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Note

A new approach to deriving Bäcklund transformations

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ABSTRACT

We give a new, surprisingly simple approach to the derivation of Bäcklund transformations. Motivated by the use of integrating factors to solve linear ordinary differential equations, for the nonlinear case this new technique leads to differential relations between equations. Although our interest here is in Painlevé equations, our approach is applicable to nonlinear equations more widely. As a completely new result we obtain a matrix version of a classical mapping between solutions of special cases of the second Painlevé equation. This involves the derivation of a new matrix second Painlevé equation, for which we also present a Lax pair. In addition, we give a matrix version of the Schwarzian second Painlevé equation, again a completely new result. In this way we also discover a new definition of matrix Schwarzian derivative.

1. Introduction

It is a classical result [3] that solutions of the second Painlevé equation (P_{II}),

$$u_{xx} - 2u^3 - xu - \alpha = 0, \tag{1.1}$$

for parameter value $\alpha = 1/2$, can be obtained as $u = -\frac{w_x}{w}$, where w is a solution of

$$w_{xx} + \frac{1}{2}xw + Cw^3 = 0. \tag{1.2}$$

This equation, when the constant $C \neq 0$, corresponds to a rescaled version of P_{II} for parameter value $\alpha = 0$. In this way an auto-Bäcklund transformation (auto-BT) is obtained from P_{II} for $\alpha = 0$ to P_{II} for $\alpha = 1/2$. By Bäcklund transformation (BT) we refer to a mapping between solutions of ordinary differential equations (ODEs), called an auto-BT when the ODEs are the same. Our aim in this paper is to understand how the above result may be derived. We show that this can be done using a remarkably simple integration-based technique. This stems from the fact that P_{II} can be written using a certain differential operator, which then

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permits a process similar to the use of an integrating factor for linear ODEs to be used, but without any resulting reduction of order. This then allows a mapping between ODEs of the same order to be established. Our interest here is in Painlevé equations, but the simplicity of our new approach means that it will be much more widely applicable.

In the case of P_{II}, the differential operator mentioned above is (minus) the adjoint of the Fréchet derivative of the Miura map relating the Korteweg-de Vries (KdV) equation and the modified KdV (mKdV) equation. A similar observation holds for the other examples considered in this paper, and is a consequence of their relationship to completely integrable partial differential equations admitting Hamiltonian Miura maps. For such examples, this relationship may be understood as fundamental to our approach. However, in the general case, the differential operator of which we make use need not have such a Miura map as its origin.

In Section 2 we show how the above auto-BT for P_{II} may be derived, along with the Schwarzian P_{II} equation and a special integral. As a further example, we apply our technique in Section 3 to the fourth Painlevé equation (P_{IV}), for which it yields a special case of a known auto-BT, as well as a Schwarzian P_{IV} equation. In Section 4 we use our approach to derive an analogue of the mapping given above for the case of a matrix P_{II} equation. This requires the introduction of a new matrix P_{II} equation, which we show to be integrable by presenting a Lax pair. In Section 4 we also obtain the first known matrix version of the Schwarzian P_{II} equation. A summary and consideration of future perspectives is given in the final Conclusions and Discussion section.

2. Our new approach: the second Painlevé equation

Let us begin by observing that P_{II} (1.1) can be written

$$(\partial_x + 2u)K + \left(\frac{1}{2} - \alpha\right) = 0, \qquad K = u_x - u^2 - \frac{1}{2}x. \tag{2.1}$$

A comparison with linear ODEs suggests that the use of an "integrating factor" $e^{\int 2u\,dx}$ might be useful. The operator $(\partial_x + 2u)$ in (2.1) is minus the adjoint of the Fréchet derivative of the KdV-mKdV Miura map $\omega = u_x - u^2$, and writing P_{II} using this operator can be traced back to the formulation in [8] of its well-known derivation from these PDEs. The use of this integrating factor may alternatively be effected using a function z defined by $z_{xx} = 2uz_x$ (so $z_x = e^{\int 2u\,dx}$): since $\partial_x z_x = z_x\partial_x + z_{xx} = z_x(\partial_x + 2u)$, we see that the result of multiplying P_{II} (2.1) by z_x and integrating is

$$z_x K + \left(\frac{1}{2} - \alpha\right) z = C, \qquad K = u_x - u^2 - \frac{1}{2}x, \tag{2.2}$$

for some arbitrary constant C. Note that this process does not, of course, result in any reduction of order, as that would be impossible for P_{II}. In equation (2.2), i.e.,

$$z_x\left(u_x - u^2 - \frac{1}{2}x\right) + \left(\frac{1}{2} - \alpha\right) z = C, \tag{2.3}$$

the change of variables $u = -w_x/w$, and then taking, without loss of generality, $z_x = 1/w^2$, yields, for $\alpha = 1/2$,

$$\frac{1}{w^2}\left(-\frac{w_{xx}}{w} - \frac{1}{2}x\right) = C, \quad \text{or} \quad w_{xx} + \frac{1}{2}xw + Cw^3 = 0. \tag{2.4}$$

