





Energy-based stochastic resetting can avoid noise-enhanced stabilityJulia Cantisán ^{*}, Alexandre R. Nieto , Jesús M. Seoane , and Miguel A. F. Sanjuán *Nonlinear Dynamics, Chaos and Complex Systems Group, Departamento de Física, Universidad Rey Juan Carlos, Tulipán s/n, 28933 Móstoles, Madrid, Spain*

(Received 27 October 2023; accepted 18 January 2024; published 7 February 2024)

The theory of stochastic resetting asserts that restarting a stochastic process can expedite its completion. In this paper, we study the escape process of a Brownian particle in an open Hamiltonian system that suffers noise-enhanced stability. This phenomenon implies that under specific noise amplitudes the escape process is delayed. Here, we propose a protocol for stochastic resetting that can avoid the noise-enhanced stability effect. In our approach, instead of resetting the trajectories at certain time intervals, a trajectory is reset when a predefined energy threshold is reached. The trajectories that delay the escape process are the ones that lower their energy due to the stochastic fluctuations. Our resetting approach leverages this fact and avoids long transients by resetting trajectories before they reach low-energy levels. Finally, we show that the chaotic dynamics (i.e., the sensitive dependence on initial conditions) catalyzes the effectiveness of the resetting strategy.

DOI: [10.1103/PhysRevE.109.024201](https://doi.org/10.1103/PhysRevE.109.024201)**I. INTRODUCTION**

Stochastic resetting is a strategy used in search problems to decrease the search time [1]. Imagine a simple search problem in which a particle, the searcher, is looking for a target in a plane without any previous information, and there are no boundaries or restrictions on its movement. The easiest strategy for the particle is just to freely diffuse until it finds the target. However, this method is not very efficient as the search time diverges. Stochastic resetting is a much more suited strategy as the search time becomes finite. This technique implies resetting the particle to its initial position at specific time intervals and letting it evolve freely between the resetting events. By resetting, we avoid the trajectories that take too long and are far away from the target, and we give them another chance to complete the search rapidly.

Search processes in nature are often accompanied by a resetting to an initial state. An example of this is animal foraging [2], where animals repeatedly return to the shelter during the search for resources. Another example where a resetting strategy has also been observed to occur naturally is human visual search [3]. The fact that resetting expedites the search may explain its prevalence in nature. Furthermore, the resetting strategy may be used purposely in different contexts, such as chemical reactions [4] and other queueing processes [5] that work like stochastic processes. The goal is to avoid the metastable states that are causing a delay. In the case of chemical reactions, for instance, the process is reset by unbinding the catalyst. For more applications, see Refs. [6,7].

Stochastic resetting is a hot topic nowadays in the community and much has been published regarding multiple searchers [8], multiple targets [9], interacting particle systems

[10], and nondiffusive processes [11,12], to name a few. The case of diffusion when the searcher can move in a bounded region due to a potential has been less studied, although it is a more realistic situation. Brownian particles subjected to various potentials have been studied, including linear [13], logarithmic [14], V-shaped [15], and higher-order potentials [16]. Only recently has stochastic resetting in an escape problem been studied [17]. In this case, the search process ends when the particle finds the exit and escapes. The authors considered the Kramers problem, where a particle moving in one dimension has to overcome a potential barrier in order to escape.

Here, we aim to broaden current knowledge of stochastic resetting in escape processes. For that purpose, we study a two-dimensional open Hamiltonian system, the Hénon-Heiles Hamiltonian. We chose this system as it presents a phenomenon called noise-enhanced stability, which is also referred to as noise-enhanced trapping in the context of chaotic scattering [18]. This phenomenon delays the escape process for certain noise intensities. We explore if stochastic resetting can counterbalance noise-enhanced stability and expedite the escape process. This interplay was previously explored in Ref. [19], where time-based resetting in a one-dimensional system was considered.

To expedite an escape process from periodic potentials may be useful in many physical situations. We would like to mention Josephson junctions, which can be used for detection of axionlike particles [20] or logical devices [21]. In that case, the dynamics can be modeled by a particle moving in the washboard potential and the escape of the particle implies a switch between the superconductive and the resistive states. The noise can delay the escape process by the same noise-enhanced stability phenomenon that was mentioned, producing a noise-delayed switching [22]. In this context, stochastic resetting may increase performance by reducing the switching time.

^{*}julia.cantisán@urjc.es

Furthermore, we propose a protocol for resetting that we call energy-based resetting. Previous studies have considered resetting at fixed intervals of time, or intervals drawn from a certain distribution (typically a Poisson distribution). Also, a protocol with a space-dependent resetting rate has been explored [23,24]. Here, we show that tracking the energy and resetting when certain thresholds are reached is a strategy capable of expediting the escape process and even suppressing the noise-enhanced stability effect.

This paper is organized as follows. In Sec. II, we describe the Hénon-Heiles system and the noise-enhanced stability phenomenon. In Secs. III and IV, we implement the strategy of stochastic resetting based on time and based on energy, respectively. Later, in Sec. V, we compare both methods. Finally, in Sec. VI, we present our concluding remarks.

II. NOISE-ENHANCED STABILITY IN THE HÉNON-HEILES SYSTEM

To study the counterbalance of stochastic resetting and the noise-enhanced stability effect, we use as a model the paradigmatic Hénon-Heiles Hamiltonian [25]. Ever since its emergence in scientific literature as a model of star motion around a galactic center, this system has garnered extensive attention as a prototype of Hamiltonian chaos. The Hamiltonian is given by

$$\mathcal{H}(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3, \quad (1)$$

where x and y are the spatial coordinates, and \dot{x} and \dot{y} are the components of the velocity.

