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Characteristic curves and the exponentiation in the Riordan Lie group: A connection through examples

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A B S T R A C T

We point out how to use the classical characteristic method, that is used to solve quasilinear PDE's, to obtain the matrix exponential of some lower triangle infinite matrices. We use the Lie Fréchet structure of the Riordan group described in [4]. After that, we describe some linear dynamical systems in $\mathbb{K}[[x]]$ with a concrete involution being a symmetry or a time-reversal symmetry for them. We take this opportunity to assign some dynamical properties to Pascal's triangle.

1. Introduction

Several phenomena may be modelled using systems of differential equations and a key notion to get their solutions is the exponential of (finite) square matrices. Therefore, the problem of computing these exponentials is relevant. Moreover, some crucial developments of Linear Algebra have been motivated by the study of the matrix exponential. We recommend [13] for a survey on this topic.

Furthermore, the matrix exponential lies in the core of Lie theory and connects the natural Lie algebra of all $n \times n$ matrices with the general linear group $\mathrm{GL}(n, \mathbb{K})$. We think that [15] is a very good introductory text for this point of view.

By allowing concepts as manifolds modelled in infinite dimensional spaces (e.g., Banach spaces, Fréchet spaces etc.), the theory of Lie groups has been extended to the infinite dimensional framework [12]. See

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[5] for some coherence relationships between infinite dimensional Lie group structures and pro-Lie group structures when both are shared by a group. This is the case of the Riordan group. The previous facts motivated us to introduce and develop in [4] a structure of infinite dimensional Lie group on the Riordan group.

In the following section we recall some basic facts about the Riordan group and its Lie group and pro-Lie group structures described in [4]. Notice that, although this group appeared under this name more or less recently, the group structure and many of its elements are latent in many developments of classical mathematics (special sequences of numbers and polynomials, Umbral Calculus and much more). It is clear that, historically, the first (and surely the best) known Riordan matrix is Pascal's triangle. Besides this, now, there are a lot of historical names of mathematicians related to some Riordan matrices.

The aim of this note is to describe the matrix exponential, for some infinite lower triangular matrices, from the solution of certain partial differential equations, which we find using the method of characteristics, see [7]. We do all of this within the framework of the Riordan Lie group and the corresponding Lie algebra. Then, motivated by general symmetry properties of dynamical systems [8,14], we describe some of those systems in $\mathbb{K}[[x]]$ that come from the Riordan group with a very special Riordan involution as a symmetry or time-reversal symmetry. We consider a sequence of linear ordinary differential equations in Euclidean spaces as problems approaching certain partial differential equations, using for that the pro-structures of both: the Riordan group and the corresponding Lie algebra. We propose a geometric analysis of these problems and we enumerate symmetric properties of a special sequence of finite matrices related to Pascal's triangle.

Apart from the general description of the Riordan group contained in Section 2, we also need some specific results from [4]. We recall these results in Section 3 and 4 to make this note as self-contained as possible.

Since our first approach to the Riordan group in [11] is different from the usual one in the literature, we have also different notation. This is the reason why we recommend our previous works [11,9,10] and, of course, [4] (ordered chronologically) for information about basic results and notation used herein. We still maintain the notation $\mathbb{K}$ for a field although in this paper we are only considering $\mathbb{K}$ as the real numbers. This is because many of the results and ideas can be translated, at least, to the case $\mathbb{K} = \mathbb{C}$, the field of complex numbers.

2. Basic facts on the differentiable structure of the Riordan group

The results of this section can be found in [4,9–11].

2.1. Riordan matrices and the Riordan group

The definition of *Riordan matrix* and the related concept of *Riordan group* appeared in the foundational paper [16] due to Shapiro, Getu, Woan, and Woodson. The original definition of a Riordan matrix given in [16] is more restrictive than that used currently in the literature and in this paper.

A Riordan matrix is an infinite matrix $D = (d_{i,j})_{i,j\in\mathbb{N}}$ whose columns are the coefficients of successive terms of a geometric progression in $\mathbb{K}[[x]]$ where the initial term is a formal power series of order 0 and the common ratio is a formal power series of order 1 (so D needs to be lower triangular and its diagonal needs to be a geometric progression in $\mathbb{K}$).

We represent a Riordan matrix D by $T(f \mid g)$, where $f(x) = \sum_{k=0}^{\infty} f_k x^k$ and $g(x) = \sum_{k=0}^{\infty} g_k x^k$ are formal power series in $\mathbb{K}[[x]]$ with $f(0) \neq 0$ and $g(0) \neq 0$, so that $d_{i,j} = [x^i]\dfrac{x^j f(x)}{g^{j+1}(x)}$. Consequently, the first term of the geometric progression is $\dfrac{f(x)}{g(x)}$ and the common ratio is $\dfrac{x}{g(x)}$. In these terms, Pascal's triangle is $T(1 \mid 1 - x)$.

The above definition can be reinterpreted saying that the generating function of the j-th column (starting at $j = 0$) of D is the formal power series $\frac{x^j f(x)}{g^{j+1}(x)}$, which makes sense because $g(0) \neq 0$. Hence, D is a lower triangular matrix and it is invertible because $f(0) \neq 0$.

In [16], one of the main results about Riordan matrices was stated. Currently many authors call it the *Fundamental Theorem for Riordan matrices (FTRM). Let $D = T(f \mid g)$ be a Riordan matrix and let $\gamma(x) = \sum_{k=0}^{\infty} \gamma_k x^k$ be a power series in $\mathbb{K}[[x]]$. Consider the column vector $\mathbf{c} = (\gamma_0, \gamma_1, \gamma_2, \ldots)^{\mathrm{T}}$. Then, the generating function of the matrix product $D\mathbf{c}$ is $\frac{f(x)}{g(x)}\gamma(\frac{x}{g(x)})$.* This fact is represented by $T(f \mid g)(\gamma) = \frac{f(x)}{g(x)}\gamma(\frac{x}{g(x)})$. A proof of this result using a special ultrametric space $(\mathbb{K}[[x]], d)$ can be found in [11, Proposition 19].

The Riordan group (i.e., the set of all Riordan matrices with the usual product of matrices), denoted by $\mathcal{R}(\mathbb{K})$ or shortly $\mathcal{R}$, is a subgroup of the group of invertible infinite lower triangular matrices with the usual product of matrices as the operation. The product is given by

$$T(f \mid g)T(l \mid m) = T\left(fl\left(\frac{x}{g}\right) \,\middle|\, gm\left(\frac{x}{g}\right)\right),$$

where $fl\left(\frac{x}{g}\right) \equiv f(x) \cdot l\left(\frac{x}{g(x)}\right)$ and analogously for the second term, and the inverse is given by

$$(T(f \mid g))^{-1} \equiv T^{-1}(f \mid g) = T\left(\frac{1}{f(\frac{x}{A})} \,\middle|\, A\right),$$

where $\left(\frac{x}{A}\right) \circ \left(\frac{x}{g}\right) = \left(\frac{x}{g}\right) \circ \left(\frac{x}{A}\right) = x$. See [11, Proposition 20] for more details.

