Strictly stable families of aggregation operators

Karina Rojas

Facultad de Ciencias Matemáticas, Universidad Complutense, Plaza de las Ciencias 3, 28040 Madrid, Spain,
E-mail: krpatuelli@mat.ucm.es

Daniel Gómez[1,*]

Facultad de Estudios Estadísticos, Universidad Complutense, Avenida Puerta de Hierro s/n 28040, Madrid, Spain,
E-mail: dagomez@estad.ucm.es

Javier Montero[*]

Facultad de Ciencias Matemáticas, Universidad Complutense, Plaza de las Ciencias 3, 28040 Madrid, Spain,
E-mail: monty@mat.ucm.es

J. Tinguaro Rodriguez[*]

Facultad de Ciencias Matemáticas, Universidad Complutense, Plaza de las Ciencias 3, 28040 Madrid, Spain,
E-mail: jtrodrig@mat.ucm.es

Abstract

In this paper we analyze the notion of *family of aggregation operators* (FAO), also refereed to as *extended aggregation functions* (EAF), i.e., a set of aggregation operators defined in the unit interval which aggregate several input values into a single output value. In particular, we address the key issue of the relationship that should hold between the operators in a family in order to understand they properly define a *consistent* FAO. We focus on the idea of *strict stability* of a family of aggregation operators in order to propose an operative notion of *consistency* between operators of such a family. In this way, robustness of the aggregation process can be guaranteed. Some strict stability definitions for $FAOs$ are proposed, leading to a classification of the main aggregation operators in terms of the properties they satisfy. Furthermore, we apply this approach to analyze the stability of some families of aggregation operators based on weights.

Keywords: Aggregation operators; stability; consistency; self-identity; fuzzy sets.

[*]Corresponding author
Email addresses: krpatuelli@yahoo.com (Karina Rojas), dagomez@estad.ucm.es (Daniel Gómez), monty@mat.ucm.es (Javier Montero), jtrodriguez@mat.ucm.es (J. Tinguaro Rodriguez)

[1]Facultad de Estudios Estadísticos, Universidad Complutense, Avenida Puerta de Hierro s/n, 28040, Madrid, Spain
email. dagomez@estad.ucm.es

Preprint submitted to Fuzzy Sets and Systems *February 21, 2017*

1. Introduction.

Aggregation of information appears in a natural way in all kinds of knowledge based systems (see, e.g., [2, 6, 11]). Usually, the aggregation-fusion process produces a reduction in the dimension of the original data, a need whenever the decision maker can not manage the excessive complexity of a problem. The main aim of such aggregation is to simplify information. Without loss of generality we can say that an *aggregation operator* is basically defined as a real function A_n that, from n data items $x_1,\ldots,x_n$ in $[0,1] \subseteq [+\infty, -\infty]$, produces an aggregated value $A_n(x_1,\ldots,x_n)$ in [0,1] [4].

Nevertheless, some information can frequently get lost or deleted, and even added. In these cases, a data cardinality change occurs, and each time it happens a different aggregation operator A_m has to be used to aggregate the new collection of m elements. This rather simple issue, known as the *dimensionality problem*, has however at least two deep implications.

In first place, instead of just a single operator, to effectively solve the aggregation problem, it is rather needed to count with a *family* of operators, which enables to aggregate collections of items with different dimension. This has led to the current standard definition [4, 15] of a *family of aggregation operators* (FAO) as a set $\{A_n : [0,1]^n \to [0,1], n \in \mathbb{N}\}$, providing instructions on how to aggregate collections of items of any dimension n. This sequence of aggregation functions $\{A_n\}_{n\in\mathbb{N}}$ is also called *extended aggregation functions* (EAF) by other authors [5, 15].

In second place, the aggregation process as a whole has to possess some kind of *consistency* despite the inevitable cardinality changes. This is, the operators that compose a FAO have to be somehow related, so the aggregation process remains *the same* throughout the possible changes in the dimension n of the data. For example, it would seem quite strange to propose a FAO using the minimum for $n = 2$, the arithmetic mean for $n = 3$, the geometric mean for $n = 4$ and the median for $n = 5$. And though it could seem that a formal approach could solve this problem by demanding a conceptual unity through a mathematical formula, it should be noted that this last example allows a trivial compact mathematical formulation. Therefore, it seems logical to study properties giving sense to the sequences $A(2)$, $A(3)$, $A(4), \ldots$, because otherwise we may have only a bunch of disconnected operators.

However, note that the current definition of a FAO does not demand any additional property regarding this last point. In this sense, many properties have been studied in relation to single aggregation operators A_n, such as continuity, commutativity, monotonicity and associativity, just to mention a few. In contrast, few efforts have been dedicated to study the relations between these operators as members of a family of aggregation functions. As it has been pointed out in some previous works (see for example [1, 7–9, 12]), most commonly assumed properties (e.g. continuity) represent desirable characteristics related to each aggregation function A_n, but they do not provide any condition regarding the *consistency* of the FAO as a whole. In this sense, no relation is being imposed among the members of a given family of operators.

Thus, as it is shown in [12], a debate about the importance of considering $FAOs$ as a consistent whole is a necessary to-do task. For example, the recursive rules in [8, 9] suggest a notion of *consistency* based on the construction of aggregation operators in a sequential way. The key idea of recursiveness is that, in order to be consistent, an aggregation rule should be based upon an iterative application of binary operators taking advantage of previous aggregations. This idea is also studied in [1, 12], in which the recursive rules are generalized in a more flexible way. In addition the concept of *operativity* of a FAO is defined and analyzed in [17, 18], trying to capture the notion of consistency from a computational point of view.

Nevertheless, the notion of consistency, in the above-exposed sense of a necessary relation between the members of a FAO, is perhaps too wide. Many facets have to be taken into account. For example, consistency is indeed a more general concept than that of recursiveness, since some non-recursive operators, like the median, fulfil such idea of consistency. Hence, it seems more plausible and convenient, at least as a first step, to define properties expressing such a notion from different perspectives, allowing different kinds of consistency instead of pursuing a single definition of such general notion of consistency.

Particularly, in this paper we study a notion of consistency based on the robustness of the aggregation process. In this sense, we introduce the property of *strict stability* for a family of aggregation functions extending the *self identity* property defined in [24]. Such strict stability property tries to force a family to have a stable/continuous definition, in the sense that an operator defined for n items should not differ too much of an operator of the same family defined for $n-1$ elements, when the last-added n-th item of information is the aggregation of the previous $n-1$ ones. Therefore, this property gives us some restrictions to be considered in order to maintain the logical consistency of operators of a given family, in such a way that the robustness of the aggregation process when a data cardinality change occurs is guaranteed.

In the second section of this paper, the concept of *strict stability* is formalized, and different possible levels of strict stability for a family of aggregation operators are considered. In the third section, we study the strict stability level of some standard families of operators. In the fourth section, we analyze the weights of the weighted mean FAO, with the aim of giving conditions that guarantee the robustness of an aggregation process based on weighted operators. On the fifth section, the convergence of some $FAOs$ in terms of the stability of its behavior is represented by means of additional computational experiments. Finally, this paper is concluded with some final comments.

2. Strict stability of a family of aggregation operators.

As pointed out in [4], *stability* of any mathematical model for engineering/applied problems means, roughly speaking, that "small input errors" does not gives us "big output errors". The *stability* property for an aggregation function A_n is defined in a similar way to a Lipschitz condition, in the sense that small changes in the vector x should not produce big changes in $A_n(x)$.

Therefore, stability is a concept that has been already studied in the framework of aggregation operators. For example, in [2], a definition of $p-stability$ for a family of aggregation operators $\{A_n\}_{n\in\mathbb{N}}$ was proposed. A FAO is considered p-stable if all the operators that define such a family are p-stable.

In this way, if we take a FAO using the maximum if n is even, and the minimum if n is odd, then we would get a p-stable FAO since these two functions are p-stable. But clearly an aggregation process with this definition is not robust. In our opinion, this definition of stability can not guarantee a stable behavior in terms of the outputs of the aggregation process when cardinality changes are possible. In other words, it can not guarantee the existence of a unifying concept linking the members of a family of aggregation operators.

Similarly, in [15] it is shown that the aggregation functions of a family can be related by means of certain grouping properties. However, again it is assumed that a FAO fulfills a property if $\forall\, n \in \mathbb{N}$ the $n-ary$ function fulfills such a property.

The notion of *strict stability* of a family of aggregation operators proposed here is also inspired in continuity, though our approach focuses in the cardinality of the data rather than in the data itself. In this way, we shall be able to assure some robustness in the result of the aggregation process. Particularly, let $A_n(x_1,\dots,x_n)$ be the aggregated value of the n-dimensional data $x_1,\dots,x_n$. Now, let us suppose that a new element x_{n+1} has to be aggregated. If x_{n+1} is close to the aggregation result $A_n(x_1,\dots,x_n)$ of the previous n-dimensional data $x_1,\dots,x_n$, then the result of aggregating these $n+1$ elements should not differ too much with the result of aggregating such n items. Following the idea of stability for any mathematical tool, if $|x_{n+1}-A_n(x_1,\dots,x_n)|$ is small, then $|A_{n+1}(x_1,\dots,x_n,x_{n+1})-A_n(x_1,\dots,x_n)|$ should be also small. Thus, given a stable family, it should be reasonable to assume that if $|x_{n+1}-A_n(x_1,\dots,x_n)|=0$, then $|A_{n+1}(x_1,\dots,x_n,x_{n+1})-A_n(x_1,\dots,x_n)|$ should be also zero. Therefore, our approach is obviously partially gathered in the $self-identity$ definition given in [24].

Definition 2.1. *(Yager 1997). Let $\{A_n : [0,1]^n \to [0,1], n \in \mathbb{N}\}$ be a family of aggregation operators. Then, it is said that the family $\{A_n\}$ satisfy the self-identity property if, $\forall n \in N$ and $\forall x_1,\dots,x_n \in [0,1]$, the following holds:*

$$A_n(x_1,x_2,...x_{n-1},A_{n-1}(x_1,x_2...,x_{n-1}))=A_{n-1}(x_1,x_2,...,x_{n-1})$$

Let us observe that self-identity is close to the stability idea in the sense that if the new item that has to be aggregated coincides with the aggregation value of the previous data, then the new result should not change. Nevertheless, in the self-identity definition it is implicitly imposed the fact that the information has to be aggregated in some order, specifically from left to right, so we have to put the last data in the $n-th$ position of the aggregation function.

It is important to note that if the family $\{A_n\}_{n\in\mathbb{N}}$ is not symmetric (i.e. there exist a n for which the aggregation operator A_n is not symmetric), then the position of the new data is relevant in the final output of the aggregation process.

For example, let us analyze self-identity in the *backward inductive extension* $\{A_n^b\}_{n\in\mathbb{N}}$ and *forward inductive extension* $\{A_n^f\}_{n\in\mathbb{N}}$ [4] of any binary aggregation operator, defined for $n>2$ as $A_n^b = L_2\left(x_1, L_2(\dots, L_2(x_{n-1},x_n)\dots\right)\ \ for\, n>2$, and $A_n^f = L_2\left(\dots,(L_2(L_2(x_1,x_2),x_3)),\dots,x_n\right)\ \ for\, n>$

2 , where L_2 is a binary aggregation operator, i.e. $L_2 : [0,1]^2 \to [0,1]$.