Since the Hamiltonian has no explicit time dependence, the energy is a conserved quantity [i.e., $\mathcal{H}(x, y, \dot{x}, \dot{y}) = E$]. Beyond the critical energy level $E_e = 1/6$, which is often referred to as the “escape energy,” the potential has three symmetrical exits separated by an angle of $2\pi/3$ rad, as illustrated in Fig. 1. Therefore, in this situation particles can escape towards $\pm\infty$. Conversely, below the escape energy the particles’ motion remains bounded.

In order to account for the presence of noise in the system, we include the influence of a random force, which we simplify as an uncorrelated additive Gaussian white noise. Under these considerations, the equations of motion can be expressed as

$$\begin{aligned} \ddot{x} &= -x - 2xy + \sqrt{2\xi}\eta_x(t), \\ \ddot{y} &= -y - x^2 + y^2 + \sqrt{2\xi}\eta_y(t), \end{aligned} \quad (2)$$

where $\eta_x(t)$ and $\eta_y(t)$ are Gaussian white noise processes $X_t \sim \mathcal{N}(0, 2\xi)$ of amplitude ξ and autocorrelation function $\langle \eta(t')\eta(t) \rangle = \sqrt{2\xi}\delta(t' - t)$.

We have solved numerically the system of stochastic differential equations (2) using a stochastic second-order Heun method [26]. The robustness of this numerical scheme in the stochastic Hénon-Heiles system has been tested in Ref. [27].

The phenomenon of noise-enhanced stability occurs in the Hénon-Heiles system for certain noise conditions. This phenomenon implies that the average escape time suffers a peak for a specific noise amplitude. To explore this, we launch trajectories from $(x, y) = (0, 0)$ using a random shooting angle $\theta \in [0, 2\pi/3]$, which is measured counterclockwise with respect to the y axis. Note that, given the symmetry of the

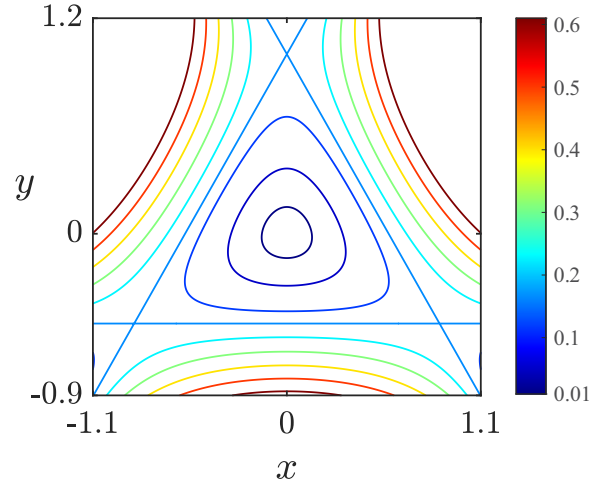


FIG. 1. Isopotential curves of the Hénon-Heiles system for various values of the potential $V(x, y) = \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3$. The color of the curves indicates the different values of the potential, which are marked in the color bar. For energy values below the escape energy $E_e = 1/6$, the isopotential curves are closed, while the potential exhibits three symmetrical exits for energy values above E_e . Particles can escape through these exits towards $\pm\infty$.

system, it is not necessary to consider the complete circumference. Without loss of generality, henceforth we use an initial energy value of $E_0 = 0.25$. Due to the effects of noise, the energy value exhibits fluctuations, so E_0 only defines the initial conditions for the shooting. Once a shooting angle is defined, the components of the initial velocity are given by $(\dot{x}, \dot{y}) = \sqrt{2E_0}(\sin\theta, \cos\theta)$.

We represent the average escape time \bar{T} for different noise levels ξ in Fig. 2. The blue shaded region around \bar{T} corresponds to a 99.7% confidence interval. The peak

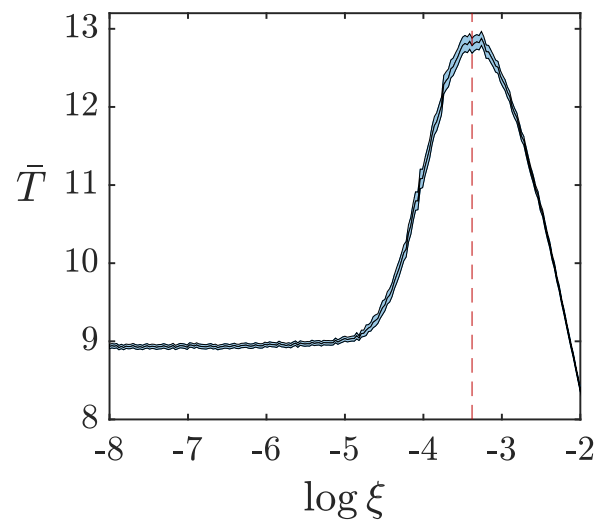


FIG. 2. Average escape time for 200 different noise amplitudes. We represent the logarithm in base 10 of the noise amplitudes for a better visualization. The escape time of 10^6 initial conditions is averaged for each value of ξ . The blue shaded region represents a 99.7% confidence interval and the peak, emphasized by a red dashed line, corresponds to the noise-enhanced stability phenomenon.

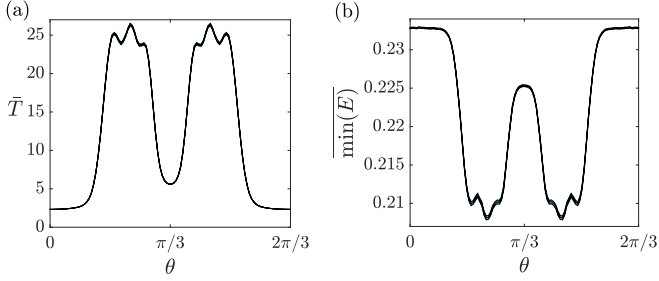


FIG. 3. (a) Average escape times and (b) average minimum energy during the escape path for 100 different values of θ . In both panels, 10^6 initial conditions are computed for each angle.

corresponding to the noise-enhanced stability phenomenon, indicated by a red dashed line in Fig. 2, can be clearly distinguished in our system. In particular, the noise amplitude that enhances the stability corresponds to the value $\xi_s = 10^{-3.38}$. This is precisely the effect that we try to counterbalance by applying the resetting, so henceforth we fix $\xi = \xi_s$. For this noise amplitude, the escape time averaged over all the initial angles is $\bar{T} = 12.97$.