The sequence of the coefficients of the previous formal power series, denoted by A, is the so-called A-sequence of $T(f \mid g)$. Obviously, the A-sequence of $T(f \mid g)$ depends only on the power series g. Moreover, if $A = \sum_{k \geq 0} a_k x^k$, then

$$d_{i,j} = \sum_{k=0}^{i-j} a_k d_{i-1,j-1+k} \qquad i, j \geq 1.$$

2.2. The Lie and pro-Lie group structures on the Riordan group

Suppose that $\mathbb{K}$ is the field of real or complex numbers, denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. Let us consider a natural way to give a completely metrizable topology in $\mathbb{K}[[x]]$ by means of the identification $\mathbb{K}^{\mathbb{N}} \equiv \mathbb{K}[[x]]$ obtained by passing from sequences to ordinary generating functions and vice versa.

The topology considered in $\mathbb{K}^{\mathbb{N}}$ is always the product topology for the usual topology in $\mathbb{K}$. Therefore, we convert $\mathbb{K}[[x]]$ into a Fréchet space, that is, a completely metrizable locally convex linear topological space.

This is the starting point to describe a natural Fréchet–Lie group structure on the Riordan group. Beside this, the Riordan group can be described as the inverse limit of an inverse sequence of groups of finite matrices obtaining a pro-Lie group structure on the Riordan group.

It is well known that any Riordan matrix is completely determined by its first column and its A-sequence. In this way, any Riordan matrix $D = (d_{i,j})_{i,j \in \mathbb{N}}$ is defined by a sequence $\mathbf{u} = (u_k)_{k \in \mathbb{N}}$ with $u_0 \neq 0$, $u_1 \neq 0$, and $u_{2k} = x_k$, $u_{2k+1} = a_k$, being $x_k = d_{k,0}$, $A(x) = \sum_{n \geq 0} a_n x^n$ and $d_{i,j} = \sum_{k=0}^{i-j} a_k d_{i-1,j-1+k}$ for $j \geq 1$. We denote the matrix D described above by $\varphi_\infty(\mathbf{u})$.

Let us consider $\mathbb{K}$ with the usual Euclidean topology, the product topology in $\mathbb{K}^{\mathbb{N}}$ and the basic open set

$$\mathcal{U}_\infty = \left\{\mathbf{u} = (u_k)_{k \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}} \;\middle|\; u_0 \neq 0,\ u_1 \neq 0\right\}$$

in $\mathbb{K}^{\mathbb{N}}$. Set

$$\begin{array}{rcccl} \varphi_\infty & : & \mathcal{U}_\infty & \longrightarrow & \mathcal{R}(\mathbb{K}), \\ & & \mathbf{u} & \longmapsto & \varphi_\infty(\mathbf{u}). \end{array}$$

Obviously, φ_∞ is a bijective function. So, we consider the unique topology on $\mathcal{R}(\mathbb{K})$ that makes φ_∞ a homeomorphism. Note that the topological space $\mathcal{R}(\mathbb{K})$, the locally convex vector space $\mathbb{K}^{\mathbb{N}}$ and the map $\varphi_\infty : \mathcal{U}_\infty \to \mathcal{R}(\mathbb{K})$ fit all conditions to get:

Theorem 1. $(\mathcal{R}(\mathbb{K}), (\mathcal{U}_\infty, \varphi_\infty))$ *is a smooth manifold modelled on the locally convex vector space* $\mathbb{K}^{\mathbb{N}}$. *Moreover,* $\mathcal{R}(\mathbb{K})$ *with this smooth structure is a Lie group.*

One of the main tools that we have used to get results on the Riordan group is to consider it as the inverse limit of an inverse sequence of groups of finite matrices. A natural way to do this is as follows. For every $n \in \mathbb{N}$, consider the general linear group $\mathrm{GL}(n+1, \mathbb{K})$ formed by all $(n+1) \times (n+1)$ invertible matrices with coefficients in $\mathbb{K}$. Since every Riordan matrix is lower triangular, we can define a natural homomorphism $\Pi_n : \mathcal{R}(\mathbb{K}) \to \mathrm{GL}(n+1, \mathbb{K})$ given by

$$\Pi_n((d_{i,j})_{i,j \in \mathbb{N}}) = (d_{i,j})_{i,j=0,\dots,n}.$$

For obvious reasons, we will refer to this homomorphism as the projection of the corresponding Riordan matrix.

To describe the Riordan group as an inverse limit of an inverse sequence of groups of finite matrices we use results of [10]. We first consider the subgroup of $\mathrm{GL}(n+1, \mathbb{K})$ defined by $\mathcal{R}_n = \Pi_n(\mathcal{R})$.

Definition 2. Let $D = (d_{i,j})_{i,j=0,\dots,n+1} \in \mathcal{R}_{n+1}$. We define $P_n : \mathcal{R}_{n+1} \to \mathcal{R}_n$ by

$$P_n((d_{i,j})_{i,j=0,1,\dots,n+1}) = (d_{i,j})_{i,j=0,\dots,n}.$$

$P_n(D)$ is obtained from D by deleting its last row and its last column. P_n is a group homomorphism for every n because the matrices are lower triangular. Moreover, the following diagram is commutative:

$$\begin{tikzcd} \mathcal{R} \arrow[d, "\Pi_n"'] \arrow[dr, "\Pi_{n+1}"] & \\ \mathcal{R}_n & \mathcal{R}_{n+1}. \arrow[l, "P_n"] \end{tikzcd}$$

From this, we get:

Theorem 3. *The Riordan group* $\mathcal{R}$ *is isomorphic to* $\varprojlim\{(\mathcal{R}_n)_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}}\}$. *Consequently,* $\mathcal{R}$ *is a pro-Lie group.*

See [10] for more details and notation.

2.3. The Lie algebra of the Riordan group

Again, using results of [4], we obtain a full and faithful representation of the Lie algebra $\mathcal{L}(\mathcal{R}(\mathbb{K}))$ of the Lie group $\mathcal{R}(\mathbb{K})$. We have [4, Theorem 16]:

Theorem 4.

(1) $L = (\ell_{i,j})_{i,j\in\mathbb{N}} \in \mathcal{L}(\mathcal{R}(\mathbb{K}))$ *if and only if* L *is lower triangular and there are two sequences* $(\chi_i)_{i\in\mathbb{N}}$ *and* $(\alpha_i)_{i\in\mathbb{N}}$ *such that*

$$L = \begin{pmatrix} \chi_0 & & & & & & \\ \chi_1 & \chi_0+\alpha_0 & & & & & \\ \chi_2 & \chi_1+\alpha_1 & \chi_0+2\alpha_0 & & & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ \chi_{n-1} & \chi_{n-2}+\alpha_{n-2} & \chi_{n-3}+2\alpha_{n-3} & \cdots & \chi_0+(n-1)\alpha_0 & & \\ \chi_n & \chi_{n-1}+\alpha_{n-1} & \chi_{n-2}+2\alpha_{n-2} & \cdots & \chi_1+(n-1)\alpha_1 & \chi_0+n\alpha_0 & \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

After denoting the above matrix L *by* $L(\chi,\alpha)$, *where* $\chi(x) = \sum_{n\geq 0}\chi_n x^n$, $\alpha(x) = \sum_{n\geq 0}\alpha_n x^n$, *we have*

$$\mathcal{L}(\mathcal{R}(\mathbb{K})) = \{L(\chi,\alpha) \mid \chi,\alpha \in \mathbb{K}[[x]]\}.$$

(2) *If* $L \in \mathcal{L}(\mathcal{R}(\mathbb{K}))$ *and* $t \in \mathbb{K}$, *then* $e^{tL} \in \mathcal{R}(\mathbb{K})$.

