}{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}}a^{m-1}(ax-ax^*-(-1)^{|x|}(xa-x^*a))a^m=\\
&= \binom{m-1}{r-1}(-1)^{r-1}a^{2r-2}(ax-ax^*-(-1)^{|x|}(xa-x^*a))a^{2r-1}=\\
&= \binom{m-1}{r-1}(-1)^{r-1}a^{2r-1}(x-x^*)a^{2r-1}.
\end{aligned}$$

Therefore, since R is free of $\binom{m-1}{r-1}$-torsion, $a^{2r-1}Ra^{2r-1}=0$, and by semiprimeness of R, $a^{2r-1}=0$ and a is an ad-nilpotent element of R of index $n=2m-1$. So $n\equiv_8 1$ and for $s=\frac{n+1}{2}$ we have that $a^s=0$, $a^{s-1}\neq 0$. By Proposition 3.1, a is ad-nilpotent of R of index n, case (2).

• $m\equiv_4 2$ is not possible.

• If $m\equiv_4 3$ and $r=\frac{m+1}{2}$, let us first see that $(a^2)^r=0$. Suppose otherwise that $(a^2)^r\neq 0$. Then $(a^2)^{r+1}=0$ and $a^{2r}ka^{2r-2}-a^{2r-2}ka^{2r}=0$ for every $k\in K$. Let x be any homogeneous element in R and let us consider $ax+(-1)^{|x|}x^*a\in K_{1+|x|}$:

$$\begin{aligned}
0 &= \mathrm{ad}_{a^2}^{m}(ax+(-1)^{|x|}x^*a)a^3=\binom{m}{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}}a^{m+1}(ax+(-1)^{|x|}x^*a)a^{m+2}+\\
&+\binom{m}{\frac{m+1}{2}}(-1)^{\frac{m+1}{2}}a^{m-1}axa^{m+4}+\binom{m}{\frac{m+1}{2}}(-1)^{\frac{m+1}{2}}a^{m-1}(-1)^{|x|}x^*aa^{m+4}=\\
&= \binom{m}{r-1}(-1)^{r-1}a^{2r}(ax+(-1)^{|x|}x^*a)a^{2r+1}+\binom{m}{\frac{m+1}{2}}(-1)^{\frac{m+1}{2}}a^{2r-2}axa^{2r+3}\\
&+\binom{m}{\frac{m+1}{2}}(-1)^{\frac{m+1}{2}}a^{2r-2}(-1)^{|x|}x^*aa^{2r+3}=\binom{m}{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}}a^{2r+1}xa^{2r+1}
\end{aligned}$$

and therefore, since R is free of $\binom{m}{r-1}$-torsion and semiprime, $a^{2r+1}=0$. Then for every homogeneous $x \in R$

$$\begin{aligned}
0 &= a\,\mathrm{ad}_{a^2}^m(ax+(-1)^{|x|}x^*a) = \binom{m}{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}}a^{m+2}(ax+(-1)^{|x|}x^*a)a^{m-1}+\\
&+\binom{m}{\frac{m+1}{2}}(-1)^{\frac{m+1}{2}}a^m axa^{m+1}+\binom{m}{\frac{m+1}{2}}(-1)^{\frac{m+1}{2}}a^m(-1)^{|x|}x^*aa^{m+1}=\\
&=\binom{m}{r-1}(-1)^{r-1}a^{2r+1}(ax+(-1)^{|x|}x^*a)a^{2r-2}+\\
&+\binom{m}{r}(-1)^r a^{2r-1}axa^{2r}+\binom{m}{r}(-1)^r a^{2r-1}(-1)^{|x|}x^*aa^{2r}=\binom{m}{r}(-1)^r a^{2r}xa^{2r}
\end{aligned}$$

and therefore, since R is free of $\binom{m}{r}$-torsion and semiprime, $a^{2r}=0$, a contradiction. Thus $(a^2)^r=0$, $(a^2)^{r-1}\neq 0$ and $\mathrm{ad}_{a^2}^m(R)=0$.