It is clear that for a certain noise level the escape time depends on the launching angle. To observe this, we represent the average escape times for different values of θ in Fig. 3(a). As one can see, trajectories initially pointing directly to the exits take less time to escape than others. The same occurs for trajectories launched with angles close to $\pi/3$, which are initially perpendicular to the potential. It is also remarkable that the standard deviation increases for the angles for which \bar{T} increases. This can be observed in the blue shaded region representing the confidence intervals, which is widened for those angles.

Furthermore, in Fig. 3(b) we represent the average minimum energy that the trajectories reach before escaping. Interestingly, it presents mirror symmetry compared to Fig. 3(a). This implies that fast-escaping particles do not have the possibility to lower their energy. On the contrary, the particles that remain in the scattering region during long transients can wander in the potential well at low-energy states.

Since our purpose is to study if stochastic resetting is a valid strategy to avoid the noise-enhanced stability effect, we shall compute the coefficient of variation (CV) beforehand. This metric measures the ratio of the standard deviation, σ , and the mean value of the escape times, i.e., $CV = \sigma(T)/\bar{T}$. In Ref. [28], the authors show that whenever $CV > 1$, stochastic resetting can expedite the search process. The reasoning behind this is that $CV > 1$ implies a wide distribution of search times, with certain trajectories taking a large time to complete the process. These are the trajectories that delay the whole process and that are given a second chance by resetting. Taking that into consideration, we explore which angles are more beneficial for resetting in Fig. 4.

We see that $CV > 1$ for all angles, except the ones close to the exits. The fundamental reason for this is that initial conditions launched directly towards the exit generate nonchaotic trajectories, which are not worth resetting. In Hamiltonian systems, chaos occurs due to the existence of a nonattracting chaotic set (a chaotic saddle) in phase space. The pattern that

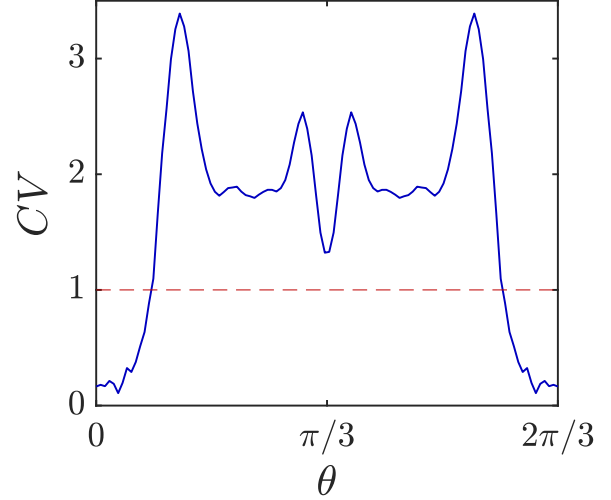


FIG. 4. Coefficient of variation, $CV = \sigma(T)/\bar{T}$, for 100 values of θ . For each value of θ , the average escape time and the corresponding standard deviation have been estimated from 10^6 realizations of the process.

can be observed for $CV > 1$ is explained by the structure of the stable manifold of the chaotic saddle in the noiseless system. This stable manifold is formed by the same set of points as the basin boundary. This implies that orbits that start close to the basin boundary are the ones that take longer times to escape as they follow the stable manifold to the chaotic saddle, spend long times in its vicinity, and finally escape following the unstable manifold. If now we consider the noise, an orbit starting close to the stable manifold (i.e., the basin boundary) can jump to different basins and consequently different realizations of the process can have drastically different escape times. Ultimately, this leads to $CV > 1$. In this sense, it is worth noting that the chaotic dynamics (i.e., the sensitive dependence on initial conditions) catalyzes the effectiveness of the resetting strategy.

All of the above results can be better understood by looking at the distribution of escape times for two characteristic ranges of angles: one with high CV ($\theta \in [0.37, 1.71]$) and one with low CV ($\theta \in [0, 0.24] \cup [1.85, 2\pi/3]$). The escape time is averaged for 10^7 initial conditions in those intervals and represented in the form of histograms. In Fig. 5(a), the relative frequency f (normalized to unity) of the escape time for the trajectories with low CV is depicted. The distribution is narrow, trajectories escape fast and are not worth resetting. Meanwhile, in Fig. 5(b) we can see trajectories from a range of angles with a high CV. In this case, the distribution shows a long tail and trajectories are worth resetting.

III. RESETTING BASED ON TIME

We start applying resetting to our system at deterministic time intervals. This technique has also been called sharp restart [29]. We fix the resetting position to the given initial condition $(x, y, \dot{x}, \dot{y}) = (0, 0, \sqrt{2E_0} \sin \theta, \sqrt{2E_0} \cos \theta)$, where $E_0 = 0.25$. This implies that a particle launched with a certain angle is left to evolve freely in the potential until a certain time t_r , which is the resetting time interval. Then the particle

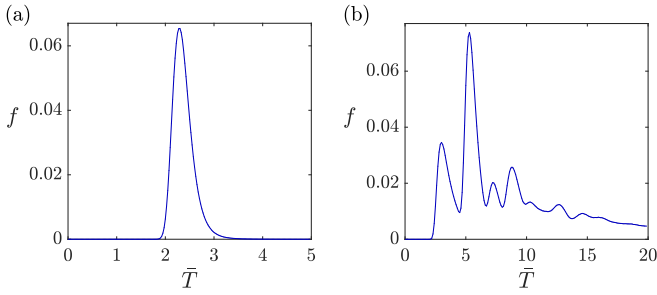


FIG. 5. Escape time distribution based on initial conditions with (a) low CV and (b) high CV. The escape time is averaged over 10^7 initial conditions in the intervals $\theta \in [0, 0.24] \cup [1.85, 2\pi/3]$ for low CV and $\theta \in [0.37, 1.71]$ for high CV. The relative frequency f is normalized to unity. The distribution for high CV shows a long tail of escape times, which we aim to avoid by resetting.

