Bogdanov-Takens resonance in time-delayed systems

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Abstract

We analyze the oscillatory dynamics of a time-delayed dynamical system subjected to a periodic external forcing. We show that, for certain values of the delay, the response can be greatly enhanced by a *very small forcing amplitude*. This phenomenon is related to the presence of a *Bogdanov-Takens* bifurcation and displays some analogies to other resonance phenomena, but also substantial differences.

I. INTRODUCTION

Different resonance phenomena play a key role in the sciences. Examples, beyond the simplest case of a linear system forced at its natural frequency, include stochastic resonance [1, 2], chaotic resonance [3], coherence resonance [4] and vibrational resonance (VR) [5]. For a recent monograph dealing with all these phenomena, see [6]. The stochastic resonance of a bistable system is triggered by the cooperation between a weak periodic forcing and noise. The noise can be replaced by a chaotic signal to obtain chaotic resonance. It is also possible to have noise–induced resonance in absence of external periodic forces, a phenomenon called coherence resonance. A nonlinear system driven by a biharmonic forcing with different frequencies can show VR. Resonances appear not only in systems described by standard differential equations, but also in time-delayed systems. Time-delay effects arise frequently in practical problems and have received much attention in recent years [7–11]. Hereditary effects are sometimes unavoidable and may easily turn a well-behaved system into one displaying very complex dynamics. A simple example is provided by Gumowski and Mira [12], who demonstrate that the presence of delays may destroy stability and cause periodic oscillations in systems governed by differential equations. Vibrational resonance occurs in time-delayed systems with two harmonic forcings of different frequencies [13–16]. Furthermore delay systems often possess oscillatory behavior even in the absence of forcing and for this reason VR and related phenomena may occur even in the presence of only one external excitation [17, 18].

In this work we present a new resonance phenomenon that may appear in systems with delay. The addition of a *very small external forcing* may result in the solution changing from a damped, small amplitude oscillation to a sustained, large amplitude oscillation. The sustained response takes place for a range of values of the frequency Ω of the external forcing (as distinct from phenomena that require well-defined values of Ω). The resonance occurs for a (narrow) interval of values of the delay and is related to the presence of a Bogdanov-Takens bifurcation in the model. We therefore will refer to it as *Bogdanov-Takens* resonance.

II. THE SYSTEM

The model that we use to describe and analyze the Bogdanov-Takens resonance is the apparently simple system called delayed action oscillator [19]. It is a single variable system with a double-well potential and a linear delayed feedback term with a constant time delay $\tau \geq 0$. The oscillator can be written as:

$$\dot{x} = \alpha x_\tau + x - (1+\alpha)x^3 + F \sin \Omega t \tag{1}$$

$$x_\tau = x(t-\tau), \tag{2}$$

where α measures the influence of the returning signal relative to that of the local feedback and represents a negative feedback, τ is the time delay, and F and Ω are the amplitude and frequency of the external periodic forcing. The constants α, τ, F and Ω are real and the interest is in the case $\alpha \in (-1, 0)$. Without the delayed term, this system would be a one-dimensional ODE and could not oscillate, but the linear delayed feedback converts the system into an infinite-dimensional one, allowing oscillatory dynamics. The system is interesting, among other things, for its analogy with the El Niño Southern Oscillator (ENSO) [20, 21] and the well-known Duffing oscillator $\ddot{x} + \gamma \dot{x} + x(x-1) = 0$, as discussed in [19].

We begin by studying the unforced system with $F = 0$:

$$\dot{x} = x + \alpha x_\tau - (1+\alpha)x^3. \tag{3}$$

This has the equilibrium points $x = 0$ and $x = \pm 1$. The equilibrium $x = 0$ is always unstable as may be easily shown by studying the corresponding linearization of Eq. (3). The equilibria at ± 1 are stable in the absence of delay ($\tau = 0$), but undergo Hopf bifurcations [22] in the delayed system. For $x = 1$ (and for symmetry reasons for $x = -1$), the characteristic equation of the linearization is