• If $\mathrm{ad}_a^{2m-1}(K)\neq 0$, then a is ad-nilpotent of K of index $n=2m$ and there exists $k\in K$ homogeneous such that

$$\begin{aligned}
0 \neq \mathrm{ad}_a^{2m-1}(k) &= \mathrm{ad}_a^{2m-2}\mathrm{ad}_a(k) = \mathrm{ad}_{a^2}^{m-1}\mathrm{ad}_a(k) =\\
&=\binom{m-1}{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}}(a^m k a^{m-1}-(-1)^{|k|}a^{m-1}ka^m)=\\
&=\binom{m-1}{r-1}(-1)^{r-1}(a^{2r-1}ka^{2r-2}-(-1)^{|k|}a^{2r-2}ka^{2r-1}).
\end{aligned}$$

Therefore, since R is free of $\binom{m-1}{r-1}$-torsion, $a^{2r-1}\neq 0$ so a is nilpotent of index $2r$. So $n\equiv_8 6$ and with $s=\frac{n}{2}$, $a^{s+1}=0$, $a^s\neq 0$ and by Proposition 3.1 a is ad-nilpotent of R of index n, case (5).

• If $\mathrm{ad}_a^{2m-1}(K)=0$, then a is ad-nilpotent of K of index $n=2m-1$. Let x be any homogeneous element in R and let us consider $ax+(-1)^{|x|}x^*a\in K_{1+|x|}$:

$$\begin{aligned}
0&=\mathrm{ad}_a^{2m-1}(ax+(-1)^{|x|}x^*a)=\mathrm{ad}_a^{2m-2}\mathrm{ad}_a(ax+(-1)^{|x|}x^*a)=\\
&=\mathrm{ad}_{a^2}^{m-1}\mathrm{ad}_a(ax+(-1)^{|x|}x^*a)=\\
&=\binom{m-1}{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}}a^{m-1}(a^2x+(-1)^{|x|}ax^*a-(-1)^{1+|x|}(axa+(-1)^{|x|}x^*a^2))a^{m-1}=\\
&=\binom{m-1}{r-1}(-1)^{r-1}a^{2r-2}(a^2x+(-1)^{|x|}ax^*a-(-1)^{1+|x|}(axa+(-1)^{|x|}x^*a^2))a^{2r-2}=\\
&=\binom{m-1}{r-1}(-1)^{r-1+|x|}a^{2r-1}(x^*+x)a^{2r-1},
\end{aligned}$$

and

$$\begin{aligned}
0&=\mathrm{ad}_a^{2m-1}(x-x^*)a=\mathrm{ad}_a^{2m-2}\mathrm{ad}_a(x-x^*)a=\mathrm{ad}_{a^2}^{m-1}\mathrm{ad}_a(x-x^*)a=\\
&=\binom{m-1}{\frac{m-1}{2}}(-1)^{\frac{m-1}{2}}a^{m-1}(ax-ax^*-(-1)^{|x|}(xa-x^*a))a^m=\\
&=\binom{m-1}{r-1}(-1)^{r-1}a^{2r-2}(ax-ax^*-(-1)^{|x|}(xa-x^*a))a^{2r-1}=\\
&=\binom{m-1}{r-1}(-1)^{\frac{m-1}{2}}a^{2r-1}(x-x^*)a^{2r-1}.
\end{aligned}$$

Therefore, since R is free of $\binom{m-1}{r-1}$-torsion, $a^{2r-1}Ra^{2r-1}=0$, and by semiprimeness of R, $a^{2r-1}=0$. So in this case $n \equiv_8 5$. For $s=\frac{n+1}{2}$ we have that $a^s=0$, $a^{s-1}\neq 0$ and, by Proposition 3.1, a is an ad-nilpotent element of R of index n, case (4). □

5. Examples

In this section we are going to construct examples of all types of homogeneous ad-nilpotent elements appearing in Theorem 3.2, and in Theorems 4.3 and 4.4. The examples of even ad-nilpotent elements of R and of K are based on the examples of ad-nilpotent elements in the non-super setting, see [2].