For $C \neq 0$ we thus recover the mapping $u = -w_x/w$, as discussed in the Introduction, from solutions of P_{II} for $\alpha = 0$ to solutions of P_{II} for $\alpha = 1/2$. Indeed, from the above equations we obtain the pair of relations [3]

$$w_x + uw = 0, \qquad u_x - u^2 - \frac{1}{2}x = Cw^2, \tag{2.5}$$

which provide an auto-BT between (1.1) for $\alpha = 1/2$ and (2.4).

We now make some further remarks. First we note that if in equation (2.3) we assume $\alpha \neq 1/2$ then we may set $C = 0$. Substituting $u = z_{xx}/(2z_x)$ then yields Schwarzian P_{II} [15], valid in fact also for $\alpha = 1/2$,

$$z_x \left[\frac{1}{2}\{z; x\}\right] - \frac{1}{2}xz_x + \left(\frac{1}{2} - \alpha\right) z = 0, \tag{2.6}$$

where $\{z; x\} = (z_{xx}/z_x)_x - \frac{1}{2}(z_{xx}/z_x)^2$ is the Schwarzian derivative of z. Secondly, we observe that for $\alpha = 1/2$ and $C = 0$ equation (2.3) also gives the special integral $u_x - u^2 - \frac{1}{2}x = 0$ of P_{II} for this value of the parameter α. If we take into account also the case $\alpha = 1/2$ and $C \neq 0$ discussed above, which provides us with the mapping from solutions of P_{II} for $\alpha = 0$ to solutions of P_{II} for $\alpha = 1/2$, we see that equation (2.3) encapsulates three well-known results for P_{II}: this particular auto-BT, a special integral, and Schwarzian P_{II}. Whilst it is known that, making use of the relation $z_{xx}/(2z_x) = u$, P_{II} is a differential consequence of Schwarzian P_{II} (2.6) [9,15], as far as we are aware, the derivation of (2.3) — as written here in terms of u and z — from P_{II} through the use of the auxiliary variable z satisfying $z_{xx} = 2uz_x$ and integration, is new. These results are an example of:

Proposition 2.1. *Consider an ODE of the form*

$$(\partial_x + M)K + \kappa L = 0, \tag{2.7}$$

where K, L, M depend on x, u and its derivatives, and κ is a constant parameter. Let the function z satisfy the relation $z_{xx} = Mz_x$. Then solutions of (2.7) may be obtained from solutions of

$$z_xK + \kappa\rho = C, \qquad \text{where} \qquad \rho_x = z_xL \qquad \text{and } C \text{ is an arbitrary constant.} \tag{2.8}$$

Proof. Since $\partial_x z_x = z_x\partial_x + z_{xx} = z_x(\partial_x + M)$, multiplying (2.7) by z_x and integrating gives (2.8). □

Remark 2.1. We may assume $\kappa C = 0$ since, when $\kappa \neq 0$, we may set $C = 0$ using a shift on ρ (where $\rho = z$ if $L = 1$). For $\kappa = C = 0$, we see that solutions of $K = 0$ give (trivially) solutions of (2.7) for parameter $\kappa = 0$.

Remark 2.2. The above proposition may be applied to individual equations in a system of several equations.