$$\lambda = -3\alpha - 2 + \alpha e^{-\lambda\tau}. \tag{4}$$

If $\alpha < -1$ or $\alpha > -1/2$, then, for any $\tau > 0$, all roots of the this equation have negative real parts and $x = 1$ and $x = -1$ are asymptotically stable. If $-1 < \alpha < -1/2$, there is a sequence $\tau = \tau_k$, $k = 0, 1, 2, \ldots$ of values of the delay for which Eq. (4) has a pair of imaginary roots $\pm i\omega_0$, $\omega_0 = \sqrt{\alpha^2 - (3\alpha+2)^2}$. The delays τ_k and the frequency ω_0 are related by the following expression:

$$\tau_k = \frac{\sin^{-1}(-\omega_0/\alpha) + 2k\pi}{\omega_0}. \tag{5}$$

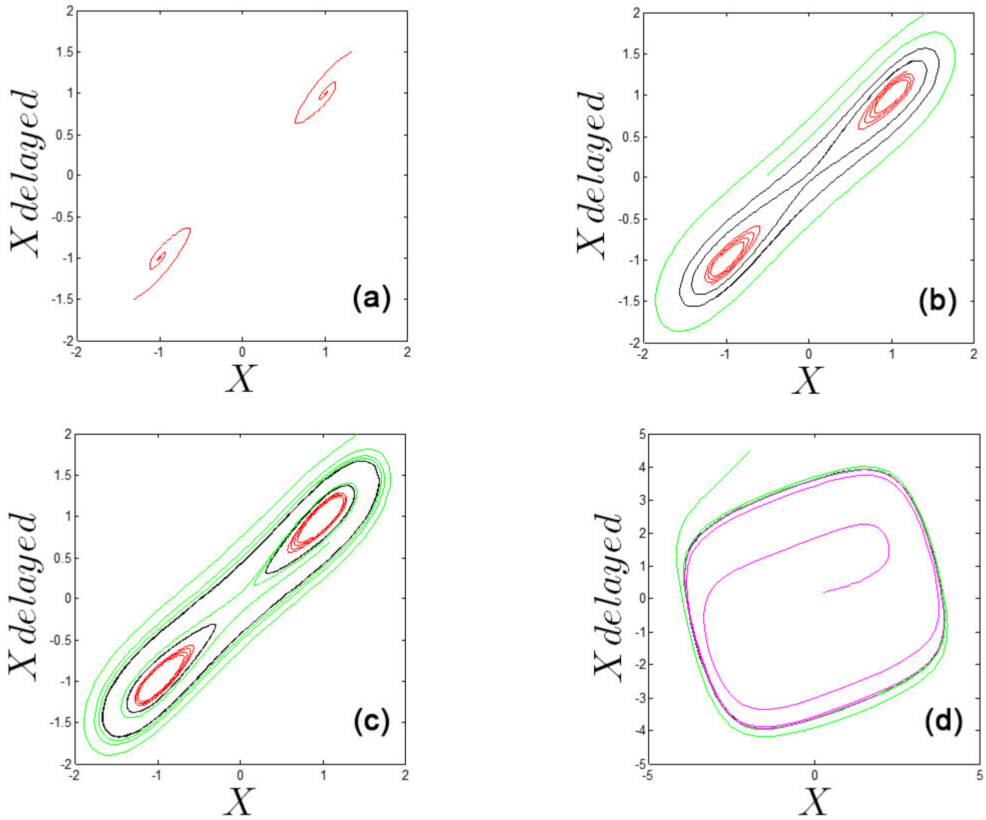

FIG. 1: Phase portrait of the system, Eq. (1), for $\alpha = -0.925$, and for $\tau = 1, 1.122, 1.13$ and 1.7, respectively. In the figures different trajectories are arise from different history functions.

If $\tau \in [0, \tau_0)$, then all roots of Eq. (4) have negative real parts. For $\tau = \tau_0$, the roots of Eq. (4) have real parts < 0, except for the pair $\pm i\omega_0$. If $\tau \in (\tau_0, \tau_1]$, Eq. (4) has one pair of complex conjugate roots with positive real parts. Thus, for $-1 < \alpha < -1/2$, the equilibria $x = 1$ and $x = -1$ undergo a Hopf bifurcation at $\tau = \tau_0$, where, as τ increases, they turn from being asymptotically stable into being unstable. Additional Hopf bifurcations occur at τ_k, $k = 1, 2, \ldots$, but we shall not be concerned with them. For the value $\alpha = -0.925$ used in [19] and in the experiments below, $\tau_0 \approx 1.1436$ and $\omega_0 \approx 0.5050$.