5.1. Let Φ be a ring of scalars and let r,s be natural numbers. Following the notation of [19], the matrix algebra $\mathcal{M}_{r+s}(\Phi)$ with

$$\mathcal{M}(r|s)_0 := \left\{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} : A \in \mathcal{M}_r(\Phi), D \in \mathcal{M}_s(\Phi) \right\} \text{ and}$$
$$\mathcal{M}(r|s)_1 := \left\{ \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} : B \in \mathcal{M}_{r,s}(\Phi), C \in \mathcal{M}_{s,r}(\Phi) \right\}$$

becomes an $\mathbb{Z}_2$-graded associative algebra. It will be denoted $\mathcal{M}(r|s)=\mathcal{M}(r|s)_0+\mathcal{M}(r|s)_1$. We will use the notation $\mathcal{M}(r)=\mathcal{M}(r|r)$.

5.2. Let r and s be two natural numbers with odd $r>1$ and even s, let $\mathbb{F}$ be a field with involution denoted by $\overline{\alpha}$ for any $\alpha \in \mathbb{F}$, and let R be the superalgebra $\mathcal{M}(r|s)$ over $\mathbb{F}$. Let $\{e_{i,j}\}$ denote the matrix units, and define

$$\begin{array}{ll} H=\sum_{i=1}^{r}(-1)^i e_{i,r+1-i} \in \mathcal{M}_r(\mathbb{F}) & (\text{notice } H=H^t=H^{-1}) \\ J=\sum_{i=1}^{s}(-1)^i e_{i,s+1-i} \in \mathcal{M}_s(\mathbb{F}) & (\text{notice } J^t=-J=J^{-1}). \end{array}$$

The map $*: R\to R$ given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} H & 0 \\ 0 & J \end{bmatrix}^{-1} \overline{\begin{bmatrix} A & -B \\ C & D \end{bmatrix}}^t \begin{bmatrix} H & 0 \\ 0 & J \end{bmatrix}$$

defines a superinvolution in R. In particular

$$e_{i,j}^* = (-1)^{j-i} e_{r-j+1,r-i+1} \text{ for every } i,j\in\{1,\dots,r\},$$

$$e_{r+i,r+j}^* = (-1)^{j-i} e_{r+s-j+1,r+s-i+1} \text{ for every } i,j\in\{1,\dots,s\} \text{ and}$$

$$e_{i,r+j}^* = (-1)^{i-j+1} e_{r+s+1-j,r+1-i} \text{ for every } i\in\{1,...,r\} \text{ and } j\in\{1,...,s\}.$$

The associative superalgebra R is a simple superalgebra with superinvolution, and its extended centroid $C(R)$, which coincides with $Z(R)$, is isomorphic to $\mathbb{F}$. Moreover, the extension of the superinvolution $*$ to $C(R)$ is isomorphic to the involution $-$ of $\mathbb{F}$.

5.3. Examples of even ad-nilpotent elements of K and of R.
Let k be an even number($k\geq 2$), let $r=3k+3$ and $s=2k$, and let us consider the associative superalgebra $R=\mathcal{M}(r|s)$ over $\mathbb{F}$ with the superinvolution defined in 5.2. Let us denote by K the skew-symmetric elements of R with respect to $*$.

Consider the following nilpotent matrices:

$$T := \sum_{i=k+2}^{2k+1} e_{i,i+1} \in R_0 \text{ (nilpotent of index } k+1)$$

$$S := \sum_{i=1}^{k-1} (e_{i,i+1} + e_{r-i,r-i+1}) \in R_0 \text{ (nilpotent of index } k)$$

$$U := \sum_{i=1}^{k-1} e_{r+i,r+i+1} + \sum_{i=k+1}^{2k-1} e_{r+i,r+i+1} \in R_0 \text{ (nilpotent of index } k).$$

By Proposition 3.1(1), T is ad-nilpotent of R and of R_0 of index $2k+1$, and S and U are ad-nilpotent elements of R and of R_0 of index $2k-1$.

Notice that $T^* = -T$, $S^* = -S$ and $U^* = -U$ so $T, S, U \in K_0$. Let us calculate their indexes of ad-nilpotence in K:

(a) If $\mathrm{Skew}(\mathbb{F}, -) \neq 0$, by Proposition 4.2 the index of ad-nilpotence of T in K coincides with its index of ad-nilpotence in R, i.e., $2k+1$.