3. A further example: the fourth Painlevé equation

The fourth Painlevé equation (P_{IV}) can be written in the form

$$\begin{pmatrix} 1 & p \\ 2 & \phi + 2p - \partial_x \end{pmatrix} \begin{pmatrix} \phi p + p^2 + p_x \\ \phi + 2p + 2x \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.1}$$

where e and f are two arbitrary constants. This then yields the system of equations

$$p_x + 2\phi p + 3p^2 + 2xp + e = 0, \tag{3.2}$$

$$\phi_x - 6\phi p - 6p^2 - \phi^2 - 2x(\phi + 2p) + 2 - f = 0, \tag{3.3}$$

and elimination of ϕ gives

$$p_{xx} = \frac{1}{2}\frac{p_x^2}{p} + \frac{3}{2}p^3 + 4xp^2 + 2\left(x^2 - \frac{1}{2}(2f - 3e - 2)\right)p - \frac{1}{2}\frac{e^2}{p}, \tag{3.4}$$

which is just P_{IV},

$$y_{xx} = \frac{1}{2}\frac{y_x^2}{y} + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha)p - \frac{1}{2}\frac{\beta^2}{p}, \tag{3.5}$$

with the identification $\alpha = \frac{1}{2}(2f - 3e - 2)$ and $\beta = e$.

In (3.1), the left-most matrix is the adjoint of the Fréchet derivative of the dispersive water wave (DWW) Miura map $(u, v)^T = (\phi + 2p, \phi p + p^2 + p_x)^T$ [10]. The appearance of this operator here is due to writing P_{IV} as derived from a modified DWW system [4]. The system (3.1) can be simplified to read

$$p_x + 2\phi p + 3p^2 + 2xp + e = 0, \tag{3.6}$$

$$(\partial_x - \phi)(\phi + 2p + 2x) + (2e - f) = 0. \tag{3.7}$$

It is to equation (3.7), following Remark 2.2, that we now apply the technique developed in Section 2.

Let us write equation (3.7) as

$$(\partial_x - \phi)K + (2e - f) = 0, \qquad K = \phi + 2p + 2x. \tag{3.8}$$

According to Proposition 2.1, multiplying by z_x, where $z_{xx} = -\phi z_x$, and integrating, yields

$$z_x K + (2e - f)z = C, \qquad K = \phi + 2p + 2x, \tag{3.9}$$

where we set $\kappa = 2e - f$, so $L = 1$ and $\rho = z$, i.e.,

$$z_x(\phi + 2p + 2x) + (2e - f)z = C. \tag{3.10}$$

Setting $\phi = q_x/q$ and solving without loss of generality for z_x as $z_x = 1/q$, we obtain, for the choice $f = 2e$,

$$\frac{1}{q}\left(\frac{q_x}{q} + 2p + 2x\right) = C. \tag{3.11}$$

This equation, together with (3.6), i.e.,

$$p_x + 2\left(\frac{q_x}{q}\right)p + 3p^2 + 2xp + e = 0, \tag{3.12}$$

forms a system of equations in the variables p and q. Solving (3.11) for p and substituting in (3.12) then gives

$$q_{xx} = \frac{1}{2}\frac{q_x^2}{q} + \frac{3}{2}C^2q^3 - 4Cxq^2 + 2\big(x^2 + (e - 1)\big)q. \tag{3.13}$$

For $C \neq 0$ we may always set $C = -1$, thus obtaining

$$q_{xx} = \frac{1}{2}\frac{q_x^2}{q} + \frac{3}{2}q^3 + 4xq^2 + 2\big(x^2 - (1 - e)\big)q. \tag{3.14}$$

This last equation is P_{IV} (3.5) for parameter values $\alpha = 1 - e$ and $\beta = 0$.

Equation (3.11) then gives the auto-BT

$$p=-\frac{1}{2}\left(\frac{q_x+q^2+2xq}{q}\right) \tag{3.15}$$

from (3.14) to (3.4) with $f=2e$, i.e., mapping from P_{IV} parameter values $\alpha_q=1-e$ and $\beta_q=0$ to $\alpha_p=\frac{1}{2}(e-2)$ and $\beta_p=e$, where we use subscripts p and q to denote the respective parameters α and β corresponding to P_{IV} in standard form (3.5) with $y=p$ and $y=q$. We thus have the relations $\alpha_p=-\frac{1}{2}(\alpha_q+1)$ and $\beta_p=1-\alpha_q$. This mapping and changes in parameter values correspond to those of the P_{IV} auto-BT [11]

$$p=-\frac{1}{2}\left(\frac{q_x+q^2+2xq+\beta_q}{q}\right), \qquad \alpha_p=-\frac{1}{4}(2+2\alpha_q-3\beta_q), \qquad \beta_p=\frac{1}{2}(2-2\alpha_q-\beta_q), \tag{3.16}$$

in the particular case $\beta_q=0$. Our technique has thus led to the recovery of a particular P_{IV} auto-BT. Let us also note that (3.11) with $C=-1$ and (3.12) yield the inverse auto-BT

$$q=\frac{1}{2}\left(\frac{p_x-p^2-2xp+e}{p}\right), \tag{3.17}$$

which provides a mapping from solutions p of the case $f=2e$ of (3.4) to solutions q of (3.14). Again this is a particular case of a result given in [11].