The panels in Fig. 1 show 'phase portraits' in the plane x, x_τ (see Ref. [24]). In each panel there are different solutions corresponding to different constant history functions $x_\tau(t) = u_0$, $t \in [-\tau, 0]$, u_0 a constant. Bear in mind that this is different from a true phase portrait of an ODE system because in the delayed case it is not true that each point in the plane defines a unique trajectory. In the panels the solutions move counterclockwise. Panel (a) corresponds to the case of "small" τ; solutions are generically attracted to a stable equilibrum

<table>
<tr><th colspan="2">Solutions as a function of τ. The Bogdanov – Takens bifurcation</th></tr>
<tr><td>$\tau < \tau_c$</td><td>The equilibrium points ± 1 attract (most) solutions.</td></tr>
<tr><td colspan="2">Periodic saddle–node bifurcation at τ_c.
A stable loop L_s and a smaller unstable loop L_u are born.</td></tr>
<tr><td>$\tau_c < \tau < \tau_h$</td><td>The equilibrium points attract solutions inside L_u.
Outside L_u solutions are attracted to L_s.</td></tr>
<tr><td colspan="2">Homoclinic bifurcation at $\tau = \tau_h$.
The loop L_u gives rise to two unstable loops $L_{\pm 1}$ around ± 1 respectively.</td></tr>
<tr><td>$\tau_h < \tau < \tau_0$</td><td>Solutions inside $L_{\pm 1}$ attracted to corresponding equilibrium.
Other solutions are attracted to L_s.</td></tr>
<tr><td colspan="2">Hopf bifurcation at $\tau = \tau_0$.
The loops $L_{\pm 1}$ merge with the corresponding equilibrium.</td></tr>
<tr><td>Large τ</td><td>L_s attracts most solutions.</td></tr>
</table>

TABLE I: Behaviour of the solutions of the unforced system (3) as a function of τ.

± 1. For τ "large", solutions are generically attracted to a single big loop L_s as in panel (d). As $\tau \to \infty$, solutions on L_s are approximately square waves where $x(t)$ jumps from $a = +\sqrt{(1-\alpha)/(1+\alpha)}$ to $-a$ and back, and simultaneously $x(t-\tau)$ jumps from $-a$ to a and back. The orbitally stable loop L_s is born at a saddle-node bifurcation at $\tau = \tau_c$ (for $\alpha = -0.0925$, $\tau_c \approx 1.119$). The saddle-node bifurcation point τ_c is smaller than the Hopf bifurcation point τ_0 discussed above, so that for $\tau \in (\tau_c, \tau_0)$ the attracting big loop L_s coexists with the attractors at $x = \pm 1$. This is the regime of interest for our purposes. The interval (τ_c, τ_0) contains two subintervals (τ_c, τ_h), (τ_h, τ_0) corresponding to different dynamics. In the first of these subintervals (panel (b)), there is an unstable loop L_u surrounding the equilibria; L_u is of course born, together with L_s, at the saddle-node bifurcation at $\tau = \tau_c$. At $\tau = \tau_h$, L_u becomes a homoclinic connection of the equilibrium at $x = 0$ and a further increase of τ turns the homoclinic connection into a couple of unstable orbits $L_{\pm 1}$, one around $x = 1$ and the other around -1 (panel (c)). These unstable orbits disappear at the subcritical Hopf bifurcation at $\tau = \tau_0$, where each of them merges with the corresponding equilibrium. The bifurcations at τ_c, τ_h and τ_0 clearly correspond to a Bogdanov-Takens scenario. A summary of the possible behaviors of the solutions as τ varies appears in Table I.