It can be proven that the family of aggregation functions $\{A_n^f\}_{n\in\mathbb{N}}$ satisfies self-identity if L_2 is idempotent, i.e., $A_n(x, ..., x) = x$, for all $n \in \mathbb{N}$ and $x \in [0,1]$ (see also [4]). Nevertheless, the family $\{A_n^b\}_{n\in\mathbb{N}}$ does not satisfy self-identity since the order in which this family aggregates the information is inverse (i.e. from right to left). In our opinion, the family $A_n^b = L_2(x_1, L_2(\ldots, L_2(x_{n-1}, x_n)))$ $for\ n > 2$ should be consistent in the sense of stability when the information is aggregated from right to left. From this observation, we propose the following definitions for stability, that extend the notion of self-identity both in the direction of allowing its application to non-symmetric operators, as well as in the direction of allowing different levels (strict, weak,...) of fulfillment.

Definition 2.2. *Let $\{A_n : [0,1]^n \to [0,1], n \in \mathbb{N}\}$ be a family of aggregation operators. Then, it is said that:*

1. $\{A_n\}_{n\in\mathbb{N}}$ *is a R-strictly stable family if*

$$A_n(x_1, x_2, ..., x_{n-1}, A_{n-1}(x_1, x_2..., x_{n-1})) = A_{n-1}(x_1, x_2, ..., x_{n-1})$$

holds $\forall n \geq 3$ *and* $\forall \{x_n\}_{n\in\mathbb{N}}$ *in* $[0,1]$

2. $\{A_n\}_{n\in\mathbb{N}}$ *is a L-strictly stable family if*

$$A_n(A_{n-1}(x_1, x_2, ..., x_{n-1}), x_1, x_2, ..., x_{n-1}) = A_{n-1}(x_1, x_2, ..., x_{n-1})$$

holds $\forall n \geq 3$ *and* $\forall \{x_n\}_{n\in\mathbb{N}}$ *in* $[0,1]$

3. $\{A_n\}_{n\in\mathbb{N}}$ *is a LR-strictly stable family if both properties hold simultaneously.*

Let us observe that in case the family $\{A_n\}_{n\in\mathbb{N}}$ is symmetric (i.e. for all n, A_n is a symmetric aggregation operator), then the three previous definitions are equivalent and coincide with the self-identity property defined by Yager. Thus, those symmetric families that satisfy the self-identity property, will be LR-strictly stable families. Among others, the following well-known FAOs belong to this group: the minimum $\{Min_n(x_1, \ldots, x_n)\}_{n\in\mathbb{N}}$, the maximum $\{Max_n(x_1, \ldots, x_n)\}_{n\in\mathbb{N}}$, the median $\{Md_n\}_{n\in\mathbb{N}}$, the arithmetic mean $\{M_n(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i/n\}_{n\in\mathbb{N}}$, the geometric mean $\{G_n(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i)^{1/n}\}_{n\in\mathbb{N}}$ and the harmonic mean $\{H_n(x_1, \ldots, x_n) = n/(\sum_{i=1}^{n} 1/x_i)\}_{n\in\mathbb{N}}$.

On the other hand, non-symmetric families, as for example the weighted mean family

$$W_n(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i^n \cdot x_i,\ n \in \mathbb{N},$$

present differences based on how its weights are built, as it will be discussed below. Also, as it has been already pointed out, any family of binary idempotent operators with inductive forward extension $\{A^f\}_{n\in\mathbb{N}}$ satisfies *R-strict stability*, and any family of binary idempotent operators with inductive backward extension $\{A^b\}_{n\in\mathbb{N}}$ satisfies *L-strict stability*.

Although the previous definition presents a reasonable approach to the idea of consistency of a FAO (i.e., from the point of view of its stability in front of cardinality changes), it is important to note that not all *consistent* families are included in this definition. Let us consider for example the family of product aggregation operators, defined as $\left\{P_n(x_1,\ldots,x_n)=\prod_{i=1}^{n}(x_i)\right\}_{n\in\mathbb{N}}$. This family defines an aggregation process that can be considered as *consistent*, but it does not satisfy any of the three previous definitions. In this way the product FAO shows the existing differences between the properties of recursion and stability, since it is a recursive operator but it is not a strictly stable one. Similarly, some weighted mean based aggregation processes can be considered as *consistent* though they do not fulfill the above definition either.

Therefore, in order to extend the proposed approach to other consistent $FAOs$, we propose the following two definitions, that express relaxed versions of the same strict stability concept: in the first one, strict stability is fulfilled in the limit, while in the second one, a weaker concept of stability is reached by demanding the operators to be, in the limit, almost sure strictly stable.

Definition 2.3. *Let* $\{A_n:[0,1]^n\to[0,1], n\in\mathbb{N}\}$ *be a family of aggregation operators. Then, it is said that:*

1. $\{A_n\}_{n\in\mathbb{N}}$ *is an asymptotically R-strictly stable family if*

$$\lim_{n\to+\infty}\left|A_n(x_1,...x_{n-1},A_{n-1}(x_1,...,x_{n-1}))-A_{n-1}(x_1,...,x_{n-1})\right|=0$$

   holds $\forall\{x_n\}_{n\in\mathbb{N}}$ *in* $[0,1]$.

2. $\{A_n\}_{n\in\mathbb{N}}$ *is an asymptotically L-strictly stable family if*

$$\lim_{n\to+\infty}\left|A_n(A_{n-1}(x_1,...,x_{n-1}),x_1,...,x_{n-1})-A_{n-1}(x_1,...,x_{n-1})\right|=0$$

   holds $\forall\{x_n\}_{n\in\mathbb{N}}$ *in* $[0,1]$.

3. $\{A_n\}_{n\in\mathbb{N}}$ *is an asymptotically strictly stable family if the two above properties simultaneously hold.*

In the previous definition, let us observe that we have a point-wise convergence of n-ary aggregation operators that has to be satisfied for any possible succession $\{x_n\}_{n\in\mathbb{N}}$. Also, note that any FAO that converges to a strictly stable FAO (and particularly any strictly stable FAO) is asymptotically strictly stable. In order to illustrate this point, let us consider the case of the family $\{W_n, n\in\mathbb{N}\}$ of weighted mean operators defined through a vector of weights $w^n=(w_1^n,...,w_n^n)\in[0,1]^n$ in such a way that $W_n(x_1,...,x_n)=\sum_{i=1}^{n}w_i^n x_i$, where $\sum_{i=1}^{n}w_i^n=1$ and $(x_1,...,x_n)\in[0,1]^n\ \ \forall n$. Thus, if for example the weights w^n are given by

$$w_i^n = \begin{cases} \dfrac{1}{(n+1)} & \text{if } 1 \le i \le n-1 \\ \dfrac{2}{(n+1)} & \text{if } i = n \end{cases}$$

then the family $\{W_n\}_{n\in\mathbb{N}}$ is *L-strictly stable*. However, it is not *R-strictly stable*: just choose for $n = 3$ the data vector $(x_1, x_2) = (0, 1)$, so it holds $W_2(0, 1) = 2/3$ but $W_3(0, 1, W_2(0, 1)) = W_3(0, 1, 2/3) = 7/12$. Nevertheless, the family $\{W_n\}_{n\in\mathbb{N}}$ is *asymptotically R-strictly stable*.

To see it, let us denote

$$dif_R(W_n, W_{n-1}) = W_n(x_1, \dots, x_{n-1}, W_{n-1}(x_1, \dots, x_{n-1})) - W_{n-1}(x_1, \dots, x_{n-1}).$$

Then, it holds that

$$dif_R(W_n, W_{n-1}) = \sum_{i=1}^{n-1}(w_i^n - (1 - w_n^n)w_i^{n-1})x_i = \sum_{i=1}^{n-2} \frac{1}{n(n+1)}x_i + \frac{2-n}{n(n+1)}x_{n-1}$$

and thus

$$\left|dif_R(W_n, W_{n-1})\right| \le \sum_{i=1}^{n-2}\left|\frac{1}{n(n+1)}x_i\right| + \left|\frac{2-n}{n(n+1)}x_{n-1}\right| \le \frac{(n-2)}{n(n+1)} + \frac{(n-2)}{n(n+1)} = 2\frac{(n-2)}{n(n+1)}$$

so it is $\left|dif_R(W_n, W_{n-1})\right| \overset{n\to\infty}{\longrightarrow} 0$, and therefore the family is stable in the limit. However, if the weights are now given by, for example,

$$w_i^n = \begin{cases} \dfrac{1}{2(n-1)} & \text{if } 1 \le i \le n-1 \\ \dfrac{1}{2} & \text{if } i = n \end{cases}$$

then the resulting weighted mean FAO is *L-strictly stable* but not *R-strictly stable*, since in this case it holds that

$$dif_R(W_n, W_{n-1}) = \sum_{i=1}^{n-1}(w_i^n - (1 - w_n^n)w_i^{n-1})x_i = \sum_{i=1}^{n-2} \frac{(2n-6)\cdot x_i}{(2n-2)(4n-8)} - \frac{(n-3)\cdot x_{n-1}}{4(n-1)}$$

and then by taking $(x_1, ..., x_{n-1}) = (1, ...1, 0)$ it follows that $dif_R(W_n, W_{n-1}) \to 1/4$ when n tends to infinity.

Therefore, the last definition properly extends the application range of the stability notion proposed in this paper for consistent, non-strictly stable $FAOs$, covering some weighted mean operators. However, again the product FAO $\{Pn\}_{n\in\mathbb{N}}$ fails to fulfill this notion of consistency. For example, if $(x_1, x_2, \dots) = (1/2, 1, 1, \dots)$, then $P_{n-1} = 1/2$, but

$$P_n(x_1, \ldots, x_{n-1}, P_{n-1}) = (P_{n-1})^2 = 1/4, \quad \forall\, n > 2,$$

so the product family is neither strictly stable nor asymptotically strictly stable. Nevertheless, since

$$dif_R(P_n, P_{n-1}) = \prod_{i=1}^{n-1} x_i \prod_{i=1}^{n-1} x_i - \prod_{i=1}^{n-1} x_i = \prod_{i=1}^{n-1} x_i \left(\prod_{i=1}^{n-1} x_i - 1 \right),$$

the product family fails to fulfill asymptotic strict stability just because of those successions $\{x_n\}_{n\in\mathbb{N}}$ such that $\prod_{i=1}^{n} x_n \overset{n\to\infty}{\longrightarrow} c \in (0,1)$, which implies that $x_n \overset{n\to\infty}{\longrightarrow} 1$ (but note that the opposite is not true, since if $\exists k \;/\; x_k = 0$ then $\prod_{n=1}^{\infty} x_n = 0$.