Note that the j^{th} column of $L(\chi,\alpha)$ has the generating function $x^j(\chi + j\cdot\alpha)$. Thus, the multiplication by $L(\chi,\alpha)$ leads to the following result:

Proposition 5. *Any* $L = L(\chi,\alpha) \in \mathcal{L}(\mathcal{R}(\mathbb{K}))$ *induces a linear continuous map, denoted again by* $L\colon \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$, *and given by* $L(h) = \chi(x)h(x) + x\alpha(x)h'(x)$, *where* $h(x) = \sum_{n\geq 0} h_n x^n$.

The continuous linear map $L(\chi,\alpha)$ can be viewed as a C^∞ vector field in the Fréchet space $\mathbb{K}^{\mathbb{N}}$ under the canonical identification $T_h\mathbb{K}^{\mathbb{N}} = \mathbb{K}^{\mathbb{N}}$ in the tangent space at any h. From this point of view, we have the following proposition [4, Proposition 4].

Proposition 6. *The initial value problem*

$$\begin{aligned} &\gamma'(t) = L(\chi,\alpha)(\gamma(t)),\\ &\gamma(0) = h \end{aligned}$$

in $\mathbb{K}^{\mathbb{N}}$ *has a unique solution given by*

$$\gamma(t) = e^{tL(\chi,\alpha)}(h).$$

Consequenly, there is a one-parameter group of Riordan matrices $T(f(x,t) \mid g(x,t)) = e^{tL(\chi,\alpha)}$ *such that*

$$\gamma(t) = T(f(x,t) \mid g(x,t))(h) = \frac{f(x,t)}{g(x,t)} h\left(\frac{x}{g(x,t)}\right).$$

In particular, $\{T(f(x,t) \mid g(x,t))\}_{t\in\mathbb{R}}$ *is an abelian subgroup of the Riordan group because* $e^{t_1L}e^{t_2L} = e^{t_2L}e^{t_1L}$. *It defines a continuous dynamical system (or flow) in* $\mathbb{K}[[x]]$.

Due to the special patterns followed by the matrices in the Lie algebra and those in the Riordan group, we named the corresponding section in [4] as *Arithmetic vector fields and geometric flows.*

2.4. The partial differential equation induced by an element in the Lie algebra

For any $t \in \mathbb{R}$ and $L \in \mathcal{L}(\mathcal{R}(\mathbb{K}))$, we have a well-defined Riordan matrix e^{tL}. Note that if γ is the solution of the initial value problem given in Proposition 6, then $\gamma(t) \in \mathbb{K}[[x]]$ for any $t \in \mathbb{R}$. With this in mind, another way to interpret the above proposition is as follows [4, Corollary 24]:

Corollary 7. *Let $\chi, \alpha \in \mathbb{K}[[x]]$. Then the unique solution of the initial value problem*

$$\begin{aligned} &\tfrac{\partial u}{\partial t} = \chi(x)u(x,t) + x\alpha(x)\tfrac{\partial u}{\partial x}, \\ &u(x,0) = h(x) \end{aligned}$$

in $\mathbb{K}[[x,t]]$ is given by

$$u(x,t) = e^{tL(\chi,\alpha)}(h(x)) = \frac{f(x,t)}{g(x,t)} h\left(\frac{x}{g(x,t)}\right).$$

3. From the matrix exponential e^{tL} to the solutions of the PDE $\frac{\partial u}{\partial t} = \chi(x)u(x,t) + x\alpha(x)\frac{\partial u}{\partial x}$ and back

Having recalled some basic results from [4], we propose the following *strategy*.

Given any element $L = L(\chi, \alpha)$ in the Lie algebra of the Riordan group, we associate with it the partial differential equation $\frac{\partial u}{\partial t} = \chi(x)u(x,t) + x\alpha(x)\frac{\partial u}{\partial x}$. At this point, we run into the following *dichotomy*.

(a) If we are able to compute the one-parameter group e^{tL} and to recognize $e^{tL} = T(f(x,t) \mid g(x,t))$ as a Riordan matrix for any $t \in \mathbb{R}$, then this allows us to solve in $\mathbb{K}[[x,t]]$ the initial value problem

$$\begin{aligned} &\tfrac{\partial u}{\partial t} = \chi(x)u(x,t) + x\alpha(x)\tfrac{\partial u}{\partial x}, \\ &u(x,0) = h(x) \end{aligned}$$

for any $h \in \mathbb{K}[[x]]$ because the solution is given by

$$u(x,t) = e^{tL(\chi(x),\alpha(x))}(h(x)) = \frac{f(x,t)}{g(x,t)} h\left(\frac{x}{g(x,t)}\right).$$

(b) If, on the contrary, we are able to solve, in $\mathbb{K}[[x,t]]$, the initial value problem

$$\begin{aligned} &\tfrac{\partial u}{\partial t} = \chi(x)u(x,t) + x\alpha(x)\tfrac{\partial u}{\partial x}, \\ &u(x,0) = h(x) \end{aligned}$$

for any $h \in \mathbb{K}[[x]]$, then we get the one-parameter subgroup $e^{tL(\chi(x),\alpha(x))} = T(f(x,t) \mid g(x,t))$. We can compute both parameters $f(x,t)$ and $g(x,t)$ because $\frac{f(x,t)}{g(x,t)}$ is the unique solution for the initial condition $h(x) \equiv 1$ and $\frac{f(x,t)}{g(x,t)}\frac{x}{g(x,t)}$ is the unique solution for the initial condition $h(x) = x$. Evaluating now at $t = 1$, we have the corresponding matrix exponential.

3.1. From the matrix exponential to the solution of the PDE

Example 8.

(i) Consider the matrix

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ 0 & 0 & 0 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Obviously, $D = L(1,1)$ is in $\mathcal{L}(\mathcal{R}(\mathbb{K}))$. The partial differential equation associated with the matrix D is

$$\frac{\partial u}{\partial t} = u(x,t) + x\frac{\partial u}{\partial x}.$$

The solution of the corresponding initial value problem

$$\begin{aligned} &\tfrac{\partial u}{\partial t} = u(x,t) + x\tfrac{\partial u}{\partial x}, \\ &u(x,0) = h(x) \end{aligned}$$

is given by $u(x,t) = e^{tD}(h)$. It is clear, by definition, that

$$e^{tD} = \begin{pmatrix} e^t & 0 & 0 & 0 & \cdots \\ 0 & e^{2t} & 0 & 0 & \cdots \\ 0 & 0 & e^{3t} & 0 & \cdots \\ 0 & 0 & 0 & e^{4t} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For any $t \in \mathbb{R}$, we have that $e^{tD} = T(1 \mid e^{-t})$ as a Riordan matrix. Therefore, the solution is $u(x,t) = T(1 \mid e^{-t})(h) = e^t h(xe^t)$.