III. RESONANCE

The forced system $\dot{x} = x - x^3 + A\cos\omega t$ shows small-amplitude sustained oscillations around one of the equilibrium points $x = \pm 1$, which are only possible due to the slow forcing $A\cos\omega t$. Then, by adding a fast forcing $B\cos\Omega t$, with $\Omega \gg \omega$ the oscillations may go from one well to the other. This is the phenomenon of vibrational resonance [5, 25]. Equations like (1) may exhibit something extremely similar [18, 26]. The autonomous system $\dot{x}+x((1+\alpha)x^2-1)-\alpha x_\tau = 0$ shows slowly damped oscillations around $+1$ or -1, induced by the delay. Then, the addition of a forcing term $F\sin\Omega t$ may give rise to sustained oscillations that go from one well to the other. The phenomenon that we study here is considerably different. We illustrate it in the case with $\alpha = -0.925$, $\tau = 1.14$ and constant history $u_0 = 1.1$. For this value of τ, the equilibria ± 1 coexist with the attractor L_s. Figure 2 corresponds to the unforced case $F = 0$. The solution is a marginally damped oscillation with angular frequency approximately equal to $\omega_n = 0.50$, as the position of the peak in Fig. 2(b) shows. Then, we add *a very small forcing value* $F = 0.01$ of angular frequency $\Omega = 0.50$ (the exact value of the forcing frequency Ω is not critical, as we will discuss later). As we show in Fig. 3(a), the solution is a sustained oscillation of large amplitude and angular frequency $\omega_n = 0.40$, as shown by the position of the peak in Fig. 3(b). Therefore, there is a huge impact of the small forcing term. The resulting sustained oscillation is triggered by the forcing, but it is not a direct response to it, because the frequency of the interwell oscillation does not match the forcing frequency Ω, as we see by comparing Figs. 2(b) and 3(b). In a phase portrait the forcing would cause the solution to jump from the neighbourhood of the equilibrium $x = 1$ to the stable loop L_s. Figure 4 shows the amplitude of the solution as a function of τ, without forcing (red line) and with $F = 0.01$ (blue line). The numerical experiments support the analysis in the previous section. In fact, it is possible to appreciate the enhancement of the amplitude A for τ in range $1.119 < \tau < 1.143$ where L_s coexists with the stable equilibria. If τ is larger than 1.143, then the equilibrium points ± 1 loose their stability so that the damped oscillations around them no longer exist, and the system shows an interwell oscillation without any need for an external forcing. On the other hand, if τ is below 1.119, the solution will eventually settle in one of the wells, even if in a transient phase it oscillates between both wells.

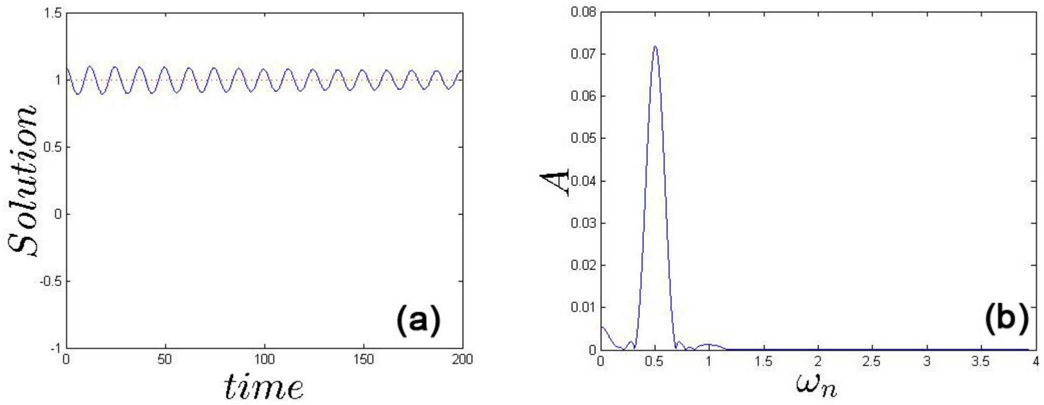

FIG. 2: The unforced ($F = 0$) system (1) with $\alpha = -0.925$, $\tau = 1.14$, and a constant history function $u_0 = 1.1$. Panel (a): The solution $x(t)$. Panel (b): Fourier analysis.