This leads to guess that those successions making the product FAO not to fulfill asymptotic strict stability constitute a very reduced subset F of the set S of all successions in $[0,1]$. In other words, the chances of gathering a collection of data potentially leading to a non-stable behavior is expected to be rather small.

In fact, such arguments can be formalized in terms of a probability measure. Let $S = \{\{x_n\}_{n\in\mathbb{N}} : x_i \in [0,1]\ \forall i\}$ be the set of successions in $[0,1]$, and let $\{A_n\}_{n\in\mathbb{N}}$ be a FAO involved in the aggregation process of a succession $s_n = (x_1, \ldots, x_n) \in S$. Consider an experiment given by "observe the stability of $\{A_n(s_n)\}_{n\in\mathbb{N}}$", or in the other words, "observe the distance between $A_n(s_{n-1}, A_{n-1}(s_{n-1}))$ and $A_{n-1}(s_{n-1})$" (for the case of the strict stability from the right). The associated sample space E is given by the set of the possible strict stability levels of $\{A_n(s_n)\}_{n\in\mathbb{N}}$. Thus, if the cardinality of S tends to infinite, it is possible to obtain the probability of the event "gathering a succession s_n leading to a strictly stable behavior of $\{A_n\}_{n\in\mathbb{N}}$ FAO", i.e.

$$\mathbb{P}[\lim_{n\to+\infty} |A_n(s_{n-1}, A_{n-1}(s_{n-1})) - A_{n-1}(s_{n-1})| = 0\ ] = 1.$$

In this way, if the value of the data items $(X_1, ..., X_n, ...)$ are assumed to be uniform $U([0,1])$ independent random variables, then it is possible to introduce a probability measure over the set of successions S through the conjoint probability distribution function:

$$\mathbb{P}(a_1 \leq X_1 \leq b_1, ..., a_n \leq X_n \leq b_n, ...) = \prod_{i=1}^{\infty} \mathbb{P}(a_i \leq X_i \leq b_i) = \prod_{i=1}^{\infty} (b_i - a_i),$$

where $a_i, b_i \in [0,1]\ \forall i$. Thus, for example, the probability of a given succession $\{x_n\}_{n\in\mathbb{N}}$ is clearly zero, and the probability of the set of all successions such that $x_i \in [0,1/2]\ \forall i \leq N$ and $x_i \in [0,1]\ \forall i > N$, for a given N, is $(1/2)^N$. Then it is possible to see that the set $F = \left\{ \{x_n\}_{n\in\mathbb{N}} / \prod_{n=1}^{\infty} x_n \overset{n\to\infty}{\longrightarrow} c \in (0,1) \right\}$ in which the product FAO fails to be strictly stable has probability zero.

Effectively, as pointed above, $F \subset S1 = \{(x_n)_{n\in\mathbb{N}} / x_n \to 1\}$. And note that, for each $\varepsilon > 0$, $S1$ can be partitioned into subsets $C_n = \left\{\{x_k\}_{k\in\mathbb{N}} / \left|x_k - 1\right| < \varepsilon, \forall k \geq n\right\}, n \in \mathbb{N}$. Since $C_n \bigcap C_m = \emptyset$ if $n \neq m$, it then holds that

$$\mathbb{P}(F) \leq \mathbb{P}(S1) = \mathbb{P}\left(\bigcup_{n\in\mathbb{N}} C_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(C_n) \leq \sum_{n=1}^{\infty} \prod_{k=n}^{\infty} \varepsilon = 0,$$

and thus it can be said that the probability of the product FAO not being strictly stable is zero.

Therefore, the product FAO verifies the notion of stability in a weaker version, that can be characterized in terms of almost sure convergence to a strictly stable FAO. This lead to introduce the notion of almost sure strict stability.

Definition 2.4. *Let $\{A_n : [0,1]^n \to [0,1], n \in \mathbb{N}\}$ be a family of aggregation operators. Then, we will say that*

1. $\{A_n\}_{n\in\mathbb{N}}$ *is an almost sure R-strictly stable family if*

   $$\mathbb{P}[\lim_{n\to+\infty} |A_n(x_1, ..., x_{n-1}, A_{n-1}(x_1, ..., x_{n-1})) - A_{n-1}(x_1, ..., x_{n-1})| = 0\ ] = 1, \forall x_i \sim U(0,1)$$

   holds $\forall \{x_n\}_{n\in N}$ *in* $[0,1]$.

2. $\{A_n\}_{n\in\mathbb{N}}$ *is an almost sure L-strictly stable family if*

   $$\mathbb{P}[\lim_{n\to+\infty} [\ |A_n(A_{n-1}(x_1, ..., x_{n-1}), x_1, ..., x_{n-1}) - A_{n-1}(x_1, ..., x_{n-1})| = 0\ ] = 1, \forall x_i \sim U(0,1)$$

   holds $\forall \{x_n\}_{n\in N}$ *in* $[0,1]$.

3. $\{A_n\}_{n\in\mathbb{N}}$ *is an almost sure LR-strictly stable family if the above two conditions hold simultaneously.*

For example, the product $\{P_n(x_1, \ldots, x_n) = \prod_{i=1}^n x_i\}_{n\in\mathbb{N}}$ and the family of operators $\{Q_n(x_1, \ldots, x_n) = \prod_{i=1}^n x_i^i\}_{n\in\mathbb{N}}$ constitute almost sure strictly stable $FAOs$.

Since asymptotically strictly stable and almost sure asymptotically strictly stable $FAOs$ converge to strictly stable $FAOs$, and asymptotically strictly stable $FAOs$ are particular cases of almost sure asymptotically strictly stable $FAOs$, the following results are immediate:

Proposition 2.1. *Let $\{A_n : [0,1]^n \to [0,1], n \in \mathbb{N}\}$ be a family of aggregation operators. Then:*

1. *If the family $\{A_n\}_{n\in\mathbb{N}}$ satisfies the property of strict stability, then it also satisfies the property of asymptotic strict stability.*

2. *If the family $\{A_n\}_{n\in\mathbb{N}}$ satisfies the property of asymptotic strict stability, then it satisfies the property of almost sure asymptotic strict stability.*

Therefore, if a FAO is not almost sure asymptotically strictly stable, then it does not verify any of the three levels of strict stability. In this case we will talk about an *unstable* FAO.

Definition 2.5. *Let* $\{A_n : [0,1]^n \to [0,1], n \in \mathbb{N}\}$ *be a family of aggregation operators. Then, we will say that:*

1. *It fulfills the property of R-instability if the family is not almost sure asymptotically R-strict stable, and it will be called R-unstable family.*

2. *It fulfills the property of L-instability if the family is not almost sure asymptotically L-strict stable, and it will be called L-unstable family.*

3. *It fulfills the property of LR-instability if* A_n *satisfies the above two points, and it will be called LR-unstable family.*

Therefore, we can summarize the definitions in this section as follows: given a data collection $x_1, ..., x_n$ of arbitrary length n, such that $x_n = A_{n-1}(x_1, ..., x_{n-1})$ for a FAO $A = \{A_n, n \in \mathbb{N}\}$,

1. If the aggregations of n and $n-1$ items are equal, then A is strictly stable.

2. If the aggregations of n and $n-1$ items converge, then A is asymptotically strictly stable.

3. If the aggregations of n and $n-1$ items almost sure converge, then A is almost sure asymptotically strictly stable.

4. Finally, if the aggregations does not almost sure converge, then A is an unstable FAO.

Thus, it is then possible to differentiate the families of aggregation operators in relation to their level of stability. In the next section, we carry out this task for some standard $FAOs$.

3. Analysis of the stability levels of some well-known families of aggregation operators.

In this section, the level of stability of some families of frequently used aggregation operators is analyzed, in order to know in advance the level of robustness of the involved aggregation process. In this way, no matter the cardinality of data, it is possible to specify the stability level of each FAO, and therefore the global robustness of the associated aggregation process.

Let us analyze the stability level of some of the most used aggregation functions, starting from the minimum family.

Proposition 3.1. *The minimum operators family* $\{Min_n : [0,1]^n \to [0,1], n \in \mathbb{N}\}$ *is a LR-strict stable family.*

Proof:

$$\begin{aligned} Min_n(x_1, ..., x_{n-1}, Min_{n-1}(x_1, ..., x_{n-1})) &= Min_n(x_1, ..., x_{n-1}, x_{(1)}) \\ &= x_{(1)} \\ &= Min_{n-1}(x_1, ..., x_{n-1}) \end{aligned}$$

Proposition 3.2. *The maximum operators family $\{Max_n : [0,1]^n \to [0,1], n \in \mathbb{N}\}$ is a LR-strict stable family.*

Proof:

$$\begin{aligned} Max_n(x_1,...,x_{n-1}, Max_{n-1}(x_1,...,x_{n-1})) &= Max_n(x_1,...,x_{n-1},x_{(n-1)}) \\ &= x_{(n-1)} \\ &= Max_{n-1}(x_1,...,x_{n-1}) \end{aligned}$$

Proposition 3.3. *The median operators family $\{Md_n : [0,1]^n \to [0,1], n \in \mathbb{N}\}$ is a LR-strict stable family.*

Proof:

If $n-1$ is odd:

$$\begin{aligned} Md_n(x_1,...,x_{n-1}, Md_{n-1}(x_1,...,x_{n-1})) &= Md_n\left(x_1,...,x_{n-1},x_{(\frac{n+1}{2})}\right) \\ &= x_{(\frac{n+1}{2})} \\ &= Md_{n-1}(x_1,...,x_{n-1}) \end{aligned}$$

If $n-1$ is even:

$$\begin{aligned} Md_n(x_1,...,x_{n-1}, Md_{n-1}(x_1,...,x_{n-1})) &= Md_n\left(x_1,...,x_{n-1},\frac{x_{(\frac{n}{2})}+x_{(\frac{n}{2}+1)}}{2}\right) \\ &= \frac{x_{(\frac{n}{2})}+x_{(\frac{n}{2}+1)}}{2} \\ &= Md_{n-1}(x_1,...,x_{n-1}) \end{aligned}$$

Proposition 3.4. *The mean operators family $\{M_n : [0,1]^n \to [0,1], n \in \mathbb{N}\}$ is a LR-strict stable family.*

Proof:

$$\begin{aligned} M_n(x_1,...,x_{n-1}, M_{n-1}(x_1,...,x_{n-1})) &= \frac{1}{n}\left(\sum_{i=1}^{n-1} x_i + \frac{1}{n-1}\sum_{i=1}^{n-1} x_i\right) \\ &= \frac{1}{n}\left(\frac{(n-1)\sum\limits_{i=1}^{n-1} x_i + \sum\limits_{i=1}^{n-1} x_i}{n-1}\right) \\ &= \frac{1}{n(n-1)}\left(\sum_{i=1}^{n-1} x_i(n-1+1)\right) \\ &= M_{n-1}(x_1,...,x_{n-1}) \end{aligned}$$

Proposition 3.5. *The geometric mean operators family* $\{G_n : [0,1]^n \to [0,1], n \in \mathbb{N}\}$ *is a LR-strict stable family.*