We now make two final remarks. Firstly, as noted in Remark 2.1, if in (3.10) we assume $f\neq 2e$ then we may set $C=0$. In the resulting equation we then set $\phi=-z_{xx}/z_x$ to obtain

$$p=\frac{1}{2}\frac{z_{xx}}{z_x}-\frac{1}{2}(2e-f)\left(\frac{z}{z_x}\right)-x, \tag{3.18}$$

and substitution of ϕ and p in terms of z in (3.6) then yields the Schwarzian P_{IV} equation, valid also for $f=2e$,

$$\{z;x\}+\frac{3}{2}(2e-f)^2\left(\frac{z}{z_x}\right)^2+4x(2e-f)\left(\frac{z}{z_x}\right)+2x^2+f-2=0. \tag{3.19}$$

For P_{IV} in standard form (3.5) (with, as before, $p=y$, $e=\beta$ and $f=(2\alpha+3\beta+2)/2$), this equation reads

$$\{z;x\}+\frac{3}{8}(\beta-2\alpha-2)^2\left(\frac{z}{z_x}\right)^2+2x(\beta-2\alpha-2)\left(\frac{z}{z_x}\right)+2x^2+\frac{1}{2}(2\alpha+3\beta-2)=0. \tag{3.20}$$

We have been unable to find the Schwarzian P_{IV} equation (3.20) in the literature. It is readily shown, making use of the relation

$$\frac{z_{xx}}{z_x}=2y+2x+\frac{1}{2}(\beta-2\alpha-2)\left(\frac{z}{z_x}\right) \tag{3.21}$$

(i.e., (3.18) with p replaced by y), that P_{IV} (3.5) is a differential consequence of Schwarzian P_{IV} (3.20).

Our second remark is that from equation (3.10) with $f=2e$ and $C=0$ we obtain $\phi+2p+2x=0$, which, substituting for ϕ from (3.6), then also gives the special integral $p_x-p^2-2xp+e=0$ of (3.4) for $f=2e$. This corresponds to the well-known special integral $y_x-y^2-2xy+2(\alpha+1)=0$ of P_{IV} in standard form, i.e., (3.5), for parameter relation $\beta=2(\alpha+1)$. The above results for P_{IV} are analogous to those for P_{II} in Section 2.

4. A matrix second Painlevé equation

To obtain a matrix analogue of Proposition 2.1, we need to consider both left and right "integrating factors":

Proposition 4.1. *Consider a matrix ODE of the form*

$$(\partial_x + L_M + R_N)K + \kappa L = 0, \tag{4.1}$$

where the square matrices K, L, M, N depend on x, the square matrix function u and its derivatives, κ is a constant scalar parameter, and the left and right multiplication operators L_M and R_N are defined by $L_M(K) = MK$ and $R_N(K) = KN$. Let the square matrix functions z and y satisfy the relations $z_{xx} = z_x M$ and $y_{xx} = Ny_x$. Then solutions of (4.1) may be obtained from solutions of[1]

$$z_x K y_y + \kappa\rho = C, \qquad \text{where} \qquad \rho_x = z_x L y_x \qquad \text{and } C \text{ is an arbitrary constant square matrix.} \tag{4.2}$$

Proof. Since $\partial_x L_{z_x} R_{y_x} = L_{z_x} R_{y_x}\partial_x + L_{z_{xx}} R_{y_x} + L_{z_x} R_{y_{xx}} = L_{z_x} R_{y_x}(\partial_x + L_M + R_N)$, multiplying (4.1) on the left by z_x, on the right by y_x, and integrating gives (4.2). □

Remark 4.1. For $y_x = I$, so $N = 0$, (4.1) and (4.2) reduce to $(\partial_x + L_M)K + \kappa L = 0$ and $z_x K + \kappa\rho = C$ where $\rho_x = z_x L$; and for $z_x = I$, so $M = 0$, to $(\partial_x + R_N)K + \kappa L = 0$ and $K y_x + \kappa\rho = C$ where $\rho_x = L y_x$.