(b) If $\mathrm{Skew}(\mathbb{F}, -) = 0$, for any $B = \sum_{i,j} \lambda_{i,j} e_{i,j} \in K$ we have that $\lambda_{2k+2,k+2} \in \mathrm{Skew}(\mathbb{F}, -) = 0$ and $\lambda_{2k+1,k+2} - \lambda_{2k+2,k+3} \in \mathrm{Skew}(\mathbb{F}, -) = 0$, so

$$\mathrm{ad}_T^{2k-1}(B) = \binom{2k-1}{k}(T^{k-1}BT^k - T^kBT^{k-1}) =$$
$$= \binom{2k-1}{k}((e_{k+2,2k+1} + e_{k+3,2k+2})B(e_{k+2,2k+2})) -$$
$$- \binom{2k-1}{k}(e_{k+2,2k+2})B(e_{k+2,2k+1} + e_{k+3,2k+2})) =$$
$$= \binom{2k-1}{k}(\lambda_{2k+1,k+2}\, e_{k+2,2k+2} + \lambda_{2k+2,k+2}\, e_{k+3,2k+2}) -$$
$$- \binom{2k-1}{k}(\lambda_{2k+2,k+2}\, e_{k+2,2k+1} + \lambda_{2k+2,k+3}\, e_{k+2,2k+2}) = 0.$$

Furthermore,

$$\mathrm{ad}_T^{2k-2}(e_{2k+1,k+2} - e^*_{2k+1,k+2}) = \mathrm{ad}_T^{2k-2}(e_{2k+1,k+2} + e_{2k+2,k+3}) \neq 0.$$

Thus T is ad-nilpotent of K of index $2k-1$.

(c) S is ad-nilpotent of K of index $2k-1$: by its ad-nilpotence in R, we have $\mathrm{ad}_S^{2k-1}(K) = 0$. Moreover, $0 \neq C = e_{k,1} - e^*_{k,1} = e_{k,1} + e_{r,r-k+1} \in K$ and

$$\mathrm{ad}_S^{2k-2}(C) = -\binom{2k-2}{k-1} S^{k-1}(e_{k,1} + e_{r,r-k+1})S^{k-1} =$$
$$= -\binom{2k-2}{k-1}(e_{1,k} + e_{r-k+1,r})(e_{k,1} + e_{r,r-k+1})(e_{1,k} + e_{r-k+1,r}) =$$
$$= -\binom{2k-2}{k-1}(e_{1,k} + e_{r-k+1,r}) \neq 0,$$

so S is also ad-nilpotent of K of index $2k-1$.

(d) U is ad-nilpotent of K of index $2k-1$: by its ad-nilpotence in R, we have $\mathrm{ad}_U^{2k-1}(K) = 0$. Moreover, $0 \neq C = e_{r+k,r+1} - e^*_{r+k,r+1} = e_{r+k,r+1} +$

$e_{r+2k,r+k+1} \in K$ and

$$
\begin{aligned}
&\mathrm{ad}_U^{2k-2}(C) = \mathrm{ad}_U^{2k-2}(e_{r+k,r+1} + e_{r+2k,r+k+1}) = \\
&= -\binom{2k-2}{k-1} U^{k-1}(e_{r+k,r+1} + e_{r+2k,r+k+1})U^{k-1} = \\
&= -\binom{2k-2}{k-1}(e_{r+1,r+k} + e_{r+k+1,r+2k}) \neq 0.
\end{aligned}
$$

Let us use these matrices T, S and U to get examples of any of models of even ad-nilpotent elements in Theorems 3.2 and 4.3.

(i). Suppose $\mathrm{Skew}(\mathbb{F}, -) \neq 0$. For any $\lambda \in \mathrm{Skew}(\mathbb{F}, -)$, the element $T + \lambda\,\mathrm{id}$ is ad-nilpotent of R of index $2k+1$, and by Proposition 4.2 its index in K is again $n = 2k+1$. This is an example that fits case (2) of Theorem 4.3 (a skew element a in K_0 with nilpotent $(a - \lambda)$ of index $k+1$ such that a is ad-nilpotent of index $n \equiv_4 1$ in K and the same index in R). It also provides an example of case (1) in Theorem 3.2.