Proof:

$$\begin{aligned}
G_n(x_1, ..., x_{n-1}, G_{n-1}(x_1, ..., x_{n-1})) &= \left(\prod_{i=1}^{n-1} x_i \left(\prod_{i=1}^{n-1} x_i\right)^{\frac{1}{n-1}}\right)^{\frac{1}{n}} \\
&= \left(\left(\prod_{i=1}^{n-1} x_i\right)^{\frac{n}{n-1}}\right)^{\frac{1}{n}} \\
&= G_{n-1}(x_1, ..., x_{n-1})
\end{aligned}$$

Proposition 3.6. *The harmonic mean operators family* $\{H_n : [0,1]^n \to [0,1], n \in \mathbb{N}\}$ *is a LR-strict stable family.*

Proof:

$$\begin{aligned}
H_n(x_1, ..., x_{n-1}, H_{n-1}(x_1, ..., x_{n-1})) &= \frac{n}{\sum_{i=1}^{n-1} \frac{1}{x_i} + \frac{1}{\frac{n-1}{\sum_{i=1}^{n-1} \frac{1}{x_i}}}} \\
&= \frac{n}{\sum_{i=1}^{n-1} \frac{1}{x_i} + \frac{\sum_{i=1}^{n-1} \frac{1}{x_i}}{n-1}} \\
&= \frac{n(n-1)}{(n-1)\sum_{i=1}^{n-1} \frac{1}{x_i} + \sum_{i=1}^{n-1} \frac{1}{x_i}} \\
&= \frac{n(n-1)}{n\sum_{i=1}^{n-1} \frac{1}{x_i}} \\
&= H_{n-1}(x_1, ..., x_{n-1})
\end{aligned}$$

Proposition 3.7. *The family of binary idempotent operators with inductive extension forward* $\{A_n^f : [0,1]^n \to [0,1], n \in \mathbb{N}\}$ *is a R-strict stable family.*

Proof:

$$\begin{aligned}
A_n^f(x_1, ..., x_{n-1}, A_{n-1}^f(x_1, ..., x_{n-1})) &= A_2(..(A_2(A_2(x_1, x_2)..), x_{n-1}), A_{n-1}^f) \\
&= A_2(A_{n-1}^f, A_{n-1}^f) \\
&= A_{n-1}^f(x_1, ..., x_{n-1})
\end{aligned}$$

This last equation only holds if A_2 is a binary idempotent operator. In general, $\{A_n^f\}_{n\in\mathbb{N}}$ is not a *L-strict stable* family. Let us consider, for example, the *L-strict condition*, for $n = 3$:

$$A_3^f(A_2^f(x_1, x_2), x_1, x_2)$$

should coincide with $A_2^f(x_1, x_2)$, but

$$A_3^f(A_2^f(x_1, x_2), x_1, x_2) = A_2\,(A_2(A_2(x_1, x_2), x_1), x_2),$$

which in general is not equal to $A_2(x_1, x_2)$.

Proposition 3.8. *The family of binary idempotent operators with inductive extension backward* $\{A_n^b : [0,1]^n \to [0,1], n \in \mathbb{N}\}$ *is a L-strict stable family.*

Proof:

$$\begin{aligned} A_n^b(A_{n-1}^b, x_2, ..., x_n) &= A_2(A_{n-1}^b, A_2(..., A_2(x_{n-2}, x_{n-1}))) \\ &= A_2(A_{n-1}^f, A_{n-1}^b) \\ &= A_{n-1}^b(x_1, ..., x_{n-1}) \end{aligned}$$

This last equation only holds if A_2 is a binary idempotent operator. With a similar analysis we can conclude that, in general, the family $\{A_n^b\}_{n\in\mathbb{N}}$ is not a *R-strictly stable* family.

The stability of the product aggregation operator family $\{P_n\}_{n\in\mathbb{N}}$ was analyzed in the previous section in order to introduce Definition 2.5. In particular, it was shown that $\{P_n\}_{n\in\mathbb{N}}$ is neither a *strictly stable* family nor an *asymptotically strictly stable* family, but it indeed is an *almost sure asymptotically strictly stable* family. In a similar way, it can be proven that the family $\{Q_n\}_{n\in\mathbb{N}}$ is an *almost sure asymptotically strictly stable* family but not an *asymptotically strictly stable* family.

Proposition 3.9. *The weighted mean FAOs* $\{W_n\}_{n\in\mathbb{N}}$ *are unstable families in general (i.e., meanwhile no further restrictions are imposed on their weights).*

Proof:

In *Section* 2 a family of weights was defined in such a way that the correspondent weighted mean FAO shows a non-stable behavior. Now we show with an example that, in general (i.e. if no further restrictions over the weights are considered), the Weighted Mean is not an *almost sure asymptotically strictly stable* FAO, or equivalently that such a family is *unstable.*

Let us consider the weights given by

$$w_i^n = \begin{cases} \dfrac{1}{2(n-2)} & \text{if } 1 < i < n \\ 1/4 & \text{if } i = 1 \\ 1/4 & \text{if } i = n \end{cases}$$

Note that $\sum_{i=1}^{n} w_i^n = 1 \; \forall n$. Thus,

$$dif_R(W_n, W_{n-1}) = \sum_{i=1}^{n-1} \left(w_i^n - (1-w_n^n) w_i^{n-1} \right) x_i = \frac{1}{16}\, x_1 + \sum_{i=2}^{n-2} \frac{n-6}{8n^2 - 40n + 48}\, x_i - \frac{3n-2}{16n-32}\, x_{n-1}$$

Assuming that all values of x_2,...,x_{n-1} are being generated by independent random uniform $U[0,1]$ variables, in the limit we have that $\lim \sum_{i=2}^{n-2} \frac{n-6}{8n^2-40n+48}\, x_i \in [0, 1/8]$ and $\frac{3n-2}{16n-32}\, x_{n-1} \in [0, 3/16]$. Therefore, if for instance it is $\varepsilon = 0.01$, and assuming n big enough, it then follows that

$$\left| dif_R(W_n, W_{n-1}) \right| \;\geq\; \frac{1}{16} \left| x_1 - 3x_{n-1} \right| > \varepsilon = 0.01$$

holds if and only if it is either $1 \geq x_1 > 3x_{n-1} + 0.16$ or $0 \leq x_1 < 3x_{n-1} - 0.16$. As any value $x_{n-1} \in [0,1]$ leads to an interval with positive length for x_1, in terms of the probability measure introduced in *Section* 2 it follows that

$$P\left(\left\{ (x_n)_{n \in N} \subset [0,1] \;/\; \lim_{n \to \infty} \left| dif_R(W_n, W_{n-1}) \right| > 0.01 \right\} \right) > 0,$$

and thus this weighted mean FAO is not *almost sure asymptotically R-strictly stable*. Also, since it is

$$dif_L(W_n, W_{n-1}) = \sum_{i=1}^{n-1} \left(w_{i+1}^n - (1-w_1^n) w_i^{n-1} \right) x_i = \frac{2-3n}{16n-32} x_1 + \sum_{i=2}^{n-2} \frac{n-6}{8n^2-40n+48} x_i + \frac{1}{16} x_{n-1},$$

a similar reasoning can be carried out in order to conclude that $\{W_n\}_{n \in \mathbb{N}}$ is not *almost sure asymptotically L-strictly stable*. Therefore, the weighted mean FAO is unstable in general.

In the following result we analyze the stability of the family of OWA operators $\{O_n : [0,1]^n \to [0,1], n \in \mathbb{N}\}$, where O_n is an OWA operator function.

Proposition 3.10. *The OWA operators family $\{O_n : [0,1]^n \to [0,1], n \in \mathbb{N}\}$ is unstable FAOs in general (i.e., meanwhile no further restrictions are imposed on their weights).*

Proof:

In order to prove that $\{O_n\}_{n \in \mathbb{N}}$ is an unstable $FAOs$, let us consider the family $\{IA_n\}_{n \in \mathbb{N}}$ defined as

$$IA_n(x_1, \ldots, x_n) = \begin{cases} \text{Max}(x_1, \ldots, x_n) & \text{if n is odd} \\ \text{Min}(x_1, \ldots, x_n) & \text{if n is even} \end{cases}$$

The previous family can be viewed as a particular case of OWA family by taking the weights $w_n = (1, 0, 0, ...)$ if n is odd and $w_n = (0, ..., 0, 1)$ if n is even. To prove that $\{IA_n\}_n$ is unstable, we will see that given $\varepsilon > 0$, it is possible to find a family of successions R_ε with positive probability, such that for any $(x_n)_n \in R_\varepsilon$, there exist $n_0 \geq 2$ with

$$\left|IA_n(x_1, \ldots, x_{n-1}, IA_{n-1}(x_1, \ldots, x_{n-1})) - IA_{n-1}(x_1, \ldots, x_{n-1})\right| \geq \epsilon \text{ if } n \geq n_0.$$

Effectively, given $\varepsilon \in (0, 1)$, let

$$R_\varepsilon = \left\{(x_n)_{n\in N},\ x_1 \in \left[0, \frac{1}{2} - \frac{\varepsilon}{2}\right],\ x_2 \in \left[\frac{1}{2} + \frac{\varepsilon}{2}, 1\right],\ x_k \in [0, 1]\ \forall k \geq 3\right\}$$

be the set of successions in which the first element belongs to $\left[0, \frac{1}{2} - \frac{\varepsilon}{2}\right]$, the second element belongs to $\left[\frac{1}{2} + \frac{\varepsilon}{2}, 1\right]$, and no more constraints are imposed.

Now, it is easy to see that $P(R_\varepsilon) = \left(\frac{1}{2} - \frac{\varepsilon}{2}\right)^2 > 0$ and also that for any $(x_n)_n \in R_\varepsilon$, it is

$$\left|IA_n(x_1, \ldots, x_{n-1}, IA_{n-1}(x_1, \ldots, x_{n-1})) - IA_{n-1}(x_1, \ldots, x_{n-1})\right| \geq \varepsilon \text{ if } n \geq 2,$$

so the family $\{IA_n\}_{n\in\mathbb{N}}$, which can be viewed as a particular case of OWA operator, is unstable, and thus the proposition is proved.

Another important family of aggregation operators is that known as the *projection operators* family. Although different definitions could be made for a projection operator, we follow the definition proposed in [4, 5]. Given an axis i, the projection operator over the axis i is defined as $P_n^i(x_1, \ldots, x_n) = x_i$. Thus, the family of the projection operators can be defined as $\{P_n^i\ n \in N\ /\ i \leq n\}$, and it is possible to see that when $i = 1$ such a family is L and R strictly stable. But for $i \geq 2$ the family of projection operators is R strictly stable and L unstable.

Another possibility is to define the family of projection operators family as $\{P_n^i\ n \in N\ /\ i \leq n\}$, where now the projection changes depending on n. For example, $i(n) = n$, so $P_n^i(x_1, \ldots, x_n) = x_n$ or $i(n) = n - 1$. Obviously, in this case stability will be strongly dependent on the stability of the function $i(n)$. For example, if $i(n) = n$, then the family is strictly stable. But if $i(n) = \left\{\begin{array}{cc} n-k & if\ n \geq k \\ n & otherwise \end{array}\right.$, then the corresponding family is L unstable.