Remark 4.2. We may assume $\kappa C = 0$ since, when $\kappa \neq 0$, we may set $C = 0$ using a matrix shift on ρ (where $\rho = z$ if $L y_x = I$, and $\rho = y$ if $z_x L = I$). For $\kappa = 0$ and $C = 0$, and z_x and y_x nonsingular, we get $K = 0$, and so obtain that solutions of $K = 0$ give (trivially) solutions of (4.1) for parameter $\kappa = 0$.

Remark 4.3. Proposition 4.1 may be applied to individual equations in a system of several matrix equations.

Our interest in this section is in a matrix P_{II} equation, with matrix coefficients, i.e.,

$$u_{xx} - 2u^3 + uE + Eu - xu - \alpha I, \tag{4.3}$$

as first derived in [7] (the case $E = 0$ was considered in [1,12]). Here u is a square matrix function of x, E is an arbitrary constant square matrix and α is an arbitrary scalar parameter. In the scalar reduction we may take $E = 0$, and thus recover P_{II} in standard form (1.1). Similarly to the scalar case, we may write (4.3) as

$$(\partial_x + A_u)K + \left(\frac{1}{2} - \alpha\right) I = 0, \qquad K = u_x - u^2 + E - \frac{1}{2}xI, \tag{4.4}$$

where $A_u = L_u + R_u$. The formulation (4.4) of matrix P_{II} may be obtained as a result of its derivation from matrix KdV and mKdV equations [7], with the operator $(\partial_x + A_u)$ being (minus) the adjoint of the Fréchet derivative of the Miura map $\omega = u_x - u^2$ between these matrix PDEs (see also [5,6]). From Proposition 4.1 we then see that, multiplying (4.4) on the left by z_x and on the right by y_x, where $z_{xx} = z_x u$ and $y_{xx} = u y_x$, and then integrating, we obtain

$$z_x \left(u_x - u^2 + E - \frac{1}{2}xI\right) y_x + \left(\frac{1}{2} - \alpha\right)\rho = C, \tag{4.5}$$

[1] For $K = u$, $M = M(x)$, $N = N(x)$, $L = L(x)$, i.e., the linear matrix equation $u_x + Mu + uN + \kappa L = 0$, the relationship between its solutions and those of equations (equivalent to) $s_x = sM$ and $t_x = Nt$ is well-known (here we set $z_x = s$, $y_x = t$).

where $\rho_x = z_x y_x$. We now set $u = -w_x w^{-1}$ and take, without loss of generality, $z_x = w^{-1}$, which for $\alpha = 1/2$ then leads to $w^{-1}\left(-w_{xx}w^{-1} + E - \frac{1}{2}xI\right) y_x = C$. The substitution $y_x = Aw^{-1}$ then yields the system

$$w_{xx} - Ew + \frac{1}{2}xw + wCwA^{-1}w = 0, \qquad A_x = Aw^{-1}w_x - w_x w^{-1}A. \tag{4.6}$$

For $C \neq 0$ this system represents a new parameter-free matrix P_{II} equation. We have thus obtained the result that solutions of this parameter-free matrix P_{II} give solutions of matrix P_{II} (4.3) for parameter value $\alpha = 1/2$ via the mapping $u = -w_x w^{-1}$. Under the scalar reduction, where we set $E = 0$, we may take the now scalar function $A = 1$ and thus recover (with C now a nonzero scalar constant) the result for P_{II} described in the Introduction and rederived in Section 2. Corresponding to (2.5) we obtain the matrix system

$$w_x + uw = 0 \qquad \text{and} \qquad u_x - u^2 + E - \frac{1}{2}xI = wCwA^{-1}, \;\; A_x = Aw^{-1}w_x - w_x w^{-1}A, \tag{4.7}$$

these relations providing a BT between (4.3) for $\alpha = 1/2$ and the system (4.6).