is reset to the initial position and launched again with the same velocity and angle as in the beginning. This process is repeated as many times as needed until the particle escapes. The whole process is represented in Fig. 6, where the black dot denotes the initial position and the black curves account for the potential. The trajectory in blue wanders until the resetting time interval t_r . At that time, the position is marked by a red dot. Then the trajectory is reset (see red segment). The process is then restarted and now the particle escapes through the lower right exit after a short time (see green trajectory). We recall that both trajectories are not the same due to the noise. Furthermore, their paths are completely different because they are chaotic.

In the remainder of this section, we show that time-based stochastic resetting can expedite the escape and we find the optimal resetting time interval t_r^* to that end. As we have seen,

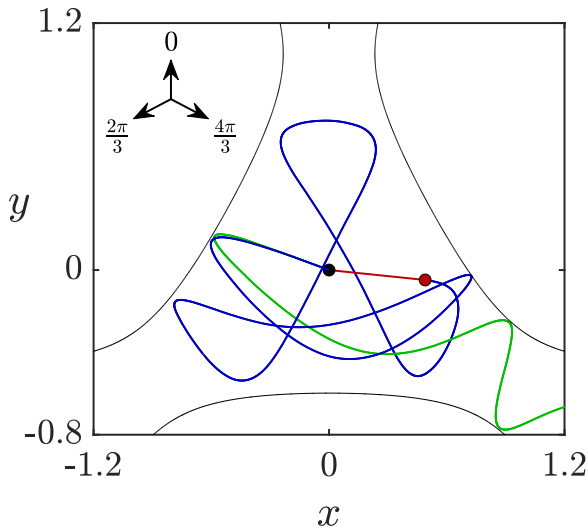


FIG. 6. Stochastic resetting for a trajectory in the Hénon-Heiles Hamiltonian system. The angles θ for the exits are marked in the upper left corner. We show a trajectory (blue curve) that starts at the origin (black dot). The position of the particle at t_r is marked with a red dot, while the resetting process is represented with a red segment. After resetting, the new trajectory (green curve) escapes in a time $t < t_r$.

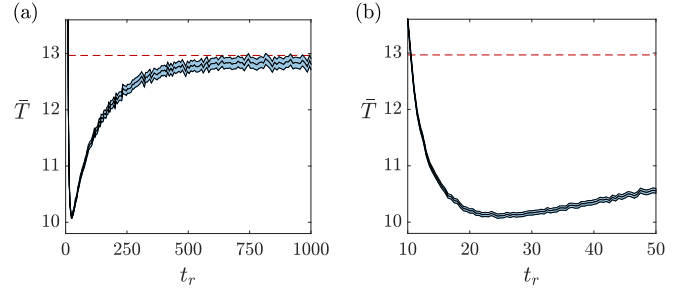


FIG. 7. Escape time for 200 values of the resetting interval t_r . For each value of t_r , \bar{T} has been calculated by averaging the escape time of 10^6 random initial conditions. In panel (b) we show a zoom-in around the optimum t_r . The horizontal red dashed line marks the average escape time of the system without stochastic resetting ($\bar{T} = 12.97$).

not all trajectories are worth resetting. In particular, for initial conditions with $CV < 1$, stochastic resetting is detrimental. Therefore, the resetting strategy should be focused on initial conditions with $CV > 1$ (i.e., the chaotic ones). This already points out that t_r^* should be larger than the escape time of the initial conditions with $CV < 1$ [recall Fig. 5(a)].

In Fig. 7, we show how the escape process is affected by different values of the resetting interval t_r . Figure 7(b) is a magnification of Fig. 7(a), focused on the interval of values of t_r where stochastic resetting is more effective. In both panels the horizontal dashed red line marks the average escape time of the system without resetting ($\bar{T} = 12.97$). For $t_r < 10.5$ stochastic resetting hinders the escape process. We have already mentioned that t_r^* should be larger than the average escape time for trajectories with $CV < 1$, as these trajectories should not be reset. For the initial conditions with $CV > 1$, the optimum strategy is the one that does not reset the trajectories that escape fast, but only the ones that belong to the tail of escape times. Stochastic resetting starts to be effective for $t_r > 10.5$. If we compare this value with Fig. 5(b), it implies resetting after the big waves of escape have passed. For example, the probability that an initial condition with the initial angle $\theta = 0.7$ [maximum in Fig. 3(a)] escapes before $t = 5$ is 2×10^{-6} . The probability that it escapes before $t = 3$ is 0 after launching 10^6 simulations. This implies that resetting has to occur for larger values of t_r . Otherwise, we would be resetting again and again without allowing any trajectory to escape. On the other hand, for very large values of t_r , stochastic resetting is not beneficial either, as it is equivalent to not applying any protocol. As a matter of fact, from Fig. 7(a) it is clear that for large values of t_r the escape time converges to the natural value of the system without resetting.

The optimum resetting time interval is $t_r^* \approx 25$. For this value we find $\bar{T} = 10.11$, which is 22% less than the average escape time of the system without stochastic resetting. Furthermore, there is a wide range of values above $t_r^* = 25$ that also bring good results; thus the chosen resetting interval does not need to be extremely accurate.