We can therefore conclude that, in general, and due to the way projection works, only the *simplest* families of projection operators $\{P_n^1,\ n \in N\}$ and $\{P_n^n,\ n \in N\}$ are strictly stable families.

Finally, to conclude this analysis, we show how the different levels of stability can be extended to different transformations of the original FAO. To this aim, let us first introduce the following notations and definitions.

Definition 3.1. *Let $f : [0,1] \to A$ be a continuous and injective function, and let $\{\phi_n : A \to A,\ n \in N\}$ be a family of aggregation operators defined in the domain A. Then, the transformed aggregation operator family $\{M_f^{\phi_n}\}_{n\in\mathbb{N}}$ is defined as:*

$$M_f^{\phi_n}(x_1,\ldots,x_n) = f^{-1}\left(\phi_n\left(f(x_1),\ldots,f(x_n)\right)\right)$$

Let us observe that if f is the identity function, then the transformation family coincides with the original family. If $\{\phi_n\}_{n\in\mathbb{N}}$ is the mean or the weighted mean then $M_f^{\phi_n}$ is called quasi-arithmetic mean or weighted quasi-arithmetic mean. The quasi-arithmetic mean functions are very important in many aggregation analysis. Some well-known quasi-arithmetic aggregation families are: the geometric mean (when $f(x) = log(x)$), the harmonic mean (when $f(x) = 1/x$) and the power mean (when $f(x) = x^p$), among others. It is important to remark that some of the aggregation operators families defined in this paper (as for example $\{P_n\}_{n\in N}$), can not be transformed or extended directly. For example if $f(x) = 5x$, then $A = [0,5]$, but we can not guarantee that for all $n \in N$, $P_n\left(f(x_1),\ldots,f(x_n)\right) = \prod_{i=1}^n f(x_i)$ belong to the interval $[0,5]$.

In the following proposition, we show that strict stability remains after transformation.

Proposition 3.11. *Let $\{\phi_n\}_{n\in\mathbb{N}}$ and $\{M_f^{\phi_n}\}_{n\in\mathbb{N}}$ be a family of aggregation operators and its extension or transformed aggregation. Then:*

$\{M_f^{\phi_n}\}_{n\in\mathbb{N}}$ is a R-strictly stable family if and only if $\{\phi_n\}_{n\in\mathbb{N}}$ is a R-strictly stable family in the A domain.

Proof:

Taking into account that $M_f^{\phi_{n+1}}\left(x_1,\ldots,x_n,M_f^{\phi_n}(x_1,\ldots,x_n)\right)$ can be rewritten as

$$f^{-1}\left(\phi_{n+1}(f(x_1),\ldots,f(x_n),\phi_n(f(x_1),\ldots,f(x_n))\right),$$

strict stability condition for $\{M_f^{\phi_n}\}_{n\in\mathbb{N}}$ can be formulated as

$$f^{-1}\left(\phi_{n+1}(f(x_1),\ldots,f(x_n),\phi_n(f(x_1),\ldots,f(x_n)))\right) = f^{-1}\left(\phi_n(f(x_1),\ldots,f(x_n))\right).$$

Hence, since f is a continuous and injective function, such a condition holds if and only if $\{\phi_n\}_n$ is an strictly stable family in A. And thus, the proposition holds.

Proposition 3.12. *Let $\{\phi_n,\ n \in N\}$ and $\{M_f^{\phi_n},\ n \in N\}$ be a family of aggregation operators and its extension or transformed aggregation. Then:*

$\{M_f^{\phi_n},\ n \in N\}$ is a L-strictly stable family if and only if $\{\phi_n,\ n \in N\}$ is a L-strictly stable family in the A domain.

Proof: Similar to the proof of the previous proposition.

We obviously can put both results together.

Proposition 3.13. *Let $\{\phi_n,\ n \in N\}$ and $\{M_f^{\phi_n},\ n \in N\}$ be a family of aggregation operators and its extension or transformed aggregation. Then:*

$\{M_f^{\phi_n},\ n \in N\}$ is a strict stable family if and only if $\{\phi_n,\ n \in N\}$ is a strict stable family in the A domain.

Taking into account that the mean is a strict stable family in any domain A, and that the mean of elements in A always belongs to A, strict stability can be guaranteed for any quasi-arithmetic aggregation operator family.

Corolary 3.1. *The quasi-arithmetic aggregation operators family is a strictly stable family.*

Table 1: Level of stability of some families of aggregation operators.

Family of aggregation operators $\{A_n\}_{n\in\mathbb{N}}$	Strict stability	Asymptotic strict stability	Almost sure strict stability	Instability
$Min_n = Min(x_1, \ldots, x_n)$	$R,\ L$	$R,\ L$	$R,\ L$	–
$Max_n = Max(x_1, \ldots, x_n)$	$R,\ L$	$R,\ L$	$R,\ L$	–
$Md_n = Md(x_1, \ldots, x_n)$	$R,\ L$	$R,\ L$	$R,\ L$	–
$M_n = \sum_{i=1}^{n} \frac{x_i}{n}$	$R,\ L$	$R,\ L$	$R,\ L$	–
$G_n = (\prod_{i=1}^{n} x_i)^{1/n}$	$R,\ L$	$R,\ L$	$R,\ L$	–
$H_n = \frac{n}{\sum_{i=1}^{n} 1/x_i}$	$R,\ L$	$R,\ L$	$R,\ L$	–
$Q_n = \prod_{i=1}^{n} x_i^i$	–	–	$R,\ L$	–
$P_n = \prod_{i=1}^{n} x_i$	–	–	$R,\ L$	–
$A_n^f = A_n^f(x_1, \ldots, x_n)$	R	R	R	L
$A_n^b = A_n^b(x_1, \ldots, x_n)$	L	L	L	R
$W_n = \sum_{i=1}^{n} x_i \cdot w_i$	-	–	–	$R,\ L$
$O_n = \sum_{i=1}^{n} x_{(i)} \cdot w_i$	–	–	–	$R,\ L$

Note: R and L indicate the fulfilment of a level of stability from the right and from the left, respectively.
Note: No restrictions are imposed on the weights of the weighted FAOs.

Table1 summarizes the previous analysis, showing the stability level of some of the most used families of aggregation operators: the minimum and maximum operators, the median, as well as the arithmetic, harmonic and geometric meansconstitute strictly stable $FAOs$. Recursive extensions of binary idempotent operators only satisfy *R-strict stability* if the inductive extension is forward, and they satisfy *L-strict stability* if the inductive extension is backward. Both the product $\{P_n\}_{n\in\mathbb{N}}$ and the geometric product $\{Q_n\}_{n\in\mathbb{N}}$ $FAOs$ are almost sure strictly stable. Finally, since it is possible to choose the weights w_n in such a way that the resulting FAO does not fulfill any of the previous stability levels, those $FAOs$ based on weights, as the weighted mean or the OWA families, are considered unstable in general.

However, in the next section we will look at these weighted $FAOs$ more in detail, establishing conditions under which the weights produce a strict stable family or an asymptotically strictly stable family. Therefore, despite the whole family of weighted $FAOs$ is regarded as unstable, it is possible to state whether a specific weighted FAO involves or not a robust aggregation process, i.e. whether such a FAO fulfills each one of the stability levels introduced in this paper.

4. Analyzing the weights to guarantee stability of the weighted mean.

In the previous section it was proven that the weighted mean aggregation family $\{W_n\}_{n\in\mathbb{N}}$ is in general unstable, at least meanwhile no conditions are imposed on the weights. In this section some conditions on these weights are analyzed in order to guarantee different levels of stability for the weighted mean FAO. Recall that, in a weighted mean, the weights associated to the elements being aggregated represent the *importance* of each one of these elements in the aggregation process. For this reason, the weighted mean surely is one of the most relevant and used aggregation operators in many different areas (e.g. statistics, knowledge representation problems, fuzzy logic, multiple criteria decision making, group decision making, etc.), and one of the most studied problems in all these areas is how to determine these *importance* weights.

Recall that, for any data cardinality n, the weights are usually assumed to form a vector $w^n = (w_1^n, ..., w_n^n) \in [0,1]^n$, such that $\sum_{i=1}^{n} w_i^n = 1$. The corresponding weighted mean operator is then given by $W_n(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \cdot w_i^n$.

It is important to stress that our aim is not to propose a new method to determine these weights, which will depend on each particular problem under consideration, but simply to specify the relationships that should exist between two vectors of weights of dimension r and s in order to produce a consistent aggregation process.

In order to illustrate our point, let us introduce the following example. Suppose a multi-criteria decision problem having four criteria C_1, C_2, C_3, C_4. A jury, after some deliberations, evaluates the different alternatives on the four criteria and then uses a weighted mean operator as aggregation rule. Our objective is not to decide how the vector of weights $w^4 = (w_1^4, w_2^4, w_3^4, w_4^4)$ should be, but to guarantee some stability or consistence in the aggregation process. For example, it would seem rather inconsistent to choose $w^4 = (1/4, 1/4, 1/4, 1/4)$ if we have the four mentioned criteria, but also choosing $w^3 = (0.8, 0.2, 0)$ in case the criteria C_4 is discarded. From the point of view of consistency, this jury would not be stable. Our objective is then to determine the relation that should exist between weights of different dimension in order to guarantee a consistent aggregation process.

For example, given the usual sequence of weights $w_i^n = \frac{v_i}{\sum_{i=1}^{n} v_i}, \forall i \in \mathbb{N}, v_i \in \mathbb{R}^+$, the corresponding family of weighted means $\{W_n\}_{n\in\mathbb{N}}$ is a *R-strictly stable* family However, if the L-strict stability of the same family is analyzed, it is easy to realize that $\{W_n\}_{n\in\mathbb{N}}$ is not a *L-strictly stable* family.

In this section, we give necessary and sufficient conditions to guarantee strict stability and asymptotic stability of a weighted mean FAO.

Note that for a generic weighted mean FAO $\{W_n\}_{n\in\mathbb{N}}$ with weights w^n, $n \in \mathbb{N}$, the R-strict stability property can be restated as

$$0 = dif_R(W_n, W_{n-1}) = \sum_{i=1}^{n-1}(w_i^n - (1-w_n^n)w_i^{n-1})x_i$$

for all data collection $(x_1, \ldots, x_n) \in [0,1]^n$. Similarly, if

$$dif_L(W_n, W_{n-1}) = W_n(W_{n-1}(x_1, \ldots, x_{n-1}), x_1, \ldots, x_{n-1}) - W_{n-1}(x_1, \ldots, x_{n-1}),$$

Analogously, L-strict stability is equivalent to

$$0 = dif_L(W_n, W_{n-1}) = \sum_{i=1}^{n-1}(w_{i+1}^n - (1-w_1^n)w_i^{n-1})x_i$$

for all data collection $(x_1, \ldots, x_n) \in [0,1]^n$. This leads to propose the following propositions:

Proposition 4.1. *Let $w^n = (w_1^n, ..., w_n^n) \in [0,1]^n, n \in \mathbb{N}$, be a sequence of weights of a weighted mean family $\{W_n\}_{n\in\mathbb{N}}$ such that $\sum_{i=1}^{n} w_i^n = 1$ holds $\forall n \geq 2$. Then, the family $\{W_n\}_{n\in\mathbb{N}}$ is a R-strict stable family if and only if the sequence of weights satisfies $w_i^n = (1-w_n^n)\cdot(w_i^{n-1})$ $\forall n \in \mathbb{N}$.*

Proof:

Direct from previous considerations.