(ii) As a more interesting example, we consider the matrix [4, Example 26]

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that $H = L(x,x)$ and so $H \in \mathcal{L}(\mathcal{R}(\mathbb{K}))$. The PDE induced by H is

$$\frac{\partial u}{\partial t} - x^2\frac{\partial u}{\partial x} = xu(x,t).$$

The solution of the corresponding initial value problem

$$\begin{aligned} &\tfrac{\partial u}{\partial t} = xu(x,t) + x^2\tfrac{\partial u}{\partial x}, \\ &u(x,0) = h(x) \end{aligned}$$

is given by $u(x,t) = e^{tH}(h)$. To compute this solution, we are going to take advantage of some previous works of other authors. Particularly, we are going to follow [1], where the matrix H is called the *creation matrix*. Formula (9) in [1, p. 233] computes $e^{t\Lambda_{n-1}(H)}$ where

$$\Lambda_{n-1}(L) = \begin{pmatrix} \chi_0 & & & & \\ \chi_1 & \chi_0 + \alpha_0 & & & \\ \chi_2 & \chi_1 + \alpha_1 & \chi_0 + 2\alpha_0 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \chi_{n-1} & \chi_{n-2} + \alpha_{n-2} & \chi_{n-3} + 2\alpha_{n-3} & \dots & \chi_0 + (n-1)\alpha_0 \end{pmatrix}$$

and $L = L(\chi, \alpha) \in \mathcal{L}(\mathcal{R}(\mathbb{R}))$. Moreover, it is obvious that $P_{n-1}(e^{t\Lambda_n(H)}) = e^{t\Lambda_{n-1}(H)}$ for any $t \in \mathbb{R}$ and any integer $n \geq 2$. Hence, in the Riordan matrix notation, we get $e^{tH} = T(1 \mid 1 - xt)$ and the solution of the corresponding initial value problem is given by

$$u(x,t) = e^{tH}(h) = \frac{1}{1-xt} h\left(\frac{x}{1-xt}\right).$$

Evaluating at $t = 1$, we obtain $e^H = T(1 \mid 1 - x)$, which is *Pascal's triangle*.

3.2. From the solution of a PDE to the matrix exponential: the method of characteristics

To point out how to go from the solution of a PDE to the exponential map of the Lie algebra to the Riordan group, we are going to deal with the following family of significant examples.

For any couple of real numbers a and b and any nonnegative integer number n, consider the infinite lower triangular matrix $L_n^{a,b} = (d_{i,j}^{(n)})_{i,j\in\mathbb{N}}$, where $d_{i,j}^{(n)} = 0$ if $i - j \neq n$ and $d_{i,j}^{(n)} = a + jb$ if $i - j = n$. How can we compute or describe the exponential of each of the matrices $L_n^{a,b}$?

Note that $L_n^{a,b} = L(ax^n, bx^n)$. This means that these matrices belong to the Lie algebra of the Riordan group. Moreover,

$$L_0^{a,b} = \begin{pmatrix} a & 0 & 0 & 0 & \cdots \\ 0 & a+b & 0 & 0 & \cdots \\ 0 & 0 & a+2b & 0 & \cdots \\ 0 & 0 & 0 & a+3b & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$L_1^{a,b} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ a & 0 & 0 & 0 & \cdots \\ 0 & a+b & 0 & 0 & \cdots \\ 0 & 0 & a+2b & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and so on. Observe that $D = L_0^{1,1}$ and $H = L_1^{1,1}$, where D and H are the matrices considered in the previous examples. The PDE induced by $L_n^{a,b}$ is

$$\frac{\partial u}{\partial t} - bx^{n+1}\frac{\partial u}{\partial x} = ax^n u(x,t).$$

Note that if $b = 0$, then $e^{tL_n^{a,0}} = T(e^{atx^n} \mid 1)$ is a Toeplitz matrix (in the Riordan group) for any $t \in \mathbb{R}$. Consequently, the solution of the corresponding initial value problem

$$\begin{aligned}&\tfrac{\partial u}{\partial t} = ax^n u(x,t),\\&u(x,0) = h(x)\end{aligned}$$

is given by $u(x,t) = e^{tL_n^{a,0}}(h) = e^{atx^n}h(x)$.

From now on, assume $b \neq 0$. Consider the matrix $L_n^{a,b}$ and the corresponding initial value problem

$$\begin{aligned}&\tfrac{\partial u}{\partial t} - bx^{n+1}\tfrac{\partial u}{\partial x} = ax^n u(x,t),\\&u(x,0) = h(x).\end{aligned} \tag{1}$$

Let us use the method of characteristics to solve it, see [7, Chapter I]. We briefly recall it. Given the first-order PDE

$$a(x,t)\frac{\partial u}{\partial x} + b(x,t)\frac{\partial u}{\partial t} = c(x,t,u)$$

with initial condition $u(x,0) = h(x)$, then we set $t = t(r,s)$, $x = x(r,s)$ and $u = u(r,s)$ and we solve the system of ordinary ODEs

$$\begin{aligned}&\tfrac{dt}{ds} = a(x,t),\\&\tfrac{dx}{ds} = b(x,t),\\&\tfrac{du}{ds} = c(x,t,u)\end{aligned}$$

with initial conditions

$$\begin{aligned}&t(r,0) = 0,\\&x(r,0) = r,\\&u(r,0) = h(r),\end{aligned}$$

to get an explicit description of u. Note that $(\frac{\partial u}{\partial x}(x,t), \frac{\partial u}{\partial t}(x,t), -1)$ is a normal vector to the surface $u = u(x,t)$ and $(a(x,t), b(x,t), c(x,t,u))$ is a vector field tangent to $u = u(x,t)$. Hence, u can be obtained computing the integral curves of the previous vector field.

So in the current situation, we consider the following system of ODEs:

$$\begin{aligned}&\tfrac{dt}{ds} = 1,\\&\tfrac{dx}{ds} = -bx^{n+1},\\&\tfrac{du}{ds} = ax^n u\end{aligned}$$

with initial conditions

$$\begin{aligned}&t(r,0) = 0,\\&x(r,0) = r,\\&u(r,0) = h(r).\end{aligned}$$

Integrating the system and imposing the initial conditions, we get

$$\begin{aligned}&t = s,\\&\tfrac{1}{x^n} = \tfrac{1+nbsr^n}{r^n},\\&u = (1+nbsr^n)^{\frac{a}{nb}}h(r)\end{aligned}$$

and

$$\begin{aligned} s &= t, \\ r &= \tfrac{x}{\sqrt[n]{1-nbtx^n}}, \\ u &= \left(\tfrac{1}{1-bntx^n}\right)^{\frac{a}{nb}} h\left(\tfrac{x}{\sqrt[n]{1-nbtx^n}}\right), \end{aligned}$$

which yields that the solution of (1) is given by

$$u(x,t) = \left(\frac{1}{1-bntx^n}\right)^{\frac{a}{nb}} h\left(\frac{x}{\sqrt[n]{1-nbtx^n}}\right)$$

or

$$u(x,t) = \sqrt[n]{\left(\frac{1}{1-bntx^n}\right)^{\frac{a}{b}}}\, h\left(\frac{x}{\sqrt[n]{1-nbtx^n}}\right).$$