(ii). Suppose $\mathrm{Skew}(\mathbb{F}, -) \neq 0$. For any $\lambda \in \mathrm{Skew}(\mathbb{F}, -)$, $S + \lambda\,\mathrm{id}$ is an ad-nilpotent element of R and of K of index $n = 2k-1$. This is an example that fits case (1) of Theorem 3.2 and case (4.1) of Theorem 4.3 (a skew element in K_0, which is ad-nilpotent of index $n \equiv_4 3$ in K_0 and in K, and ad-nilpotent of the same index in R).

(iii). Suppose $\mathrm{Skew}(\mathbb{F}, -) = 0$. T is an element of K_0 which is ad-nilpotent of K of index $n = 2k-1$. This is an example that fits case (4.2) of Theorem 4.3 (an element in K_0 which is ad-nilpotent of index $n \equiv_4 3$ in K and in K_0, and ad-nilpotent of index $n+2$ in R).

(iv). Suppose $\mathrm{Skew}(\mathbb{F}, -) = 0$. The matrix $A = T + S$, which is an orthogonal sum of T and S, is nilpotent of index $t+1$ and ad-nilpotent of R and of R_0 of index $2k+1$. Let us see that it is ad-nilpotent of K of index $2k$: from the indexes of nilpotence of T and S, their indexes of ad-nilpotence in K and the fact that $TS = 0 = ST$ we get that $\mathrm{ad}_A^{2k}(K) = 0$. Moreover, $C = e_{k,k+2} - e^*_{k,k+2} = e_{k,k+2} - e_{2k+2,2k+4} \in K$ and one can check that $\mathrm{ad}_A^{2k-1}(C) = -\binom{2k-1}{k}(e_{1,2k+2} + e_{k+2,3k+3}) \neq 0$. This is an example that fits case (1.1) of Theorem 4.3 (a skew element in K_0 which is ad-nilpotent of index $n \equiv_4 0$ in K_0 and in K, and ad-nilpotent of index $n+1$ in R).

(v). Suppose $\mathrm{Skew}(\mathbb{F}, -) = 0$. Let us consider $A = T + U$, which is an orthogonal sum of T and U. The nilpotence of $T + U$ implies that the index of ad-nilpotence of A in R (and in R_0) is $2k+1$ (by Proposition 3.1(1)). Since both T and U are ad-nilpotent elements of K_0 of indexes $2k-1$, A is ad-nilpotent of K_0 of index $2k-1$. Nevertheless, its index of ad-nilpotence in K is higher: for any $B = \sum \lambda_{i,j} e_{i,j} \in K$ we have that

$$
\begin{aligned}
\mathrm{ad}_A^{2k}(B) &= \binom{2k}{k} A^k B A^k = \binom{2k}{k} e_{k+2,2k+2}\, B\, e_{k+2,2k+2} = \\
&= \binom{2k}{k} \lambda_{2k+2,k+2} e_{k+2,2k+2} = 0
\end{aligned}
$$

because $\lambda_{2k+2,k+2} \in \mathrm{Skew}(\mathbb{F},-) = 0$. Moreover, if we consider the element $C = e_{2k+2,r+1} - e^*_{2k+2,r+1} = e_{2k+2,r+1} - e_{r+s,k+2} \in K$ one can check that

$$\begin{aligned}\mathrm{ad}_A^{2k-1}(C) &= \binom{2k-1}{k}(A^{k-1}CA^k - A^kCA^{k-1}) = \\ &= -\binom{2k-1}{k}(e_{r+k+1,2k+2} + e_{k+2,r+k}) \neq 0\end{aligned}$$

because

$$A^{k-1} = T^{k-1} + U^{k-1} = e_{k+2,2k+1} + e_{k+3,2k+2} + e_{r+1,r+k} + e_{r+k+1,r+s}.$$

This means that the index of ad-nilpotence of A in K is $n = 2k$. This gives an example of an element in the conditions of Theorem 4.3(1) (a skew element in K_0, which ad-nilpotent of K of index $n \equiv_4 0$, ad-nilpotent of K_0 of index $n-1$, and ad-nilpotent of R index $n+1$).

5.4. Examples of odd ad-nilpotent elements of K and of R.

Let $\mathbb{F}$ be a field with identity involution, let $r > 1$ be an odd number, let $s = r-1$, and consider the superalgebra $R = \mathcal{M}(r|s)$ with the superinvolution given in 5.2. Again, let us denote by K the skew-symmetric elements of R with respect to $*$.