We note that for nonsingular C, we may simplify the above result as follows. For nonsingular constant square matrices P and Q, and constant square matrix F, we set

$$w = PWQ, \qquad u = PUP^{-1}, \qquad E = PFP^{-1}, \qquad A = PBQ. \tag{4.8}$$

Choosing $Q = P^{-1}C^{-1}$, the system (4.6) then becomes

$$W_{xx} - FW + \frac{1}{2}xW + W^2B^{-1}W = 0, \qquad B_x = BW^{-1}W_x - W_x W^{-1}B, \tag{4.9}$$

and we find that solutions of this system are mapped via $U = -W_x W^{-1}$ to solutions of

$$U_{xx} - 2U^3 + UF + FU - xU - \frac{1}{2}I = 0. \tag{4.10}$$

In addition, the relations (4.7) are transformed to

$$W_x + UW = 0 \qquad \text{and} \qquad U_x - U^2 + F - \frac{1}{2}xI = W^2B^{-1}, \;\; B_x = BW^{-1}W_x - W_x W^{-1}B, \tag{4.11}$$

which then provide a BT between (4.10) and the system (4.9). That is, we obtain the same results as before (i.e., in variables u, w, A) but with $C = I$. We may also choose the matrix F, which is similar to the given matrix E, to be of a required form, e.g., to be in Jordan canonical form, or upper-triangular, or symmetric.

The system (4.6) can be obtained as the compatibility condition $\mathbf{F}_\lambda - \mathbf{G}_x + [\mathbf{F}, \mathbf{G}] = 0$ of the Lax pair

$$\mathbf{\Psi}_x = \mathbf{F}\mathbf{\Psi}, \qquad \mathbf{\Psi}_\lambda = \mathbf{G}\mathbf{\Psi}, \tag{4.12}$$

where

$$\mathbf{F} = \begin{pmatrix} \lambda I & -w_x w^{-1} \\ -w_x w^{-1} & -\lambda I \end{pmatrix}, \qquad \mathbf{G} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & -g_{11} \end{pmatrix} \tag{4.13}$$

and

$$g_{11} = -4\lambda^2 I + 2(w_x w^{-1})^2 - 2E + xI, \tag{4.14}$$

$$g_{12} = 4\lambda w_x w^{-1} - 2(w_x w^{-1})^2 + 2E - xI - 2wCwA^{-1} - \frac{1}{2\lambda}I, \tag{4.15}$$

$$g_{21} = 4\lambda w_x w^{-1} + 2(w_x w^{-1})^2 - 2E + xI + 2wCwA^{-1} - \frac{1}{2\lambda}I. \tag{4.16}$$

We thus conclude that the system (4.6) is integrable, and so does indeed represent a new parameter-free matrix P_{II} equation. (This Lax pair may be obtained by suitable modification of that given in [1] for (4.3) with $E = 0$.)

Let us now return to equation (4.5). For $\alpha = 1/2$ we now take $u = -\sigma^{-1}\sigma_x$ and, without loss of generality, $y_x = \sigma^{-1}$. Setting $z_x = \sigma^{-1}D$, we are then led to the system

$$\sigma_{xx} - \sigma E + \frac{1}{2}x\sigma + \sigma D^{-1}\sigma C\sigma = 0, \qquad D_x = \sigma_x\sigma^{-1}D - D\sigma^{-1}\sigma_x. \tag{4.17}$$

Solutions of this system thus give solutions of matrix P_{II} (4.3) for parameter value $\alpha = 1/2$ via the mapping $u = -\sigma^{-1}\sigma_x$. With $\sigma^T = w$, $C^T = K$, $D^T = A$, $u^T = v$ and $E^T = F$, this then implies that solutions of the system

$$w_{xx} - Fw + \frac{1}{2}xw + wKwA^{-1}w = 0, \qquad A_x = Aw^{-1}w_x - w_x w^{-1}A \tag{4.18}$$

give rise to solutions of the matrix P_{II} equation

$$v_{xx} - 2v^3 + vF + Fv - xv - \frac{1}{2}I = 0 \tag{4.19}$$

via the mapping $v = -w_x w^{-1}$. Making the alternative choice $u = -\sigma^{-1}\sigma_x$ in equation (4.5) with $\alpha = 1/2$ thus leads to results equivalent to those already obtained by making the choice $u = -w_x w^{-1}$.

We return once again to equation (4.5) in order to make two final remarks. First of all, as noted in Remark 4.2, if we assume $\alpha \neq 1/2$ then we may set $C = 0$. From the relations $z_{xx} = z_x u$, $y_{xx} = uy_x$ and $\rho_x = z_x y_x$ we then obtain the following expressions for ρ_{xx}:

$$\rho_{xx} = 2z_x u y_x, \qquad \rho_{xx} = 2z_{xx} y_x, \qquad \rho_{xx} = 2z_x y_{xx}. \tag{4.20}$$