All above in this subsection may be summarized as:

Theorem 9. *Suppose that a and b are two real numbers and that n is a nonnegative integer number. Consider the infinite lower triangular matrix $L_n^{a,b} = (d_{i,j}^{(n)})_{i,j\in\mathbb{N}}$, where $d_{i,j}^{(n)} = 0$ if $i-j \neq n$ and $d_{i,j}^{(n)} = a + jb$ if $i-j = n$. Then, $L_n^{a,b}$ is in the Lie algebra of the Riordan group and for any $t \in \mathbb{R}$ the Riordan matrix $e^{tL_n^{a,b}}$ is given by*

$$e^{tL_n^{a,b}} = \begin{cases} T\left(\sqrt[n]{(1-bntx^n)^{\frac{b-a}{b}}} \mid \sqrt[n]{1-bntx^n}\right) & \text{if } b \neq 0, \\ T\left(e^{atx^n} \mid 1\right) & \text{if } b=0. \end{cases}$$

4. A special involution as symmetry or time-reversal symmetry for flows in $\mathbb{K}[[x]]$ related to the Riordan group

There is an involution M in the Riordan group playing a special role in such group. This matrix is

$$M = T(-1|-1) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Considered as an endomorphism in $\mathbb{K}[[x]]$, M is continuous and has exactly two eigenvalues 1 and -1. The corresponding eigenspaces are $E(1) = \{h \in \mathbb{K}[x]]/h(x) = h(-x)\}$ and $E(-1) = \{h \in \mathbb{K}[x]]/-h(x) = h(-x)\}$, i.e., they are the linear subspaces of even, respectively odd, formal power series. Both of them are closed subspaces in $\mathbb{K}[[x]]$ and we get $\mathbb{K}[[x]] = E(1) \oplus E(-1)$ and $(M-I)\circ(M+I) = (M+I)\circ(M-I) \equiv 0$, where $I = T(1 \mid 1)$ is the identity.

The reason of the interest about M is that it is used for defining what is known as a pseudo-involution in the Riordan group. Following [3], we say that a Riordan matrix R is a *pseudo-involution* if the product RM is an involution, that is, $RMRM = I$.

In group theory, we have the definitions of *reversible* and *strongly reversible* elements (see [14]). We recall it here for completeness.

Definition 10.

(i) An element g of a group G is said to be reversible in G if there is another element h of G such that

$$hgh^{-1} = g^{-1}.$$

In this situation, we also say that h reverses g and h is a reverser of g.

(ii) An element g of a group G is said to be strongly reversible in G if there is an involution h of G such that

$$hgh^{-1} = g^{-1}.$$

Proposition 11. *If R is a pseudo-involution, then R is strongly reversible in the Riordan group and is the product of two involutions.*

Proof. Since $RMRM = I$, we directly obtain, multiplying on the left by the inverse R^{-1}, that $MRM = R^{-1}$, i.e., R is strongly reversible. Moreover, $R^{-1}M$ is an involution because $(R^{-1}M)^{-1} = MR = MMR^{-1}M = R^{-1}M$. Consequently, we have that $R = (M)(R^{-1}M)$ is a product of two involutions. □

The above proposition tells us that the pseudo-involutions are particular examples of *strongly reversible* elements in the Riordan group and M is a reverser for any of them.

Given $S \in \mathcal{R}(\mathbb{K})$, consider the left and right translations L_S and R_S in the Lie group $\mathcal{R}(\mathbb{K})$ given by

$$\begin{array}{rcccc} L_S & : & \mathcal{R}(\mathbb{K}) & \longrightarrow & \mathcal{R}(\mathbb{K}), \\ & & X & \longmapsto & L_S(X) = SX, \end{array} \qquad \begin{array}{rcccc} R_S & : & \mathcal{R}(\mathbb{K}) & \longrightarrow & \mathcal{R}(\mathbb{K}), \\ & & X & \longmapsto & R_S(X) = XS. \end{array}$$

Since the product is a C^∞-function, both L_S and R_S are diffeomorphisms. We need to recall the following facts from [4].

Proposition 12. *Let $T(f \mid g)$ be a Riordan matrix. The tangent space $T_{T(f|g)}\mathcal{R}(\mathbb{K})$ to $\mathcal{R}(\mathbb{K})$ at $T(f \mid g)$ is given by*

$$T_{T(f|g)}\mathcal{R}(\mathbb{K}) = \{T(f \mid g)L(\chi,\alpha)/\chi,\alpha \in \mathbb{K}[[x]]\} = \{L(\chi,\alpha)T(f \mid g)/\chi,\alpha \in \mathbb{K}[[x]]\}.$$

Moreover, the conjugation by $T(f \mid g)$ defined by

$$\begin{array}{rcccc} \mathrm{conj}_{T(f|g)} & : & \mathcal{R}(\mathbb{K}) & \longrightarrow & \mathcal{R}(\mathbb{K}), \\ & & X & \longmapsto & T(f \mid g)XT^{-1}(f \mid g) \end{array}$$

is a C^∞-diffeomorphism and its tangent (or differential) map at the identity

$$(D\mathrm{conj}_{T(f|g)})_I \quad : \quad \mathcal{L}(\mathcal{R}(\mathbb{K})) \quad \longrightarrow \quad \mathcal{L}(\mathcal{R}(\mathbb{K})),$$

is given by

$$(D\mathrm{conj}_{T(f|g)})_I(L(\chi,\alpha)) = T(f \mid g)L(\chi,\alpha)T^{-1}(f \mid g).$$

Finally, for any $t \in \mathbb{K}$ we have

$$e^{t(D\mathrm{conj}_{T(f|g)})_I(I)(L(\chi,\alpha))} = \mathrm{conj}_{T(f|g)}(e^{tL(\chi,\alpha)}).$$

From this, we deduce:

Corollary 13. *Given $T(f \mid g) \in \mathcal{R}(\mathbb{K})$ and $L(\chi, \alpha) \in \mathcal{L}(\mathcal{R}(\mathbb{K}))$, there exists a unique $L(\tilde{\chi}, \tilde{\alpha}) \in \mathcal{L}(\mathcal{R}(\mathbb{K}))$ such that*

$$T(f \mid g)L(\chi, \alpha) = L(\tilde{\chi}, \tilde{\alpha})T(f \mid g)$$

or, equivalently,

$$(D\mathrm{conj}_{T(f|g)})_I(L(\chi, \alpha)) = L(\tilde{\chi}, \tilde{\alpha}).$$

Moreover,

$$\tilde{\chi} = \frac{\chi\left(\frac{x}{g}\right)(g - xg')f - x\alpha\left(\frac{x}{g}\right)(f'g - g'f)}{f(g - xg')}, \qquad \tilde{\alpha} = \frac{g\alpha\left(\frac{x}{g}\right)}{g - xg'}. \tag{2}$$

As a consequence, we obtain:

Proposition 14. *Consider the involution M and the C^∞-diffeomorphism*

$$\begin{array}{rcccc} \mathrm{conj}_M & : & \mathcal{R}(\mathbb{K}) & \longrightarrow & \mathcal{R}(\mathbb{K}), \\ & & X & \longmapsto & MXM. \end{array}$$

Then

(i) *the tangent (or differential) map at identity*

$$(D\mathrm{conj}_M)_I \quad : \quad \mathcal{L}(\mathcal{R}(\mathbb{K})) \quad \longrightarrow \quad \mathcal{L}(\mathcal{R}(\mathbb{K}))$$

is a linear involution;

(ii) *for any couple of nonzero real numbers a and b the matrix $L_n^{a,b}$ is an eigenvector of $(D\mathrm{conj}_M)_I$ corresponding to the eigenvalue -1 if n is odd and an eigenvector of $(D\mathrm{conj}_M)_I$ corresponding to the eigenvalue 1 if n is even (including $n = 0$).*

Proof. It is clear that $(D\mathrm{conj}_M)_I \circ (D\mathrm{conj}_M)_I = I_{\mathcal{L}(\mathcal{R}(\mathbb{K}))}$ because M is an involution in $\mathcal{R}(\mathbb{K})$, where $I_{\mathcal{L}(\mathcal{R}(\mathbb{K}))}$ represents the identity map in $\mathcal{L}(\mathcal{R}(\mathbb{K}))$. To prove (ii), recall that $M = T(-1 \mid -1)$ and that $L_n^{a,b} = L(ax^n, bx^n)$. From the previous corollary we get

$$(D\mathrm{conj}_M)_I(L(\chi, \alpha)) = L(\tilde{\chi}, \tilde{\alpha})$$

where $\tilde{\chi}$ and $\tilde{\alpha}$ are given by (2). In this case, $f = g = -1$ and $\chi(x) = ax^n$, $\alpha(x) = bx^n$. For the case that n is even, the announced result is obvious. The same also holds for the case n is odd because $-L(\chi, \alpha) = L(-\chi, -\alpha)$ for any χ, α in $\mathbb{K}[[x]]$. □

Finally, we obtain the following (here we use [8] for some of the definitions appearing below).

Theorem 15. *Let a and b two real numbers such that $a = b = 0$ does not hold. Suppose also that n is a nonnegative integer number. Consider the infinite lower triangular matrix $L_n^{a,b}$. Then we have the following:*

(i) *If n is even, the one-parameter subgroup $\{e^{tL_n^{a,b}}\}_{t\in\mathbb{R}}$ of $\mathcal{R}(\mathbb{R})$ is contained in the centralizer of the involution M. In other words, M is a symmetry of the flow $\{e^{tL_n^{a,b}}\}_{t\in\mathbb{R}}$ in $\mathbb{K}[[x]]$.*

(ii) *If n is odd, any element in the one-parameter subgroup $\{e^{tL_n^{a,b}}\}_{t\in\mathbb{R}}$ of $\mathcal{R}(\mathbb{R})$ is a pseudo-involution and the involution M is a time-reversal symmetry of the flow $\{e^{tL_n^{a,b}}\}_{t\in\mathbb{R}}$ in $\mathbb{K}[[x]]$.*

Proof. Suppose n is even. Using Proposition 14, we have $(D\mathrm{conj}_M)_I(L_n^{a,b}) = L_n^{a,b}$. Hence

$$e^{tL_n^{a,b}} = e^{t(D\mathrm{conj}_M)_I(L_n^{a,b})} = Me^{tL_n^{a,b}}M.$$

The last equality above is a consequence of Proposition 12. Consequently, $e^{tL_n^{a,b}}$ is in the centralizer of the involution M for every $t \in \mathbb{R}$.

Suppose now that n is odd. By analogous arguments, we first have that $(D\mathrm{conj}_M)_I(L_n^{a,b}) = -L_n^{a,b}$ and then

$$e^{-tL_n^{a,b}} = e^{t(D\mathrm{conj}_M)_I(L_n^{a,b})} = Me^{tL_n^{a,b}}M.$$

Therefore, the Riordan matrix $e^{tL_n^{a,b}}$ is reversible for any $t \in \mathbb{R}$ and the involution M is a reverser for all of them. This fact implies that $e^{tL_n^{a,b}}$ is strongly reversible and that $e^{tL_n^{a,b}}$ is a pseudo-involution because $e^{tL_n^{a,b}}M$ is an involution ($e^{tL_n^{a,b}}Me^{tL_n^{a,b}}M = e^{tL_n^{a,b}}e^{-tL_n^{a,b}} = I$). $\square$

Corollary 16. *The same as in the above theorem is true, word by word, changing the involution M by the involution $-M = T(1 \mid -1)$.*

5. On some dynamical properties of Pascal's triangle

Using some of the results obtained herein we can get some new information about the Riordan group. For example, in [4] we recognized the *substitution group of formal power series* as a subgroup of the Riordan group. This group was introduced in [6], see also [2] for a good survey about it, where results about topological generation are established. One can notice at once that some of the one-parameter groups $\{e^{tL_n^{a,b}}\}_{t\in\mathbb{R}}$ are involved in [6, Section 3]. In this section, we focus only on Pascal's triangle and assign to it others of the many properties related to patterns and symmetries that it has.

We will use the pro-Lie group structure of the Riordan group. So, we will consider elements of the Riordan group and of the Lie algebra as approximated by the components of the corresponding points in the inverse limit interpretation of both $\mathcal{R}(\mathbb{K})$ and $\mathcal{L}(\mathcal{R}(\mathbb{K}))$. See [4] for the pro-structure of $\mathcal{L}(\mathcal{R}(\mathbb{K}))$.

Let $L = L(\chi(x), \alpha(x))$. Recall that, related to any problem

$$\gamma'(t) = L(\gamma(t)) \tag{3}$$

or, equivalently,

$$\frac{\partial u}{\partial t} = \chi(x)u(x,t) + x\alpha(x)\frac{\partial u}{\partial x},$$

we have a sequence of finite-dimensional problems, denoted by $\{(3)_n\}_{n\in\mathbb{N}}$. We call this sequence as the sequence of *approaching* problems of (3).

Problem $(3)_n$: Approaching problems. *Consider the Euclidean space $\mathbb{R}^{n+1}$. For any $x \in \mathbb{R}^{n+1}$, let $x = (x_0, x_1, \ldots, x_n)$ with its usual components. Suppose x^{T} represents the transpose matrix of x. Let $x\colon \mathbb{R} \to \mathbb{R}^{n+1}$ be any differentiable curve given by $x(t) = (x_0(t), x_1(t), \ldots, x_n(t))$. We denote by $x'(t) = (x_0'(t), x_1'(t), \ldots, x_n'(t))$ the derivative of x at t, where x_i' is the usual derivative of a real function of a real argument for any $i = 0, \ldots, n$. When we refer to the approaching problem $(3)_n$, we refer to the linear differential equation*

$$x'^{\mathrm{T}} = (D\Pi_n)_I(L)x^{\mathrm{T}}.$$

In the above equation, $(D\Pi_n)_I$ *represents the differential of the projection at the identity* I *in the Riordan group* $\mathcal{R}(\mathbb{K})$

$$\Pi_n \colon \mathcal{R}(\mathbb{K}) \longrightarrow \mathcal{R}_n(\mathbb{K})$$

in the pro-Lie group structure of $\mathcal{R}(\mathbb{K})$*, see again [4] if needed.*