Let us consider $T := \sum_{i=1}^{r-1} e_{i,r+i} \in R_1$. Then

$$A = T - T^* = \sum_{i=1}^{r-1} e_{i,r+i} + \sum_{i=2}^{r} e_{r+i-1,i} \in K_1 \text{ (nilpotent of index } 2r-1).$$

We have that

$$\begin{aligned}
A^2 &= \textstyle\sum_{i=1}^{r-1} e_{i,i+1} + \sum_{i=2}^{r-1} e_{r+i-1,r+i},\\
A^{2r-7} &= e_{1,2r-3} + e_{2,2r-2} + e_{3,2r-1} + e_{r+1,r-2} + e_{r+2,r-1} + e_{r+3,r},\\
A^{2r-6} &= e_{1,r-2} + e_{2,r-1} + e_{3,r} + e_{r+1,2r-2} + e_{r+2,2r-1},\\
A^{2r-3} &= e_{1,2r-1} + e_{r+1,r},\\
A^{2r-2} &= e_{1,r} \text{ and}\\
A^{2r-1} &= 0.
\end{aligned}$$

By Proposition 3.1(2b) A is ad-nilpotent in R of index $m = 4r-3$. For every $B = \sum_{i,j} \lambda_{i,j} e_{i,j} \in K_0 \cup K_1$,

$$\begin{aligned}
&\mathrm{ad}_A^{4r-5}(B) = \mathrm{ad}_{A^2}^{2r-3}\,\mathrm{ad}_A(B) = \\
&= \binom{2r-3}{r-1}((A^2)^{r-2}\,\mathrm{ad}_A(B)(A^2)^{r-1} - (A^2)^{r-1}\,\mathrm{ad}_A(B)(A^2)^{r-2}) = \\
&= \binom{2r-3}{r-1}(A^{2r-3}BA^{2r-2} + (-1)^{|B|}A^{2r-2}BA^{2r-3}) = \\
&= \binom{2r-3}{r-1}((e_{1,2r-1} + e_{r+1,r})Be_{1,r} + (-1)^{|B|}e_{1,r}B(e_{1,2r-1} + e_{r+1,r})) = \\
&= \binom{2r-3}{r-1}(\lambda_{2r-1,1}e_{1,r} + \lambda_{r,1}e_{r+1,r} + (-1)^{|B|}\lambda_{r,1}e_{1,2r-1} + (-1)^{|B|}\lambda_{r,r+1}e_{1,r}) = 0
\end{aligned}$$

because when $B \in K_0$ we always have that $\lambda_{2r-1,1} = \lambda_{r,r+1} = 0$ (by grading) and $\lambda_{r,1} = 0$, and when $B \in K_1$, $\lambda_{r,1} = 0$ (by grading) and $\lambda_{2r-1,1} = \lambda_{r,r+1}$. Moreover, by Theorem 4.4, the index of ad-nilpotence of A in K can be m, $m-1$ or $m-2$, so it is $m - 2 = 4r - 5$.

(i). The element $A \in K_1$ is an example of an element in the conditions of Theorem 4.4(6) (a nilpotent element of index $2r-1$, which is ad-nilpotent of index $n = 4r-5 \equiv_8 7$ in K and ad-nilpotent of index $n+2$ in R, and such that $A^{2r-3}BA^{2r-2} + (-1)^{|B|}A^{2r-2}BA^{2r-3} = 0$ for every $B \in K_0 \cup K_1$).

To produce examples for the rest of the cases of Theorem 4.4, let us consider $A^5 \in K_1$ for some particular cases of odd $r > 1$.