Corolary 4.1. *The weighted quasi-arithmetic aggregation operators family is a R-strict stable family if and only if the sequence of weights satisfies $w_i^n = (1-w_n^n)\cdot(w_i^{n-1})$ $\forall n \in \mathbb{N}$.*

Remark 1. Let us observe that, if the family $\{W_n\}_{n\in\mathbb{N}}$ is a R-strict stable family, then any vector of weights $w^r = (w_1^r, \ldots, w_r^r)$ can be built from a given w^s, where $r \leq s$. For example, if $w^5 = (1/5, \ldots, 1/5)$, then we have that $w^r = (1/r, \ldots, 1/r)$ for any $r \leq 5$. Also, for any $r > 5$, it has to be $w^r = (w, w, w, w, w, w_6^r, \ldots, w_r^r)$ to guarantee R-strict stability (i.e. the five first items have to be equal).

On the contrary, let also us observe that, if the family $\{W_n\}_{n\in\mathbb{N}}$ is a L-strict stable family, then any vector of weights $w^r = (w_1^r, \ldots, w_r^r)$ can be built from a given w^s, where $r \leq s$. For example, if $w^5 = (0.3, 0.2, 0.1, 0, 0.4)$, then we have that $w^4 = (\frac{0.2}{0.7}, \frac{0.1}{0.7}, \frac{0}{0.7}, \frac{0.4}{0.7})$, $w^3 = (\frac{0.1}{0.5}, 0, \frac{0.4}{0.5})$, $w^2 = (0, 1)$.

Proposition 4.2. *Let $w^n = (w_1^n, ..., w_n^n) \in [0,1]^n, n \in \mathbb{N}$, be a sequence of weights of the weighted mean family $\{W_n\}_{n\in\mathbb{N}}$ such that $\sum_{i=1}^{n} w_i^n = 1$ holds $\forall n \geq 2$. Then, the family $\{W_n\}_n$ is a L-strict stable family if and only if the sequence of weights satisfies $w_{i+1}^n = (1-w_1^n)\cdot(w_i^{n-1})$ $\forall n \in \mathbb{N}$.*

Proof:

Direct from previous considerations.

Corolary 4.2. *The weighted quasi-arithmetic aggregation operators family is a L-strict stable family if and only if the sequence of weights satisfies* $w_{i+1}^n = (1 - w_1^n) \cdot (w_i^{n-1})\ \forall n \in \mathbb{N}$.

Remark 2. Let us observe again that, if the family $\{W_n\}_{n\in\mathbb{N}}$ is a L-strict stable family, then it is possible to build a vector of weights $w^r = (w_1^r, \ldots, w_r^r)$ whenever a vector of weights w^s is known, where $r \leq s$. For example if $w^5 = (1/5, \ldots, 1/5)$, then we have that $w^r = (1/r, \ldots, 1/r)$ for any $r \leq 5$. Also, for any $r > 5$, if L-strict stability is assumed then $w^r = (w_1^r, \ldots, w_{r-6}^r, w, w, w, w, w)$ (i.e. the last 5 items have to be equal).

Previously, we studied some conditions that the weights in the weighted mean family should fulfill in order to satisfy the strict stability property. Now, let us study some sufficient conditions that guarantee the asymptotically strict stability property. Thus, since $x_i \in [0, 1]$, note that, for all $n \in \mathbb{N}$,

$$\Big|dif_R(W_n, W_{n-1})\Big| = \Big|\sum_{i=1}^{n-1} \left(w_i^n - (1 - w_n^n)w_i^{n-1}\right)\ x_i\ \Big| \leq\ z(n) \cdot g(n)\ \leq\ (n-1) \cdot g(n),$$

where $z(n) = \sharp\{i = 1, ..., n-1\ /\ w_i^n - (1-w_n^n)w_i^{n-1} \neq 0\}$ and $g(n) = Max_{i<n}\{w_i^n - (1-w_n^n)\ w_i^{n-1}\}$. Therefore, if $\forall i$ the successions $w_i^n - (1 - w_n^n)\ w_i^{n-1}$ tend to zero with n, it is enough to impose the condition $\lim_{n\to+\infty} g(n) \cdot z(n) = 0$ in order to guarantee *asymptotic R-strict stability* of the resulting weighted mean FAO. A similar reasoning could be carried out in the case of *asymptotic L-strict stability.*

This leads to the following results:

Proposition 4.3. *Let* $w^n = (w_1^n, ..., w_n^n) \in [0, 1]^n, n \in \mathbb{N}$ *be a sequence of weights of the weighted mean* FAO $\{W_n\}_{n\in\mathbb{N}}$ *such that* $\sum_{i=1}^{n} w_i^n = 1$ *holds* $\forall n$. *Then, the family* $\{W_n\}_{n\in\mathbb{N}}$ *is asymptotically R-strictly stable if:*

$$\lim_{n\to+\infty} \left(w_i^n - (1 - w_n^n)w_i^{n-1}\right) = 0, \quad \forall\ 1 \leq i \leq n-1\ \text{and} \lim_{n\to+\infty} g(n) \cdot z(n)\ =\ 0\ .$$

Proof:

Direct from previous considerations.

Proposition 4.4. *Let $w^n = (w_1^n, ..., w_n^n) \in [0,1]^n, n \in \mathbb{N}$ be a sequence of weights of the weighted mean FAO $\{W_n, n \in \mathbb{N}\}$ such that $\sum_{i=1}^{n} w_i^n = 1$ holds $\forall n$. Then, the family $\{W_n\}_n$ is asymptotically L-strictly stable if:*

$$\lim_{n\to+\infty} \left(w_{i+1}^n - (1-w_n^n)w_i^{n-1}\right) = 0, \quad \forall\ 1 \leq i \leq n-1 \ \textit{and} \lim_{n\to+\infty} g(n)\cdot z(n) \ = \ 0\ .$$

Proof:

Direct from previous considerations.

5. Simulation results

In this section some simulations are carried out to study the behavior of different FAOs from an empiric point of view. Particularly, the convergence speed of some non-strictly stable families is studied, allowing to identify the minimum size of the data sequence that guarantees the stability of the aggregation process for a certain tolerance level.

In the following simulation exercises, the difference

$$\left|dif_R(A_n, A_{n-1})\right| = \left|A_n(x_1, ...x_{n-1}, A_{n-1}(x_1, .., x_{n-1})) - A_{n-1}(x_1, .., x_{n-1})\right|$$

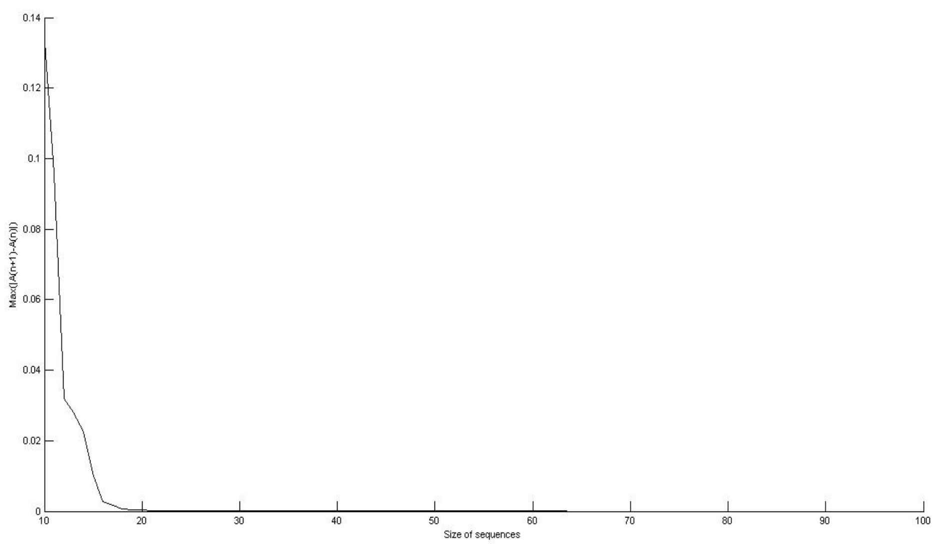
is calculated for 1.000 sequences $(x_1, \ldots, x_{n-1})$ of independent random $U[0,1]$ variables, and for the following sizes of the data sequences: $\{10, 20, 30, ..., 50000\}$. In order to study the behavior of these differences as n increases, the maximum value of $\left|dif_R(A_n, A_{n-1})\right|$ is computed for each n. Then, it is studied the fulfilment of $\mathbb{P}[\lim_{n\to+\infty} \left|dif_R(A_n, A_{n-1})\right| = 0\ ] < \varepsilon$, for each n and tolerance levels ε given by $\{10^{-1}, 10^{-2}, 10^{-3}, ..., 10^{-30}\}$.

Let us start by the product FAO. *Figure 1* shows that the aggregated values of the sequences of n and $n-1$ items are more and more similar as n increases, where it can see a zero in the maximum difference $|A_n - A_{n-1}|$ for any sequence greater than 20 in size. Regarding the convergence from the right of the product FAO, in *Table 2* it can be seen that the convergence speed is quite high, in such a way that a sequence of data with size $n = 100$ is enough to guarantee stable outputs for any tolerance level. Figure 2 depicts the results in Table 2.

Table 2: Simulation outputs of $\mathbb{P}[\lim_{n\to+\infty} \left|dif_R(P_n, P_{n-1})\right| < Tolerance]$ for the product FAO.