5.1. The approaching problems related to Pascal's triangle

Recall that (Example 8, (ii)) Pascal's triangle is the time 1 map of the dynamical system generated by the linear differential equation in $\mathbb{K}[[x]]$

$$\gamma'(t) = L_1^{1,1}(\gamma(t)), \tag{4}$$

where $L_1^{1,1} = L(x,x)$, or, equivalently,

$$\frac{\partial u}{\partial t} = xu + x^2\frac{\partial u}{\partial x}$$

in $\mathbb{K}[[x,t]]$. Recall also that $\gamma(t) \in \mathbb{R}[[x]]$ for any $t \in \mathbb{R}$ and that $L_1^{1,1}(\gamma(t)) = x\gamma(t) + x^2\frac{d\gamma(t)}{dx}$, where $\frac{d}{dx}$ is the formal derivative in $\mathbb{R}[[x]]$. While $\gamma \colon \mathbb{R} \longrightarrow \mathbb{R}[[x]]$ is a curve and $\gamma'(t)$ is the usual derivative in t, when we consider $\mathbb{R}[[x]]$ identified with $\mathbb{R}^{\mathbb{N}}$ with the product topology. Using the pro-Lie group structure in $\mathcal{R}(\mathbb{K})$, the corresponding pro-Lie algebra structure in $\mathcal{L}(\mathcal{R}(\mathbb{K}))$ and recalling that

$$L_1^{1,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

we can associate to problem (4) a countably infinite family of finite-dimensional problems $(4)_n$ in the Euclidean space $\mathbb{R}^{n+1}$ for any nonnegative integer n. As we said before, we will interpret the family of problems $\{(4)_n\}_{n\in\mathbb{N}}$ as *approaching* the problem (4) when n tends to ∞. For the first few values of n we have:

$(4)_0$ for $n = 0$, we consider the differential equation $x_0' = 0$ in the one-dimensional Euclidean space $\mathbb{R}$,
$(4)_1$ for $n = 1$, we consider the differential equation in $\mathbb{R}^2$

$$\begin{pmatrix} x_0'(t) \\ x_1'(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_0(t) \\ x_1(t) \end{pmatrix},$$

$(4)_2$ for $n = 2$, we consider the differential equation in $\mathbb{R}^3$

$$\begin{pmatrix} x_0'(t) \\ x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \end{pmatrix},$$

$(4)_3$ for $n=3$, we consider the differential equation in $\mathbb{R}^4$

$$\begin{pmatrix} x_0'(t) \\ x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix},$$

and so on.

The problem $(4)_0$ is very easy to analyze. Any point in the phase space $\mathbb{R}$ is an equilibrium point. The phase flow is trivial and nothing is moving under it.

Let us now consider $(4)_1$. The equilibrium points in this case are just the points in the x_1-axis, which means that they are of the form $(0,b)$, where $b \in \mathbb{R}$. All of them are unstable in the Lyapunov sense. The rest of the orbits are the straight lines $x_0 = a$ for a fix nonzero $a \in \mathbb{R}$. Through any orbit the motion is uniform. In the semiplane $x_0 > 0$ the sense of the motion is increasing, respect to the x_1-axis as t increases, i.e., the particle comes from the $-\infty$ part respect to the x_1-axis and goes to (positive) ∞ of such axis as t increases. In the semiplane $x_0 < 0$ the sense of the motion is the opposite one. The constant speed of the motion in the orbit $x_0 = a$ is $|a|$, the absolute value of a. Particles move quickly for large values of $|a|$ and slowly for small values $|a|$ and they do not move in the x_1-axis. Therefore, there seems to be something in the x_1-axis slowing down the motion. We can deduce all above only knowing that the corresponding velocity vector field for the equation $(4)_1$, in the Euclidean plane, is given by $X(a,b) = (0,a)$. We can also compute, quickly and easily the solution of problem $(4)_1$ with initial condition $x_0 = a$ and $x_1 = b$ because the corresponding matrix is nilpotent. It is the curve $x(t) = (a, at+b)$.

From Theorem 15 and Corollary 16, we get that $\Pi_1(M) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\Pi_1(-M) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ are time-reversal symmetries for the flow generated by the equation $(4)_1$. Consider the orbit $\theta(a,b) = \{(a, at+b), a \neq 0, b, t \in \mathbb{R}\}$ of $(4)_1$. Then it is invariant with respect to the involution $\Pi_1(M)$. This means that if we transform the orbit $\theta(a,b)$ by $\Pi_1(M)$, we get again $\theta(a,b)$ but the parametrization obtained by means of t is not a solution. However, if we finally change t by $-t$ in the obtained curve, we have a solution of $(4)_1$. In this case, the initial value at $t=0$ is $(a,-b)$. Of course, $\theta(a,b) = \theta(a,-b)$.

What is the behaviour under the action of the involution $\Pi_1(-M)$? If we transform the orbit $\theta(a,b)$ by means of $\Pi_1(-M)$, we get a *different* orbit of $(4)_1$. In fact, we obtain $\Pi_1(-M)(\theta(a,b)) = \theta(-a,b)$ and $\theta(a,b) \neq \theta(-a,b)$. Again, the parametrization obtained by means of t is not a solution of $(4)_1$. If we change t by $-t$, we get another solution but with *different* orbit. Finally, see that the composition of both time reversal symmetries is really a *symmetry* for $(4)_1$. Then we obtain that $-I$ is a symmetry and it implies that if $x(t) = (a, at+b)$ is a solution of the problem with initial condition $x(0) = (a,b)$, then $-x(t) = -I(x(t))$ is a solution with initial condition $(-a,-b)$.

Remark 17. Note that any orbit $x_0 = a \neq 0$ of $(4)_1$ is invariant with respect to the time reversal symmetry $\Pi_1(M)$. On the contrary, no orbit is invariant with respect to the time reversal symmetry $\Pi_1(-M)$ except for equilibrium points.

We propose to the reader the beautiful exercise of analysing the problem $(4)_2$ and the role of the involutions $\Pi_2(M) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\Pi_2(-M) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ as time reversal symmetries for $(4)_2$. The geometric analysis of this problem is, obviously, richer than that of $(4)_1$. In the problem $(4)_1$ the involutions $\Pi_1(M)$ and $\Pi_1(-M)$ are conjugated and they represent reflections about different axis. But in $(4)_2$, $\Pi_2(M)$ is a reflection about the plane $x_1 = 0$, which is a plane of fixed points of $\Pi_2(M)$, and $\Pi_2(-M)$ is a rotation of angle π around the x_1-axis. Then, they are not conjugated. Analysing this case, one can see first that the equilibrium points are those of the form $(0,0,c)$ and that under the flow generated by $(4)_2$, particles are moving within affine hyperplanes $x_0 = a$. When $a = 0$ we obtain the motion described in $(4)_1$ in this hyperplane. The orbits of points are parabolas for $a \neq 0$. In this case, the velocity vector field is given by $X(a, b, c) = (0, a, 2b)$ and the solution of $(4)_2$ with initial conditions $x_0(0) = a$, $x_1(0) = b$ and $x_2(0) = c$ is given by $x(t) = (a, at + b, at^2 + 2bt + c)$. As in the previous case, any orbit, which is not an equilibrium point, is symmetric with respect to the involution $\Pi_2(M)$, but any nontrivial orbit is transformed by $\Pi_2(-M)$ into another different orbit. In both cases, if after the transformation, we change t by $-t$, then we obtain new solutions of the problem $(4)_2$. Finally, $-I\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a symmetry for the problem.