An advantage of this method is that the fraction of reset trajectories is low (only 9.3%). This is convenient if we assume that the reset implies a certain cost. A simplification we have taken in our model is that the resetting is instantaneous. Of

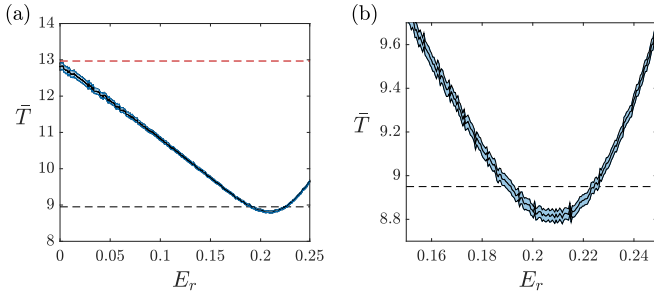


FIG. 8. Escape time averaged over 10^6 random initial conditions for 250 different values of the resetting energy E_r . In panel (b) we show a zoom-in around the optimum value of E_r . The horizontal red dashed line is located at the average escape time of the system without stochastic resetting ($\bar{T} = 12.97$). The horizontal black dashed line is located at the average escape time of the noiseless system ($\bar{T} = 8.95$). It can be seen that resetting based on energy is capable of avoiding the noise-enhanced stability effect and even reducing the escape time under the value of the noiseless system.

course, every resetting event comes with a cost and requires a certain time. This has been implemented by including refractory periods or time penalties proportional to the distance to the resetting position. Energetic and thermodynamics costs have also been studied. For a review, see Ref. [30].

IV. RESETTING BASED ON ENERGY

In this section, we introduce an alternative approach for resetting that is based on the energy of the particle. In this case, the particle launched with a certain angle is left to evolve freely in the potential, until a certain energy threshold E_r is reached. Then the trajectory is reset to the initial position and launched with the same velocity and angle as in the beginning.

The interest of this approach is the following. Noise-enhanced stability occurs due to a small amount of trajectories that decrease their energy to very low values and remain during long transients in the scattering region. This drop in the energy allows these “unusual” trajectories to wander in Kolmogorov-Arnold-Moser (KAM) regions or even to reach energy levels for which the isopotential curves are closed [31]. Thus, the more natural parameter to track seems to be the energy. This is true not only for our specific system but also for all systems in which the process gets stuck in metastable states.

Following this protocol, we show how the resetting energy E_r affects the escape times in Fig. 8(a). For the sake of comparison, the horizontal red dashed line marks the average escape time of the system without resetting. First of all, the initial energy is fixed in all cases to $E_0 = 0.25$, and we can see that, if the energy drops below this value, resetting is always beneficial in the sense that it accelerates the escape process. However, the behavior is nonmonotonic and the optimum value of E_r can be found at $E_r^* = 0.208$.

The origin of the minimum in the escape times for resetting with E_r^* can be explained as follows. On the one hand, if we set the threshold at too low energies, the escape time increases. This is because trajectories do not reach such low-energy values in short times. Thus, setting a low E_r would imply

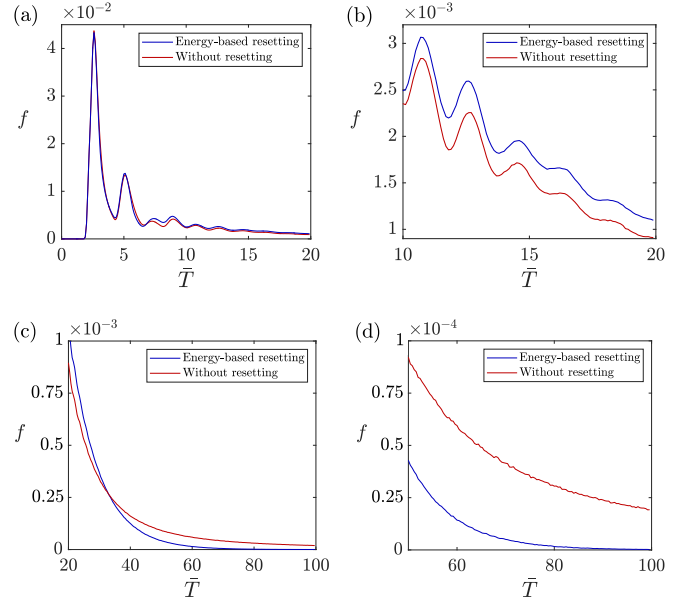


FIG. 9. Escape time distribution averaged over 10^8 initial conditions. The relative frequency f is normalized to unity. The red curve is calculated for the system without resetting, while the blue curve denotes the system with stochastic resetting with $E_r = E_r^* = 0.208$. Panels (b) and (d) are zoom-ins of panels (a) and (c), respectively. It can be seen that the resetting is capable of reducing the long tail of escape times.

resetting after large periods of time. On top of that, only a few trajectories reach energy values close to 0, so we would be resetting only these ones and not others that also take large times to escape. On the other hand, setting E_r too close to the initial energy level results in resetting a high number of particles, including those that were about to escape fast after experiencing an initial energy drop. The action of both effects leads to the nonmonotonic behavior and the presence of an optimum value for E_r . In this sense, the method predicts the trajectories that will take longer times and resets them beforehand.

In Fig. 8(b), we can see a zoom-in of Fig. 8(a) to observe that the value of \bar{T} drops slightly below the horizontal dashed black line, which denotes the average escape time of the noiseless system ($\bar{T} = 8.95$). Thus, we can conclude that energy-based resetting is able to suppress noise-enhanced stability. For this value, the average escape is $\bar{T} = 8.81$, which is 32% less than the average escape time of the system without stochastic resetting. Also, the maximum escape time is $\max(T) = 163$, while without stochastic resetting $\max(T) > 10\,000$ and after $T = 163$ still 0.8% of the trajectories did not escape. These trajectories have $\bar{T} = 295$ and constitute the 20% of the average escape time. These are the type of events that stochastic resetting is good at avoiding. We recall that we have fixed the noise level ξ to the value that maximizes the noise-enhanced effect (see Fig. 2), so we have shown that the method is able to suppress noise-enhanced stability even in the worst scenario. Needless to say, the method is also efficient for other values of ξ .