Size of the sequences	Tolerance levels 10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-10}	10^{-15}	10^{-20}	10^{-25}	10^{-30}
10	1.000	0.979	0.867	0.53	0.327	0.002	0.000	0.000	0.000	0.000
20	1.000	1.000	1.000	0.998	0.978	0.239	0.005	0.000	0.000	0.000
30	1.000	1.000	1.000	1.000	1.000	0.906	0.216	0.004	0.000	0.000
40	1.000	1.000	1.000	1.000	1.000	1.000	0.797	0.181	0.010	0.000
50	1.000	1.000	1.000	1.000	1.000	1.000	0.995	0.693	0.134	0.004
60	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.965	0.585	0.121
70	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.934	0.492
80	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.888
90	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.989
100	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
300	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
500	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

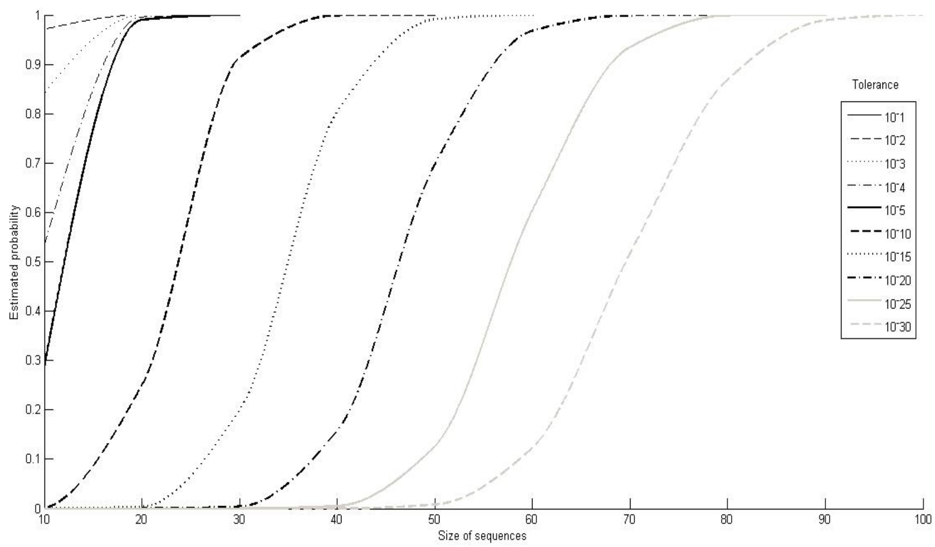
Figure 1: Stability of the product FAO as n increases.



Note: The graph was made for values of n lower than 50 to better observe the depicted behavior.

Now, let us focus on the weighted mean FAO. As shown in *Section* 3, stability for this FAO is strongly dependent on the weights chosen, and thus in general it is an unstable FAO. Particularly, some examples of a *asymptotic R-strictly stable* and an *unstable* weighted mean families were pre-

Figure 2: Speed of convergence of the product FAO



sented. Next, a similar simulation process as with the product FAO will be carried out for these examples.

Therefore, consider the weighted mean family $\{W_n(x_1, ..., x_n) = \sum_{i=1}^{n} w_i^n x_i\}_{n \in \mathbb{N}}$, having weights $w^n \in [0,1]^n$ defined by

$$w_i^n = \begin{cases} \dfrac{1}{(n+1)} & \text{if } 1 \leq i \leq n-1 \\ \dfrac{2}{(n+1)} & \text{if } i = n \end{cases}$$

As pointed out in *Section 2*, this family is *asymptotically R-strict stable*, so it has to satisfy

$$\lim_{n \to +\infty} \left| dif_R(W_n, W_{n-1}) \right| = 0.$$

Table 3: Simulation outputs of $\mathbb{P}[\lim_{n\to+\infty}\left|dif_R(W_n,W_{n-1})\right| < Tolerance]$ for an asymptotic R-strict stable weighted mean FAO.

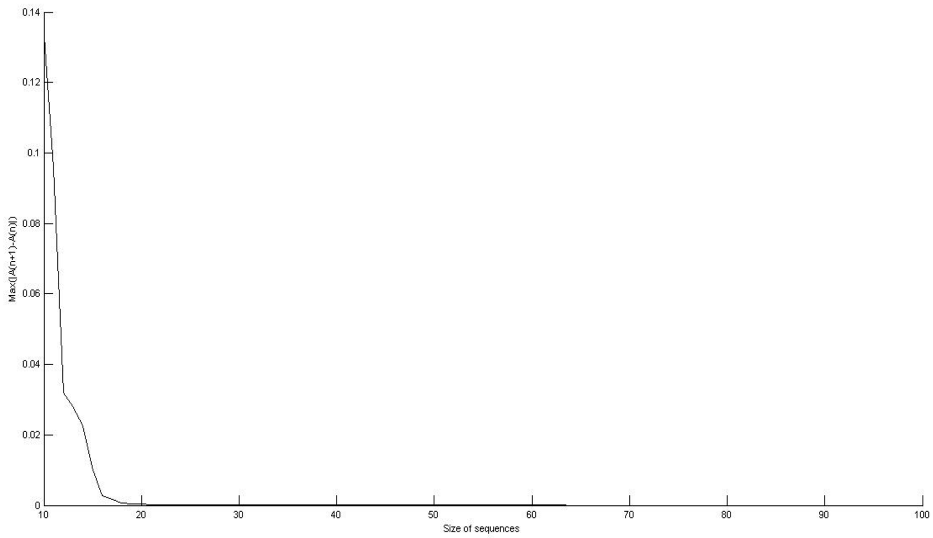
Size of the sequences	Tolerance levels 10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
10	1.000	0.289	0.029	0.003	0.000	0.000	0.000	0.000
20	1.000	0.478	0.064	0.004	0.000	0.000	0.000	0.000
30	1.000	0.673	0.067	0.008	0.000	0.000	0.000	0.000
40	1.000	0.875	0.081	0.003	0.001	0.000	0.000	0.000
50	1.000	0.992	0.109	0.017	0.002	0.000	0.000	0.000
60	1.000	1.000	0.123	0.015	0.001	0.000	0.000	0.000
70	1.000	1.000	0.135	0.017	0.001	0.000	0.000	0.000
80	1.000	1.000	0.166	0.008	0.001	0.000	0.000	0.000
90	1.000	1.000	0.196	0.021	0.003	0.000	0.000	0.000
100	1.000	1.000	0.172	0.011	0.000	0.000	0.000	0.000
300	1.000	1.000	0.600	0.054	0.004	0.000	0.000	0.000
500	1.000	1.000	0.990	0.092	0.009	0.002	0.000	0.000
1000	1.000	1.000	1.000	0.207	0.016	0.001	0.000	0.000
2000	1.000	1.000	1.000	0.403	0.043	0.003	0.000	0.000
4000	1.00	1.000	1.000	0.802	0.067	0.007	0.000	0.000
6000	1.000	1.000	1.000	1.000	0.122	0.007	0.003	0.000
8000	1.000	1.000	1.000	1.000	0.144	0.015	0.001	0.000
10000	1.000	1.000	1.000	1.000	0.220	0.019	0.001	0.000
12000	1.000	1.000	1.000	1.000	0.266	0.034	0.004	0.000
14000	1.000	1.000	1.000	1.000	0.265	0.031	0.001	0.000
18000	1.000	1.000	1.000	1.000	0.354	0.044	0.002	0.000
20000	1.000	1.000	1.000	1.000	0.389	0.036	0.003	0.000
25000	1.000	1.000	1.000	1.000	0.483	0.063	0.006	0.000
30000	1.000	1.000	1.000	1.000	0.603	0.067	0.010	0.000
35000	1.000	1.000	1.000	1.000	0.728	0.076	0.005	0.000
40000	1.000	1.000	1.000	1.000	0.821	0.087	0.003	0.000
45000	1.000	1.000	1.000	1.000	0.887	0.093	0.009	0.000
50000	1.000	1.000	1.000	1.000	1.00	0.099	0.008	0.000

Note: If we consider the three last columns' tolerance levels, the sequence respectively converges for data sizes $2x10^6$, $2x10^7$ and $2x10^8$.

Figure 3 shows that the aggregated values of the sequences of n and $n-1$ items are more and more similar as n increases, but with a convergence speed lower than that of the product family, as also shown in Figure 4 and Table 3. In this case, data sequences with size $n = 100$ produce stable aggregation results whenever a tolerance level not lower than 10^{-3} is taken.

Figure 3: Stability of a weighted mean FAO in which its weights are defined by

$$w_i^n = \frac{1}{(n+1)} \; si \; 1 \leq i \leq n-1 \;\; y \;\; w_i^n = \frac{2}{(n+1)} \; si \; i = n.$$



Finally, the following outputs show the behavior of the weighted mean FAO having weights

$$u_i^n = \begin{cases} \dfrac{1}{2(n-2)} & \text{if } 1 < i < n \\ 1/4 & \text{if } i = 1 \\ 1/4 & \text{if } i = n \end{cases}$$

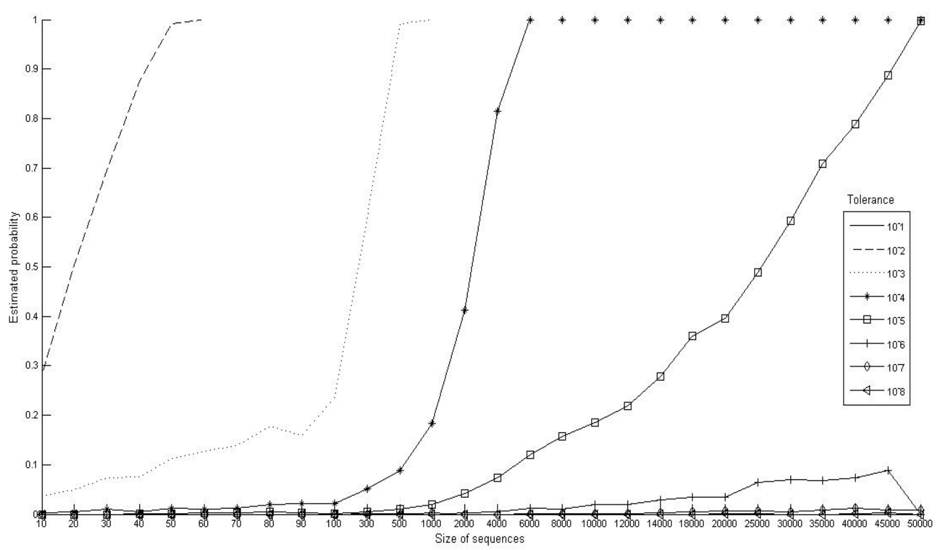
As proven in *Section 4*, this weighted mean FAO is unstable. Therefore, it holds that

$$\mathbb{P}[\lim_{n \to +\infty} \left| dif_R(W_n, W_{n-1}) \right| = 0] < 1.$$

$Figure$ 5 clearly shows this behavior, since the aggregation results of n and $n-1$ items do not converge when n increases. Also, note that the probability of being stable does not converge to 1, as it can be seen in Table 4.