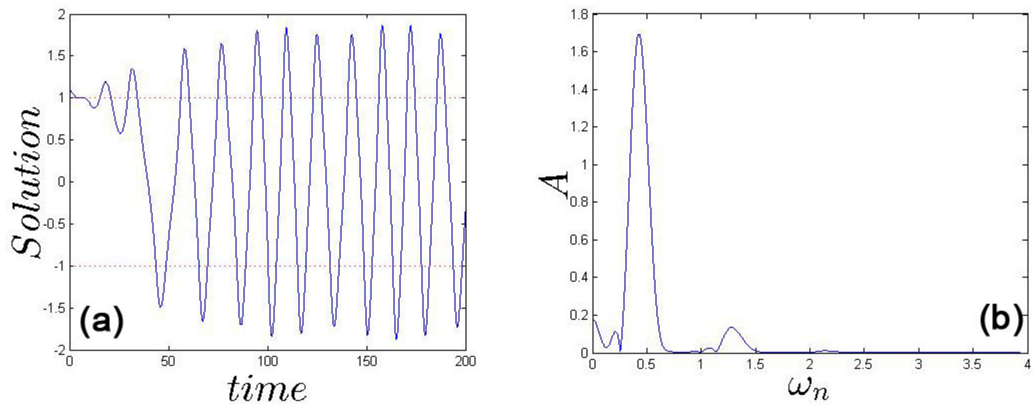

FIG. 3: As in Fig. 2, except that a small forcing $F = 0.01$, $\Omega = 0.5$ has been added; the solution is now a sustained oscillation of large amplitude. When comparing with Fig. 2(b) note the change in the vertical scale for A.

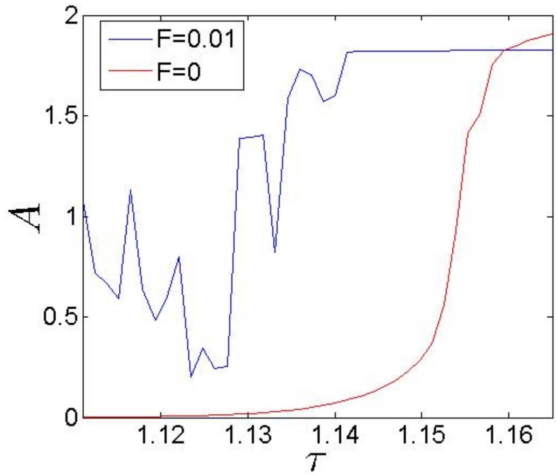

FIG. 4: The figure shows the amplitude of the oscillatory solution of the system, Eq. (1), as a function of τ for two different values of the external periodic forcing. We have considered values of τ close to the critical values where the bifurcations occur. Here $\Omega = 0.5$, $\alpha = -0.925$ and the history function is the constant $u_0 = 1.1$.

It is important to point out that the phenomenon that we are discussing is very different from well-known cases where a forcing with a moderate value of F gives a solution that, upon Fourier analysis, is seen to consist of modes $\cos(\Omega t + \phi_1)$ (the fundamental harmonic), $\cos(3\Omega t + \phi_3)$ (the third harmonic) or $\cos(\Omega t/3 + \phi_1)$ (subharmonic) (odd numbered overtones are expected in view of the cubic nonlinearity).

In order to show that the phenomenon is not specific to the particular model (1), we have also analyzed the equation

$$\dot{x} = \alpha x_\tau + x - 3(1+\alpha)x x_\tau^2 + 2(1+\alpha)x_\tau^3 + F \sin \Omega t \tag{6}$$

$$x_\tau = x(t-\tau), \tag{7}$$

that undergoes a Bogdanov-Takens bifurcation [24]. The resonance studied here also occurs, as seen in Fig. 5: the introduction of a *very small external forcing* induces again interwell oscillations.

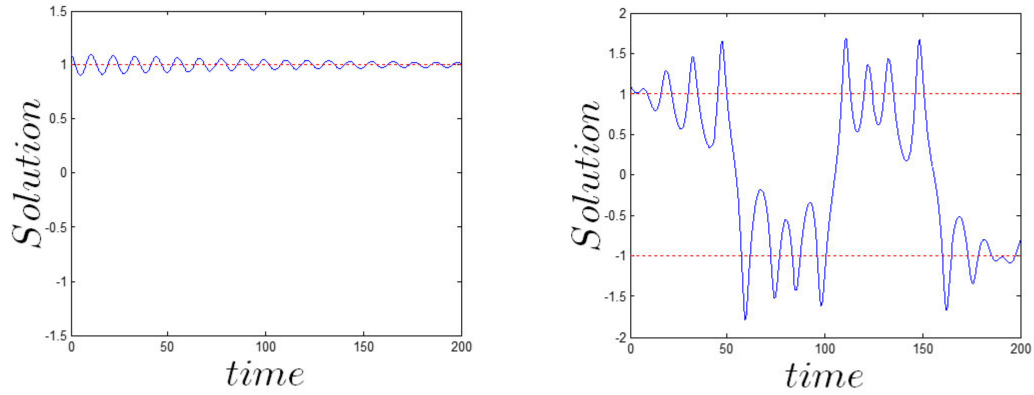

FIG. 5: Model (6) with $\tau = 1.19$, $\alpha = -0.9$, and a constant history function $u_0 = 1.1$. On the left: damped oscillations in the absence of forcing ($F = 0$). On the right: large amplitude oscillations for a small forcing, $F = 0.015$, $\Omega = 0.45$.