In Fig. 9, we depict the escape-time distribution for the case without resetting (red curve) and with energy-based resetting

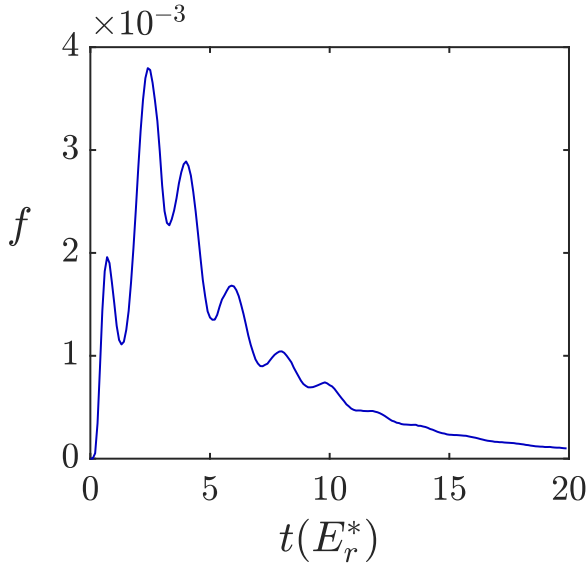


FIG. 10. Distribution of times at which the energy-based resetting with E_r^* is applied. The results of this figure are based on 10^8 initial conditions and the relative frequency f is normalized to unity. It can be seen that this protocol takes place mainly at early stages of the process.

using E_r^* (blue curve). Figures 9(b) and 9(d) are zoom-ins of Figs. 9(a) and 9(c), respectively. The method’s primary advantage lies in the reduction of escape times’ tail.

Another interesting result is that 19.5% of the trajectories are reset using E_r^* . Thus, more trajectories are reset using this method than using the sharp restart. Furthermore, the position of the particles when they are reset following the energy-based protocol is approximately uniform, so there is no relation between the position and the resetting in this case. Regarding the times at which the energy-based resetting with E_r^* is applied [$t(E_r^*)$], we have computed the distribution and the results can be seen in Fig. 10.

As previously mentioned, the noise-enhanced stability is caused by a small amount of trajectories that decrease their energy to very low values and remain during long transients in the scattering region. For this phenomenon to occur, trajectories must avoid the escape by decreasing their energy fast. If not, fluctuations would probably kick the particle out of the well. Therefore, the particles that enhance the stability reach low-energy values in short times. Energy-based resetting identifies these trajectories by detecting a fast energy drop. As we can see in Fig. 10, most of the trajectories are reset after very short times, meaning that, on average, it is not necessary to let a trajectory evolve to know that its escape time will be large. This is one of the key advantages of energy-based resetting in comparison with time-based resetting.

V. COMPARISON BETWEEN METHODS

In this section, we make a comparison between time-based and energy-based resetting to show the advantages and downsides of both methods. Furthermore, the results from the preceding two sections have been calculated after averaging different initial conditions using a random shooting angle $\theta \in [0, 2\pi/3]$. Here, we explore the effects of both methods depending on the launching angle.

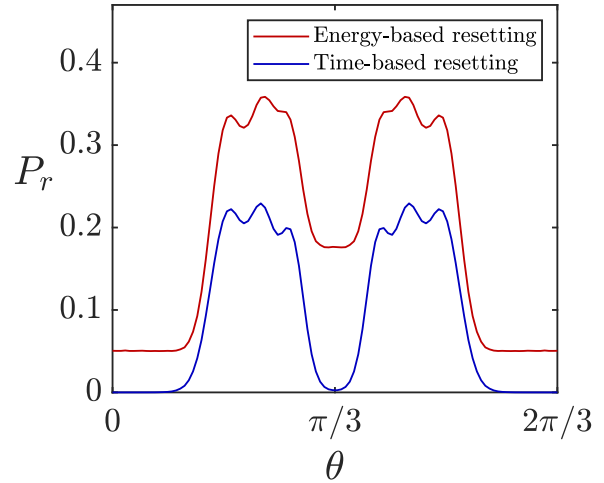


FIG. 11. Resetting probability for 100 different values of the launching angle. The time-based resetting protocol with $t_r^* = 25$ is shown in blue, and the energy-based resetting protocol with $E_r^* = 0.208$ is shown in red. For each initial condition, we perform 10^6 realizations of the process in order to obtain the resetting probability.

We define the resetting probability P_r as the probability that an initial condition is reset using a particular value of t_r or E_r . In the case of time-based resetting, we fix the resetting time to t_r^* , and for 100 values of the angle in the interval $\theta \in [0, 2\pi/3]$, we compute 10^6 realizations of the process. From these data, we calculate the fraction of trajectories that have not escaped after $t = t_r^*$. Equivalently, for energy-based resetting, we fix the resetting energy to E_r^* and we calculate the fraction of trajectories that have reached $E = E_r^*$ before escaping. With this procedure, we are calculating the fraction of initial conditions that are reset at least once using the time or energy-based protocols. Therefore, the resetting probability is a measure of the cost of each method.

The dependence of P_r with the angle is depicted in Fig. 11. For time-based resetting (blue curve), stochastic resetting is never applied for angles with $CV < 1$. This is simply because the nonchaotic trajectories generated by these initial conditions always escape before $t = t_r^*$. However, the energy-based protocol (red curve) is applied even in those cases, although it is unnecessary.