Figure 4: Speed of convergence of a weighted mean FAO in which its weights are defined by

$$w_i^n = \frac{1}{(n+1)} \; si \; 1 \le i \le n-1 \;\; y \;\; w_i^n = \frac{2}{(n+1)} \; si \; i = n.$$

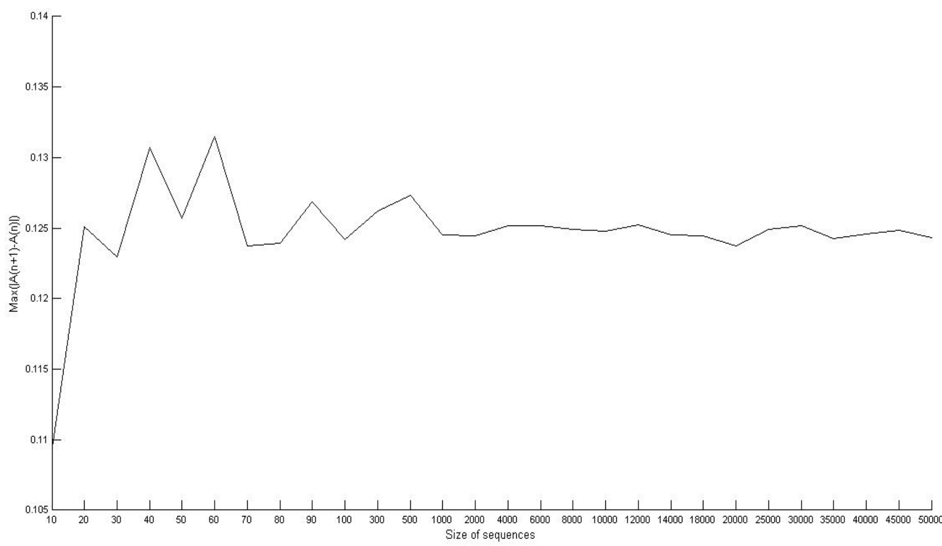


6. Conclusions and final remarks

The classical definition of an aggregation family does not impose any relation among its elements, and meanwhile such a relation does not exist it should not be properly understood as a *family*, but just as a bunch of n-ary operators. Aggregation operators within a *family* must be deeply related, following some building procedure throughout the aggregation process. Since it is clear that we should not define a family of aggregation operators $\{A_n\}_{n\in\mathbb{N}}$ in which each operator can be randomly chosen, the aggregation process demands a conceptual unit idea. With this objective we have presented here three properties that pursue a possible approach for consistency.

Few authors have studied the relationships that should exist between the different aggregation functions that compose a family of aggregation operators (at least beyond demanding properties such as continuity or derivability for each isolated aggregation function). The approach of this paper is linked to the notions of stability and robustness in an information aggregation process. Within such a process, if a new item of information is added and it corroborates our current knowledge (i.e. if it coincides with the aggregation of the information we had up to that moment), then the updated aggregation should not be far from our previous knowledge. Close to this idea, Yager [24] proposed the property of self-identity, that reflects a part of the stability notion presented in this paper. However, Yager's idea required more technical development, as it was only defined in one direction

Figure 5: Stability of an unstable weighted mean FAO as n increases.



(from left to right) and no weaker conditions were analyzed. For this reason, three stability levels (strict, asymptotically strict and almost sure asymptotically strict) have been defined in this paper, also taking into account that, in each level, information can be aggregated from left to right or from right to left (as in the common queues and stacks). In this way we have generalized and expanded the seminal work of Yager [24]. Moreover, the stability levels of some relevant aggregation operators have been analyzed.

It is also important to remark that in [24], the self-identity property was analyzed adding associativity as a constraint. It is possible to demand this property together with self-identity in order to guarantee some consistency in an aggregation family, but as pointed in [12], associativity can be viewed as a necessary restriction only once we accept that our aggregation process should be based upon a unique binary operator. Such an assumption is not so obvious even in the crisp case, and it is difficult to accept in a more general framework. In this paper we have not imposed associativity, since there are too many situations (see for example the recursive rules introduced in [1, 8, 9]) in which different binary aggregation operators are introduced depending on the cardinality of the data n, always maintaining some degree of consistency in the aggregation process.

Obviously, this work leaves also many open questions. For example, in this paper we have obtained necessary and sufficient conditions to guarantee strict stability of a weighted mean family, as well as sufficient conditions for asymptotic stability. How to extend these conditions in order to also include almost sure asymptotic strict stability of the weighted mean is a question that should be addressed. An analysis of the stability of OWA operators, in any of its levels and in relation with the conditions that have to be imposed over the weights, is another question that remains open for the future. In particular, we should address the case in which the weights are generated by a quantifier, leading to a somehow consistent aggregation process.

Table 4: Simulation output of $\mathbb{P}[\lim_{n\to+\infty} \left|dif_R(W_n, W_{n-1})\right| < Tolerance]$ for an unstable weighted mean FAO.

Size of the sequences	Tolerance levels 10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
10	1.000	0.157	0.012	0.002	0.000
20	0.986	0.123	0.006	0.001	0.000
30	0.978	0.133	0.009	0.000	0.000
40	0.965	0.120	0.015	0.000	0.000
50	0.957	0.117	0.012	0.001	0.000
60	0.960	0.117	0.014	0.000	0.000
70	0.959	0.123	0.008	0.001	0.000
80	0.971	0.121	0.014	0.003	0.000
90	0.965	0.121	0.015	0.003	0.000
100	0.950	0.109	0.009	0.000	0.000
300	0.956	0.101	0.007	0.002	0.000
500	0.935	0.097	0.014	0.001	0.000
1000	0.956	0.110	0.014	0.002	0.000
2000	0.953	0.137	0.006	0.001	0.000
4000	0.924	0.121	0.007	0.001	0.000
6000	0.947	0.105	0.008	0.000	0.000
8000	0.938	0.109	0.015	0.000	0.000
10000	0.941	0.109	0.013	0.001	0.000
12000	0.948	0.100	0.014	0.001	0.000
14000	0.962	0.115	0.008	0.000	0.000
18000	0.953	0.108	0.009	0.002	0.000
20000	0.953	0.103	0.005	0.001	0.000
25000	0.947	0.110	0.013	0.000	0.000
30000	0.958	0.098	0.013	0.003	0.000
35000	0.952	0.113	0.007	0.001	0.000
40000	0.961	0.099	0.013	0.000	0.000
45000	0.951	0.120	0.010	0.001	0.000
50000	0.943	0.123	0.009	0.002	0.000

Moreover, in this paper it has been assumed that each new item of information to be aggregated *coincides* with the aggregation of the $n-1$ previous items (i.e. $x_n = A_{n-1}(x_1, \ldots, x_{n-1})$). Based on this assumption, the similarity between A_{n-1} and A_n has been then analyzed in order to define the different stability levels. Relaxing such an assumption would lead to a more general approach to the problem, linked to Lipschitz or analogous inequalities, and should enable us to analyze in what situations a x_n *close* to $A_{n-1}(x_1, \ldots, x_{n-1})$ produces similar aggregation values.

Another issue that should be considered is the *stability speed* of the aggregation processes. Though our three stability levels give a theoretical idea of the relative consistence of the operators, the speed of convergence constitutes another relevant aspect in practice. For example, as shown in *Section 5*, though some weighted mean families are asymptotically stable and the product family is almost sure asymptotically stable, the speed of convergence in both processes is clearly different, in favor of the second one as it is shown in Tables 2 and 3.

Acknowledgment

This research has been partially supported by the Government of Spain, grant TIN2009-07901.

Bibliography

[1] A. Amo, J. Montero, E. Molina. Representation of consistent recursive rules. European Journal of Operational Research, 130, 29-53, 2001.

[2] G. Beliakov, A. Pradera, T. Calvo. Aggregation Functions, a Guide to Practitioners. Springer-Verlag, Berlin. 2007.

[3] H. Bustince, E. Barrenechea, J. Fernandez, M. Pagola, J. Montero, C. Guerra. Contrast of a fuzzy relation. Information Sciences, 180(8), 1326-1344, 2010.

[4] T. Calvo, A. Kolesarova, M. Kolesárová, R. Mesiar. Aggregation operators, properties, classes and construction methods. In T. Calvo et al. (Eds.): Aggregation Operators New trends ans Aplications. Physica-Verlag, Heidelberg, 3-104, 2002.

[5] T. Calvo, G. Mayor, J. Torrens, J. Suñer, M. Mas and M. Carbonell. Generation of weighting triangles associated with aggregation fuctions. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 8(4),417-451, 2000.

[6] C.H. Carlsson, R. Fuller. Fuzzy reasoning in decision making and optimization. Heidelberg, Springfield-Verlag, 2002.

[7] V. Cutello, J. Montero. Hierarchical aggregation of OWA operators: basic measures and related computational problems. International Journal of Uncertainty, Fuzziness and Knowledge-based systems, 3, 17-26, 1995.

[8] V. Cutello, J. Montero. Recursive families of OWA operators. Proceedings FUZZ-IEEE Conference. IEEE Press, Piscataway, 1137-1141, 1994.

[9] V. Cutello, J. Montero. Recursive connective rules. International Journal of Intelligent Systems 14, 3-20, 1999.

[10] D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk, H. Prade. Terminological difficulties in fuzzy set theory - the case of intuitionistic fuzzy sets. Fuzzy Sets and Systems 156, 485-491, 2005.

[11] L.W. Fung, K.S. Fu. An axiomatic approach to rational decision making in a fuzzy environment. In: L.A. Zadeh et at. (Eds.): Fuzzy Sets and their Applications to Cognitive and Decision Processes. Academic Press, New York, 227-256, 1975.

[12] D. Gómez, J. Montero. A discussion of aggregation functions. Kybernetika 40, 107-120, 2004.

[13] D. Gómez, J. Montero, J. Yáñez, C. Poidomani. A graph coloring algorithm approach for image segmentation. Omega 35(2),173-183, 2007.

[14] D. Gómez, J. Montero and J. Yánez. A coloring algorithm for image classification. Information Sciences, 176, 3645-3657, 2006.

[15] M. Grabisch, J. Marichal, R. Mesiar and E. Pap. Aggregation Functions. Encyclopedia of Mathematics and its Applications. Cambridge University, New York, 2009.

[16] A. Kolesárová. Sequential aggregation. In: M. González et al. (Eds.): Proceedings of the Fifth International Summer School on Aggregation Operators, AGOP. Universitat de les Illes Balears, Palma de Mallorca, 183-187, 2009.

[17] V. López, L. Garmendia, J. Montero, G. Resconi. Specification and computing states in fuzzy algorithms. International Journal of Uncertainty, Fuzziness and Knowledge-based systems, 16, 301-336, 2008.

[18] V. López, J. Montero. Software engineering specification under fuzziness. Multiple-Valued Logic and Soft Computing, 15, 209-228, 2009.

[19] V. López, J. Montero and J. Tinguaro Rodriguez. Formal Specification and implementation of computational aggregation Functions. Computational Intelligence, Foundations and Applications. Proceedings of the 9th International FLINS Conference, 523-528, 2010.

[20] J. Montero. A note on Fung-Fu‘s theorem. Fuzzy Sets and Systems, 13, 259-269, 1985.

[21] J. Montero, D. Gómez, H. Bustince. On the relevance of some families of fuzzy sets. Fuzzy Sets and Systems, 158, 2429-2442, 2007.

[22] J. Montero, V. López, D. Gómez. The role of fuzziness in decision making. Studies in Fuzziness and Soft Computing, 215, 337-349, 2007.

[23] R.R. Yager. On ordered weighted averaging aggregation operators in multi-criteria decision making. IEEE Transactions on Systems, Man and Cybernetics 18, 183-190, 1988.

[24] R.R. Yager and A. Rybalov. Nonconmutative self-identity aggregation. Fuzzy Sets and Systems, 85, 73-82, 1997.