IV. DYNAMICS OF THE RESONANCE

We now study the impact on the resonance of changes in the forcing frequency Ω, the parameter α and the history function u_0.

Figure 6(a) depicts the amplitude (maximum value of the Fourier spectrum) of the response x as a function of Ω; the resonance manifests itself for a range of values of Ω around

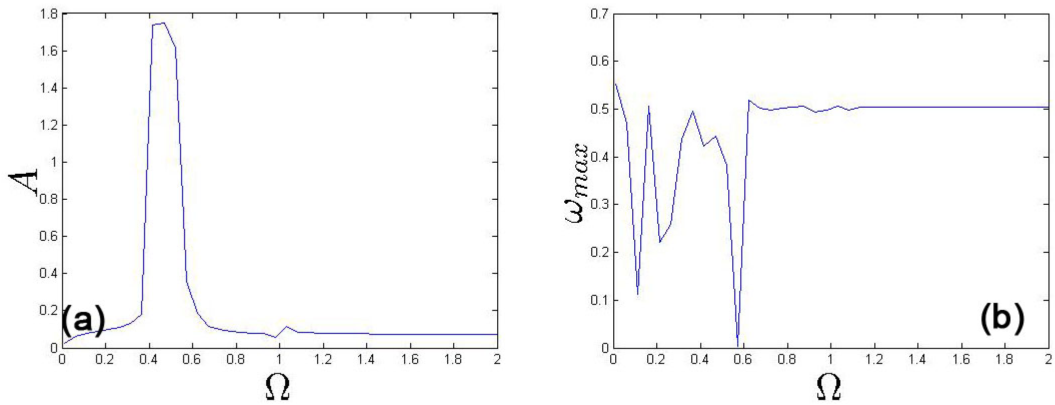

FIG. 6: The figures show (a) the amplitude of the oscillation and (b) the frequency ω_{max} as a function of Ω, corresponding to the system, Eq. (1), for $F = 0.01$, $\tau = 1.14$, $\alpha = -0.925$, and a constant history function $u_0 = 1.1$. In (a) a well-defined peak appears around $\Omega = 0.4$, that shows that the resonance phenomenon takes place for suitable values of the forcing frequency.

0.4. Panel (b) gives the frequency ω_n for which the Fourier spectrum of the signal attains its maximum value. Note the little correlation between ω_n and Ω; for Ω large, ω_n corresponds to the frequency on L_s.

In Fig. 7 we use the alternative value $\alpha = -0.8$ in order to check the occurrence of the resonance. Note that the value of the critical τ_0 for which the resonance appears increases, in agreement with Eq. (5).

It is of interest to study numerically the changes of the critical τ_0 as α varies, as shown in Fig. 8, that plots the amplitude of the oscillation as a function of Ω for different values of α, from $\alpha = -0.5$, panel (a) to $\alpha = -0.9$, panel (e). We note that for larger values of α, the critical value of τ that triggers the resonance increases. The figures also show that the shape of the peak in the (Ω, A) plane and the range of Ω leading to resonance change with α, although not as much as the value of τ_0.

Another important factor in the study of delayed systems is the history function. We have carried out numerical experiments changing the history function and found that the phenomenon is robust against the variation of the history. In fact, none of the figures shown above changes if we alternatively use linear, quadratic or sinusoidal histories.

V. CONCLUSIONS

In conclusion, we have shown the phenomenon of *Bogdanov-Takens resonance* in time-delayed systems. This resonance is produced when a periodic signal of a very small amplitude applied to a delayed system produces sustained oscillations of large amplitude. The frequency of the resulting sustained oscillation is not related to the frequency Ω of the forcing. Resonance takes places for Ω in a suitable interval, rather than at critical values of Ω. The phenomenon also appears in systems without delay as we shall describe in a forthcoming paper.