To study how effective the method is depending on the angle, we compute the escape times for different angles in Fig. 12. The blue (red) curve corresponds to time-based (energy-based) resetting. The black curve indicates the escape time of the system without stochastic resetting. It is clear that energy-based resetting is, overall, more effective than time-based resetting. In particular, the average escape time of the trajectories close to the basin boundary is reduced much more significantly than using time-based resetting. On the contrary, for initial conditions far from the basin boundary, this method is ineffective and can be even counterproductive. This is the case for $\theta = \pi/3$ and for θ close to 0 and $2\pi/3$. This suggests that a combined method (i.e., using energy-based resetting in conjunction with time-based resetting) could be beneficial. Energy-based stochastic resetting resets the trajectories in very short times (recall Fig. 10), so for initial conditions far from the boundary (nonchaotic ones) the method also resets

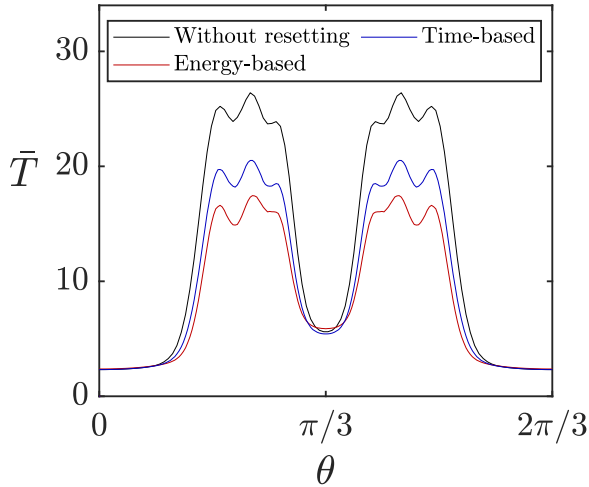


FIG. 12. Escape time for 100 different values of the launching angle. The time-based resetting protocol with $t_r^* = 25$ is shown in blue, and the energy-based resetting protocol with $E_r^* = 0.208$ is shown in red. For comparison, we show in black the escape times without applying resetting. For each initial condition, we perform 10^6 realizations in order to obtain \bar{T} .

trajectories that were going to escape very fast, making the method inefficient for such initial conditions. By forcing the reset to occur only if the time is greater than some threshold ($t_r \approx 3$) or if $CV > 1$, we could retain the advantages of the method while reducing its downsides.

Another consequence of resetting is a reduction of the survival probability, that is, the probability that a certain trajectory has not escaped until time t . This is a standard measure in scattering problems in open Hamiltonian systems, and for fully chaotic systems it roughly follows an exponential decay

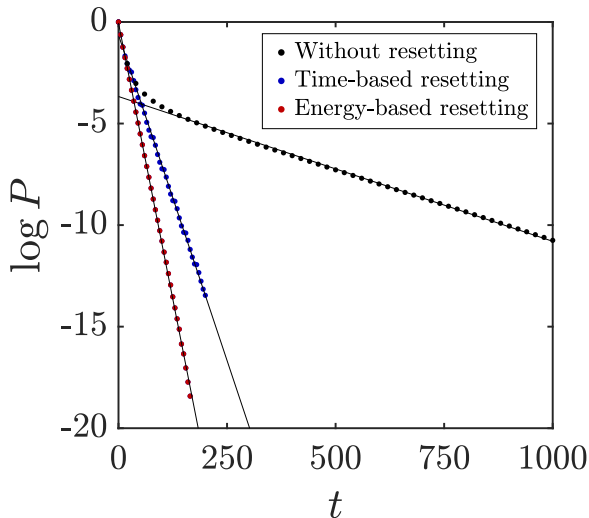


FIG. 13. Survival probability decay with time. Blue dots correspond to stochastic resetting based on time with $t_r = 25$ [$\alpha = 0.0641$ ($r = 0.9991$)], red dots to stochastic resetting based on energy with $E_r = 0.208$ [$\alpha = 0.1087$ ($r = 0.9997$)], and black dots to the system without stochastic resetting [$\alpha = 0.0071$ ($r = 0.9993$)]. The results for each data set are calculated after shooting 10^8 initial conditions.

law,

$$P(t) \propto e^{-\alpha t}, \quad (3)$$

where $1/\alpha$ is commonly referred to as the characteristic time. In Fig. 13, we see in logarithmic scale how this probability changes with time for the case without resetting (black dots), with time-based resetting (blue dots), and with energy-based resetting (red dots). In this figure, we see that α (the slope of the straight lines) is larger for energy-based stochastic resetting. Although both methods significantly increase the escape rate, energy-based resetting is a better protocol for expediting the escape process.

VI. CONCLUSIONS AND DISCUSSION

Stochastic resetting has been proven to be a very effective protocol to reduce search time in search processes, and recently the same has been found for escape processes too. This strategy consists of restarting a stochastic process in order to avoid the system getting stuck in an undesired state.

Here, we have studied an open Hamiltonian system that presents noise-enhanced stability. This implies that certain noise levels delay the escape process. For these values, some trajectories present long transients as the noise pushes them close to KAM islands. Our aim has been to explore if stochastic resetting can counterbalance this effect and expedite the escape process. This technique could be applied in many physical contexts involving the escape from periodic potentials, such as to improve the performance of Josephson junctions.

We have explored two different resetting protocols: a time-based protocol (sharp restart) and an alternative protocol based on tracking the energy of the system and resetting the process when a certain threshold is reached. We have proven that this protocol is capable of avoiding the noise-enhanced stability phenomenon. Although time-based resetting is not as effective in reducing the escape time, each protocol has its advantages and disadvantages. Resetting is expected to have a cost, either in terms of time or in terms of energy. In this sense, time-based resetting is applied to a smaller number of trajectories, which probably reduces the cost of resetting. Energy-based resetting tends to over-reset, so even for trajectories that have $CV < 1$ resetting is sometimes applied. Also, tracking the energy of the process might not be possible in all systems and time-based energy does not imply tracking any measure of the process.

Another important remark is that we have explored stochastic resetting in a system that presents transient chaos. We conclude that in this situation the effectiveness of the resetting strategy is strongly linked to chaotic dynamics. Trajectories exhibiting sensitive dependence on initial conditions can escape in a wide range of escape times under the effects of noise. This implies high values of CV and thus stochastic resetting is a more interesting strategy to be applied.