(ii). Fix $r = 10t+1$ for some $t \in \mathbb{N}$. Then

$$(A^5)^{4t+1} = A^{2r+3} = 0,$$
$$(A^5)^{4t} = A^{2r-2} \in H_0,$$
$$(A^5)^{4t-1} = A^{2r-7}.$$

In particular, A^5 is nilpotent of index $4t+1$ and ad-nilpotent of R of index $8t+1$. Notice that for every $B = \sum_{i,j} \lambda_{i,j} e_{i,j} \in K$

$$(A^5)^{4t}B(A^5)^{4t} = e_{1,r}Be_{1,r} = \lambda_{r,1}e_{1,r} = 0$$

because every $B \in K$ has $\lambda_{r,1} = 0$. Therefore, for every $B \in K$ we have

$$\mathrm{ad}^{8t}_{A^5}(B) = \mathrm{ad}^{4t}_{A^{10}}(B) = \binom{4t}{2t}(A^{10})^{2t}B(A^{10})^{2t} = 0.$$

Furthermore, considering $C = e_{r,r+1} - e^*_{r,r+1} = e_{r,r+1} + e_{2r-1,1} \in K_1$

$$\begin{aligned}
&\mathrm{ad}^{8t-1}_{A^5}(C) = \mathrm{ad}^{8t-2}_{A^5}(\mathrm{ad}_{A^5}(e_{r,r+1} + e_{2r-1,1})) = \\
&= \mathrm{ad}^{4t-1}_{A^{10}}(\mathrm{ad}_{A^5}(e_{r,r+1} + e_{2r-1,1})) = \\
&= \binom{4t-1}{2t}(A^{10})^{2t-1}(\mathrm{ad}_{A^5}(e_{r,r+1} + e_{2r-1,1}))(A^{10})^{2t} - \\
&- \binom{4t-1}{2t}(A^{10})^{2t}(\mathrm{ad}_{A^5}(e_{r,r+1} + e_{2r-1,1}))(A^{10})^{2t-1} = \\
&= \binom{4t-1}{2t}\left(A^{20t-5}(e_{r,r+1} + e_{2r-1,1})A^{20t}\right) - \left(A^{20t}(e_{r,r+1} + e_{2r-1,1})A^{20t-5}\right) = \\
&= \binom{4t-1}{2t}(e_{3,r} - e_{1,r-2}) \neq 0.
\end{aligned}$$

The element A^5 gives an example of an element in the conditions of Theorem 4.4(1) (a nilpotent element of index $4t+1$, ad-nilpotent element in K_1 of index $n = 8t \equiv_8 0$, ad-nilpotent in R of index $n+1 = 8t+1$ and such that $(A^5)^{4t}K(A^5)^{4t} = 0$).

(iii). Fix $r = 10t+3$ for some $t \in \mathbb{N}$. Then

$$(A^5)^{4t+1} = A^{2r-1} = 0$$
$$(A^5)^{4t} = A^{2r-6}.$$

In particular, A^5 is nilpotent of index $4t+1$ and ad-nilpotent of R of index $8t+1$ (see Proposition 3.1(2b)). In this case the index of ad-nilpotence of A^5 in K is the same as in R because for $C = e_{r,r+1} - e^*_{r,r+1} = e_{r,r+1} + e_{2r-1,1} \in K_1$ we have

$$\begin{aligned}
\mathrm{ad}_{A^5}{}^{8t}(C) &= \mathrm{ad}^{4t}_{A^{10}}(e_{r,r+1} + e_{2r-1,1}) = \\
&= \binom{4t}{2t}(A^{10})^{2t}(e_{r,r+1} + e_{2r-1,1})(A^{10})^{2t} = \\
&= \binom{4t}{2t}(e_{3,2r-2} + e_{r+2,r-2}) \neq 0.
\end{aligned}$$

The element A^5 gives an example of an element in the conditions of Theorem 4.4(2) (a nilpotent element in K_1 of index $4t+1$, ad-nilpotent of K and of R of the same index $n = 8t+1 \equiv_8 1$).

(iv). Fix $r = 10t+5$ for some $t \in \mathbb{N}$. Then A^5 is nilpotent of index $4t+2$. Since the index of nilpotence of A^5 is even, we know by Proposition 3.1(2a) that A^5 is ad-nilpotent of R of index $2(4t+2)-2 = 8t+2$. Moreover, from the fact that A^5 is ad-nilpotent of R of index $8t+2 \equiv_8 2$ we get from Theorem 4.4 that its index of ad-nilpotence in K is the same as in R. The element A^5 gives an example of an element in the conditions of Theorem 4.4(3) with $\lambda = 0$ (a nilpotent element of K_1 of index $4t+2$ which is ad-nilpotent of K and of R of the same index $n = 8t+2 \equiv_8 2$.)