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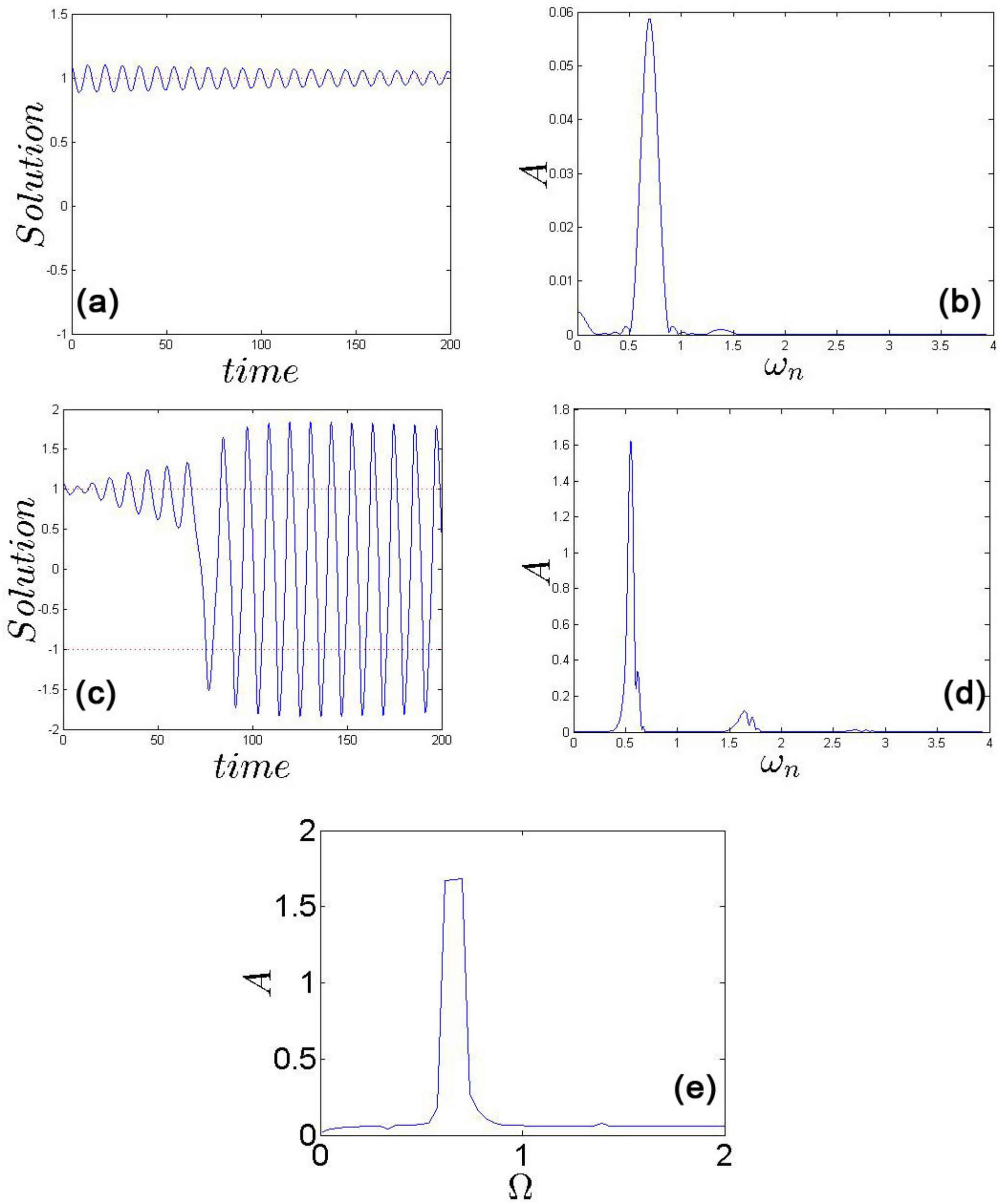

FIG. 7: The resonance for Eq. (1), but now with $\alpha = -0.8$, $\tau = 1.5$. Panels (a) and (b) show the solution and its Fourier analysis for $F = 0$. Panels (c) and (d) show the solution and its Fourier analysis for $F = 0.01$, $\Omega = 0.6$. Panel (e) shows the amplitude of the solution as a function of Ω.