ACKNOWLEDGMENT

This work has been supported by the Spanish State Research Agency (AEI) and the European Regional Development Fund (ERDF, EU) under Project No. PID2019105554GB-I00 (MCIN/AEI/10.13039/501100011033).

- [1] M. R. Evans and S. N. Majumdar, Diffusion with stochastic resetting, *Phys. Rev. Lett.* **106**, 160601 (2011).
- [2] G. Mercado-Vásquez and D. Boyer, Lotka-Volterra systems with stochastic resetting, *J. Phys. A: Math. Theor.* **51**, 405601 (2018).
- [3] J. M. Wolfe and T. S. Horowitz, What attributes guide the deployment of visual attention and how do they do it? *Nat. Rev. Neurosci.* **5**, 495 (2004).
- [4] S. Reuveni, M. Urbakh, and J. Klafter, Role of substrate unbinding in Michaelis-Menten enzymatic reactions, *Proc. Natl. Acad. Sci. USA* **111**, 4391 (2014).
- [5] O. L. Bonomo, A. Pal, and S. Reuveni, Mitigating long queues and waiting times with service resetting, *PNAS Nexus* **1**, pgac070 (2022).
- [6] M. R. Evans, S. N. Majumdar, and G. Schehr, Stochastic resetting and applications, *J. Phys. A: Math. Theor.* **53**, 193001 (2020).
- [7] S. Gupta and A. M. Jayannavar, Stochastic resetting: A (very) brief review, *Front. Phys.* **10**, 789097 (2022).
- [8] U. Bhat, C. De Bacco, and S. Redner, Stochastic search with Poisson and deterministic resetting, *J. Stat. Mech.* (2016) 083401.
- [9] P. C. Bressloff, Search processes with stochastic resetting and multiple targets, *Phys. Rev. E* **102**, 022115 (2020).
- [10] A. Nagar and S. Gupta, Stochastic resetting in interacting particle systems: A review, *J. Phys. A: Math. Theor.* **56**, 283001 (2023).
- [11] J. Masoliver, Telegraphic processes with stochastic resetting, *Phys. Rev. E* **99**, 012121 (2019).
- [12] L. Kusmierz, S. N. Majumdar, S. Sabhapandit, and G. Schehr, First order transition for the optimal search time of Lévy flights with resetting, *Phys. Rev. Lett.* **113**, 220602 (2014).
- [13] A. Pal, Diffusion in a potential landscape with stochastic resetting, *Phys. Rev. E* **91**, 012113 (2015).
- [14] S. Ray and S. Reuveni, Diffusion with resetting in a logarithmic potential, *J. Chem. Phys.* **152**, 234110 (2020).
- [15] R. K. Singh, R. Metzler, and T. Sandev, Resetting dynamics in a confining potential, *J. Phys. A: Math. Theor.* **53**, 505003 (2020).
- [16] S. Ahmad, I. Nayak, A. Bansal, A. Nandi, and D. Das, First passage of a particle in a potential under stochastic resetting: A vanishing transition of optimal resetting rate, *Phys. Rev. E* **99**, 022130 (2019).
- [17] J. Cantisán, J. M. Seoane, and M. A. F. Sanjuán, Stochastic resetting in the Kramers problem: A Monte Carlo approach, *Chaos, Solitons Fractals* **152**, 111342 (2021).
- [18] E. G. Altmann and A. Endler, Noise-enhanced trapping in chaotic scattering, *Phys. Rev. Lett.* **105**, 244102 (2010).
- [19] K. Capała, B. Dybiec, and E. Gudowska-Nowak, Interplay of noise induced stability and stochastic resetting, *Chaos* **32**, 063129 (2022).
- [20] A. L. Pankratov, L. S. Revin, A. V. Gordeeva, A. A. Yablokov, L. S. Kuzmin, and E. Il'ichev, Towards a microwave single-photon counter for searching axions, *npj Quantum Inf.* **8**, 61 (2022).
- [21] A. V. Gordeeva and A. L. Pankratov, Minimization of timing errors in reproduction of single flux quantum pulses, *Appl. Phys. Lett.* **88**, 022505 (2006).
- [22] K. G. Fedorov, A. L. Pankratov, and B. Spagnolo, Influence of length on the noise delayed switching of long Josephson junctions, *Int. J. Bifurcation Chaos* **18**, 2857 (2008).
- [23] M. R. Evans and S. N. Majumdar, Diffusion with optimal resetting, *J. Phys. A: Math. Theor.* **44**, 435001 (2011).
- [24] É. Roldán and S. Gupta, Path-integral formalism for stochastic resetting: Exactly solved examples and shortcuts to confinement, *Phys. Rev. E* **96**, 022130 (2017).
- [25] M. Hénon and C. Heiles, The applicability of the third integral of motion: Some numerical experiments, *Astron. J.* **69**, 73 (1964).
- [26] P. E. Kloeden and E. Platen, *Numerical Solutions of Stochastic Differential Equations* (Springer, Berlin, 1992).
- [27] A. R. Nieto, J. M. Seoane, and M. A. F. Sanjuán, Final state sensitivity in noisy chaotic scattering, *Chaos, Solitons Fractals* **150**, 111181 (2021).
- [28] A. Pal and S. Reuveni, First passage under restart, *Phys. Rev. Lett.* **118**, 030603 (2017).
- [29] I. Eliazar and S. Reuveni, Diversity of sharp restart, *J. Phys. A: Math. Theor.* **56**, 024003 (2023).
- [30] J. C. Sunil, R. A. Blythe, M. R. Evans, and S. N. Majumdar, The cost of stochastic resetting, *J. Phys. A: Math. Theor.* **56**, 395001 (2023).
- [31] A. R. Nieto, J. M. Seoane, and M. A. F. Sanjuán, Trapping enhanced by noise in nonhyperbolic and hyperbolic chaotic scattering, *Commun. Nonlinear Sci. Numer. Simul.* **102**, 105905 (2021).