5.2. Facts and/or conjectures and/or speculations on the problems $(4)_n$ and (4)

The flow induced by problem (4) in $\mathbb{K}[[x]]$ is given by

$$\Phi\colon \mathbb{K}[[x]] \times \mathbb{R} \longrightarrow \mathbb{K}[[x]],$$

where $\Phi(h, t) = e^{tL_1^{1,1}}(h) = \frac{1}{1-xt} h\left(\frac{x}{1-xt}\right)$, while the flow generated by the problem $(4)_n$ in $\mathbb{R}^{n+1}$ is

$$\Phi_n\colon \mathbb{R}^{n+1} \times \mathbb{R} \longrightarrow \mathbb{R}^{n+1},$$

whose matrix expression is $\Phi_n(x, t) = \Pi_n(e^{tL_1^{1,1}})x^{\mathrm{T}}$.

We now are going to state, without proofs, properties related to problems $(4)_n$ and (4). This is the reason why we entitled this subsection as we did.

Proposition 18. ***(Dynamical properties related to problems $(4)_n$)*** *Let n be a nonnegative integer number, then we have the following properties.*

(i) *The orbit of any point $a = (a_0, a_1, \ldots, a_n)$ in $\mathbb{R}^{n+1}$ is contained in the affine hyperplane $x_0 = a_0$. Moreover, if $a_0 = 0$ and if one considers $\mathbb{R}^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}/x_0 = 0\}$, the motion induced by Φ_n in $\mathbb{R}^n$ is Φ_{n-1}.*

(ii) *The equilibrium points in $(4)_n$ are those in the x_n-axis and, if $n > 0$, all of them are unstable in the Lyapunov sense.*

(iii) *The nontrivial orbits in $(4)_n$, i.e., those which are not equilibrium points, are related to the so called moment curve in the corresponding hyperplane. In particular, the solution of $(4)_n$ with initial condition $(1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$ is $x(t) = (1, t, t^2 \ldots, t^n)$, which is a copy of the corresponding moment curve in the hyperplane $x_0 = 1$.*

(iv) *Any nontrivial orbit in $(4)_n$ is invariant with respect to the time-reversal symmetry $\Pi_n(M)$, and no one of them is invariant with respect to $\Pi_n(-M)$. Anyway, if we have a nontrivial solution of $(4)_n$, i.e., a nonconstant one, we transform it by any of the involutions $\Pi_n(M)$ or $\Pi_n(-M)$ and then change t by $-t$, we get another solution of $(4)_n$.*

(v) *$-I\colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ is a symmetry for the equation $(4)_n$.*

Motivated by the previous result and knowing that Pascal's triangle is the time 1 map of the flow Φ, we can state:

Proposition 19. ***(Some dynamical properties of Pascal's triangle)***

(i) *The flow Φ has no equilibrium points except for the null power series, that is, the power series whose coefficients are all null.*

(ii) *The orbit of the formal power series constantly* 1*, by means of* Φ*, is the set of geometric progressions* $\{\frac{1}{1-tx}\}_{t\in\mathbb{R}}$*. Then one can think about* 1 *moving through the set of the germs of analytic functions, at* $x = 0$*, because for any* $t \in \mathbb{R}$ *the function* $f_t(x) = \frac{1}{1-tx}$ *for* $x \in (-\frac{1}{|t|}, \frac{1}{|t|})$ *is analytic at* $x = 0$*. Moreover, this orbit can be seen as the asymptotic behaviour, as* n *goes to* ∞*, of the moment curves.*

(iii) *The Riordan involution* M *and* $-M$ *are time-reversal symmetries for the flow* Φ*. Any orbit of problem* (4) *is invariant with respect to* M *and no orbit, except for the unique equilibrium point, is invariant with respect to* $-M$*. Anyway, if we transform any solution of* (4) *by means of* M *or* $-M$ *and then change* t *by* $-t$*, we get another solution of the problem.*

(iv) $-I\colon \mathbb{K}[[x]] \longrightarrow \mathbb{K}[[x]]$ *is a symmetry for the equation (4).*

Claim: *The* α*- and* ω*-limit sets of any nontrivial orbit of the problem* $(4)_n$ *are empty.* On the other hand, the problem $(4)_n$ has time reversal symmetries. This fact allows us to think that there should be relationships between both limit sets. We then decided to force the corresponding flows, by means of considering a compactification of the corresponding phase spaces, to get nonempty α-limit and ω-limit sets and to look for relationships between them. We proceed as follows.

Consider the one point (or Alexandroff) compactification of $\mathbb{R}^{n+1}$, which is, topologically, the $(n+1)$-dimensional sphere S^{n+1}. Let us denote by ∞ the added point. Note that any t-map of the flow Φ_n, Φ_n^t, can be continuously extended to a map $\widetilde{\Phi_n^t}\colon S^{n+1} \longrightarrow S^{n+1}$ just defining $\widetilde{\Phi_n^t}(\infty) = \infty$. In this way we get a dynamical system

$$\widetilde{\Phi_n}\colon S^{n+1} \times \mathbb{R} \longrightarrow S^{n+1}.$$

For every nonnegative integer n, we can also extend the time reversal symmetries $\Pi_n(M)$ and $\Pi_n(-M)$ to continuous maps $\widetilde{\Pi_n(M)}$ and $\widetilde{\Pi_n(-M)}$ from S^{n+1} onto itself imposing that the point ∞ is a fixed point for both of them. We can identify, topologically, $\mathbb{R}^{n+1}$ with $S^{n+1}\backslash\{\infty\}$ (which is a dense subset of S^{n+1}). With all these constructions we have:

Proposition 20. *Let* n *be a nonnegative integer number. We have the following.*

(i) *The maps* $\widetilde{\Pi_n(M)}$ *and* $\widetilde{\Pi_n(-M)}$ *are continuous involutions in* S^{n+1} *and they are time-reversal symmetries for the dynamical system* $\widetilde{\Phi_n}$*.*

(ii) *The orbits of the dynamical system* $\widetilde{\Phi_n}$ *are those of* Φ_n *(after the mentioned identification) plus* $\{\infty\}$ *which is an equilibrium point. Moreover, every nontrivial orbit of* $\widetilde{\Phi_n}$ *is homoclinic, being the point* ∞ *an attractor and a repeller of all of them.*

Consequently, the α*-limit and the* ω*-limit sets of any nontrivial orbit coincide.*

Final Remark. After reading the first version of this manuscript, the referee introduced a question related to the surjectivity of the exponential map in Lie groups. With the referee's agreement, we decided to postpone the study of this question to another occasion. We really thank the referee for drawing our attention to this interesting problem.

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