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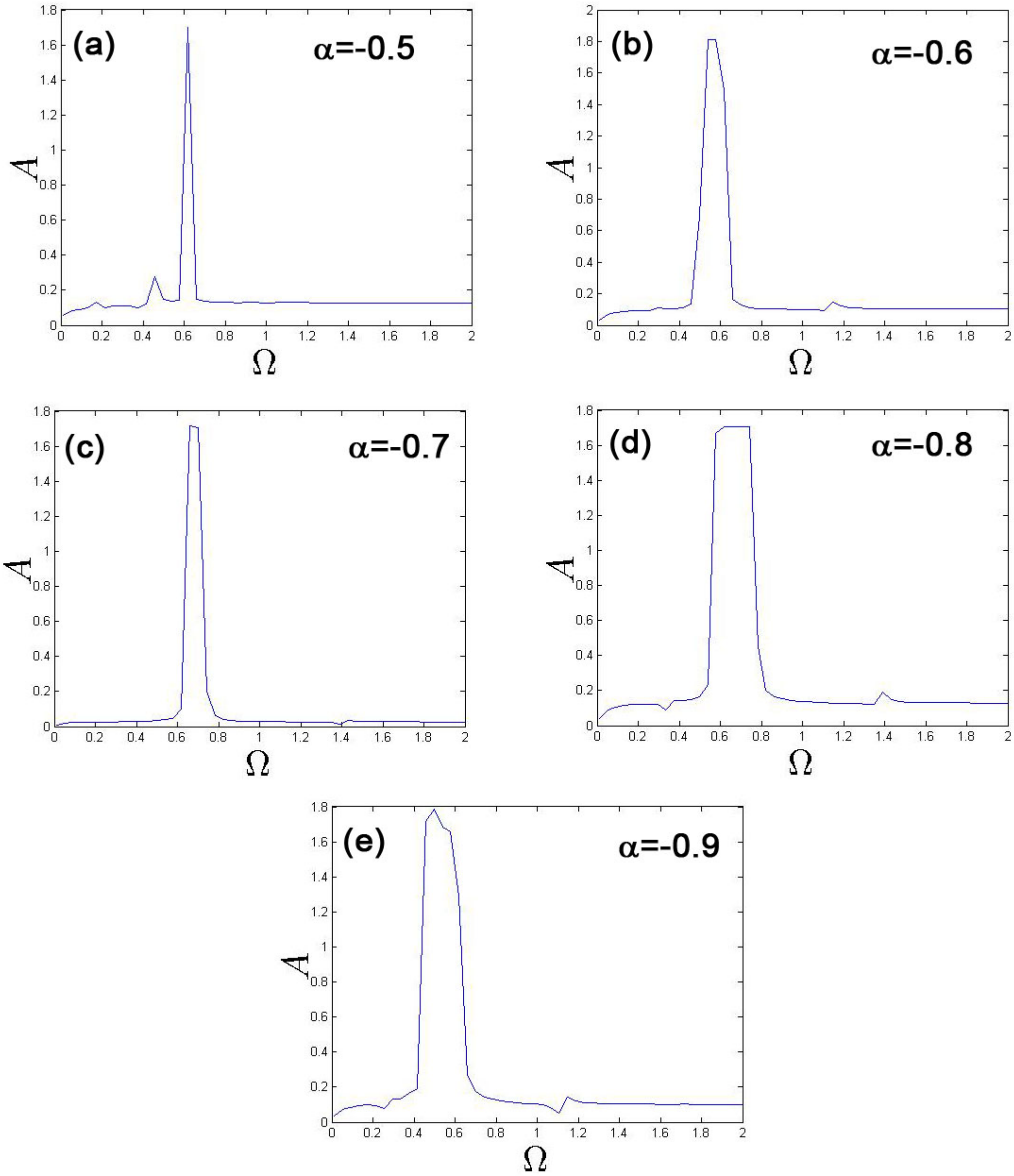

FIG. 8: Amplitude of the solution of Eq. (1) as a function of Ω for different values of the parameter α. Panel (a): $\tau = 12.8$ and $\alpha = -0.5$. Panel (b): $\tau = 3.3$ and $\alpha = -0.6$. Panel (c): $\tau = 2.01$ and $\alpha = -0.7$. Panel (d): $\tau = 1.51$ and $\alpha = -0.8$. Panel (e): $\tau = 1.2$ and $\alpha = -0.9$. As α decreases, the values of τ that trigger the resonance decrease.