(v). Fix $r = 10t+7$ for some $t \in \mathbb{N}$. Then A^5 is nilpotent of index $4t+3$. Since the index of nilpotence of A^5 is odd, we know by Proposition 3.1(2a) that A^5 is ad-nilpotent of R of index $2(4t+3)-1 = 8t+5$. Moreover, from the fact that A^5 is ad-nilpotent of R of index $8t+5 \equiv_8 5$ we get from Theorem 4.4 that its index of ad-nilpotence in K is the same as in R. The element A^5 gives an example of an element in the conditions of Theorem 4.4(4) (a nilpotent element of K_1 of index $4t+3$ which is ad-nilpotent of K and of R of the same index $n = 8t+5 \equiv_8 5$).

(vi). Fix $r = 10t+9$ for some $t \in \mathbb{N}$. Then A^5 is nilpotent of $4t+4$. Since the index of nilpotence of A^5 is even, we know by Proposition 3.1(2a) that A^5 is ad-nilpotent of R of index $2(4t+4)-2 = 8t+6$. Moreover, from the fact that A^5 is ad-nilpotent of R of index $8t+6 \equiv_8 6$ we get from Theorem 4.4 that its index of ad-nilpotence in K is the same as in R. The element A^5 gives an example of an element in the conditions of Theorem 4.4(5) with $\lambda = 0$ (a nilpotent element of K_1 of index $4t+4$ which is ad-nilpotent of K and of R of the same index $n = 8t+6 \equiv_8 6$).

The matrices given in **(i)**, **(ii)**, **(iii)** and **(v)** provide examples of (2.a) in Theorem 3.2. Moreover, the matrices of **(iv)** and **(vi)** fit in case (2.b) of Theorem 3.2 with $\lambda = 0$.

5.5. Some other examples of odd ad-nilpotent elements of K and of R.

The examples **(iv)** and **(vi)** in the previous section are ad-nilpotent elements of K of indexes $n \equiv_8 2$ and $n \equiv_8 6$, and fit in Theorem 4.4(3) and (5) with $\lambda = 0$. To get examples of such types of elements with nonzero λ's, we will work with matrices over a field with nontrivial involution.

Let r be a natural number, let $\mathbb{C}$ be the field of complex numbers with involution given by conjugation, and let us consider the simple superalgebra $R = \mathcal{M}(r)$ over $\mathbb{C}$. The map trp given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{trp} = \begin{bmatrix} D^t & -B^t \\ C^t & A^t \end{bmatrix},$$

where $A, B, C, D \in \mathcal{M}_r(\mathbb{C})$ and $(\ )^t$ denotes the usual matrix transposition, defines a superinvolution in R known as the transpose superperinvolution (see [14, Example 2.2]).

Let us denote by K the set of skew-symmetric elements of $\mathcal{M}(r)$ with respect trp. Note that any element of K_1 has the form $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ where B is a symmetric matrix and C is a skew-symmetric matrix in $\mathcal{M}_r(\mathbb{C})$ with respect to the usual transposition.

Let us consider a symmetric matrix $B \in \mathcal{M}_r(\mathbb{C})$ with $B^r = 0$ and $B^{r-1} \neq 0$ (it is shown in [20, Corollary 5] that for every r there exist symmetric nilpotent matrices in $\mathcal{M}_r(\mathbb{C})$ of rank $r-1$). Let $0 \neq \lambda \in \mathbb{R}$ and let i denote the square root of -1. Then

$$a = \begin{bmatrix} 0 & B + \mathrm{id} \\ (\lambda i)\,\mathrm{id} & 0 \end{bmatrix} \in K_1 \text{ and } a^2 = \begin{bmatrix} (\lambda i)B + (\lambda i)\,\mathrm{id} & 0 \\ 0 & (\lambda i)B + (\lambda i)\,\mathrm{id} \end{bmatrix}$$

i.e., $(a^2 - \lambda i)$ is nilpotent of index r.

When r is odd a is an example for Theorem 4.4 (3), and when r is even a is an example for Theorem 4.4 (5). Both cases are examples of elements of the form (2.b) of Theorem 3.2.

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