Thus, for nonsingular z_x and y_x, we have three equivalent expressions for u:

$$u = \frac{1}{2}z_x^{-1}\rho_{xx}y_x^{-1} = z_x^{-1}z_{xx} = y_{xx}y_x^{-1}. \tag{4.21}$$

Defining

$$S(\rho) = \left(z_x^{-1}\rho_{xx}y_x^{-1}\right)_x - \frac{1}{2}\left(z_x^{-1}\rho_{xx}y_x^{-1}\right)^2 = z_x^{-1}\left(\rho_{xxx} - \frac{3}{2}\rho_{xx}\rho_x^{-1}\rho_{xx}\right)y_x^{-1}, \tag{4.22}$$

we see that the result of substituting for u in (4.5) with $C = 0$ by any of the expressions (4.21) is equivalent to

$$z_x\left[\frac{1}{2}S(\rho) + E - \frac{1}{2}xI\right]y_x + \left(\frac{1}{2} - \alpha\right)\rho = 0, \tag{4.23}$$

this being valid also for $\alpha = 1/2$, or

$$\frac{1}{2}z_x^{-1}\left(\rho_{xxx}-\frac{3}{2}\rho_{xx}\rho_x^{-1}\rho_{xx}\right)y_x^{-1}+E-\frac{1}{2}xI+\left(\frac{1}{2}-\alpha\right)z_x^{-1}\rho y_x^{-1}=0. \tag{4.24}$$

We claim that (4.24) constitutes a matrix extension of Schwarzian P_{II} (2.6). Identifying $\frac{1}{2}z_x^{-1}\rho_{xx}y_x^{-1}=u$, matrix P_{II} is, by construction, a differential consequence of (4.24): we note in particular that $(\partial_x+A_u)(z_x^{-1}\rho y_x^{-1})=I$. In the scalar reduction, where we set $E=0$, $S(\rho)=\{\rho;x\}$ and we obtain $\frac{1}{2}\{\rho;x\}-\frac{1}{2}x+\left(\frac{1}{2}-\alpha\right)\rho/\rho_x=0$, i.e., (2.6) written in terms of ρ. Our second final remark is that for $\alpha=1/2$ and $C=0$, (4.5) also gives the special integral $u_x-u^2+E-\frac{1}{2}xI=0$ of matrix P_{II} (4.3) for this value of the parameter [7] (see also [5,6]).

5. Conclusions and discussion

We have presented a new approach to deriving BTs and auto-BTs for ODEs. Whilst our interest here has been focused on Painlevé equations, this technique is, in fact, much more widely applicable. In addition to a consideration of the second and fourth Painlevé equations, we have also applied our approach to a matrix second Painlevé equation. We have thus obtained a matrix analogue of the classical result relating solutions of the second Painlevé equation for parameter values $\alpha=0$ and $\alpha=1/2$. In obtaining this result, we have also derived a new matrix version of the second Painlevé equation, for which we have presented a Lax pair. In addition, we have derived the first known matrix analogue of the Schwarzian second Painlevé equation.

A particularly interesting consequence of our matrix results is the introduction of a new definition of matrix Schwarzian derivative, as follows. Given a square matrix function ρ of x such that ρ_x is nonsingular, we define its Schwarzian derivative $S(\rho)$ as

$$S(\rho)=\left(s^{-1}\rho_{xx}t^{-1}\right)_x-\frac{1}{2}\left(s^{-1}\rho_{xx}t^{-1}\right)^2, \tag{5.1}$$

where s and t are any two nonsingular square matrix functions of x such that $st=\rho_x$, $s_xt=st_x$. (Here s and t respectively replace z_x and y_x as used in Section 4.) We may also write $S(\rho)=\left(s^{-1}\rho_{xx}\rho_x^{-1}s\right)_x-\frac{1}{2}\left(s^{-1}\rho_{xx}\rho_x^{-1}s\right)^2$ and $s_x=\frac{1}{2}\rho_{xx}\rho_x^{-1}s$, or $S(\rho)=\left(t\rho_x^{-1}\rho_{xx}t^{-1}\right)_x-\frac{1}{2}\left(t\rho_x^{-1}\rho_{xx}t^{-1}\right)^2$ and $t_x=\frac{1}{2}t\rho_x^{-1}\rho_{xx}$. The formulation (5.1) is simpler than the Lagrange Schwarzian derivative discussed in [13], with fewer assumptions being made on ρ, and provides an alternative also to that given in [2]. The properties of this new matrix Schwarzian derivative are discussed in [14], where we also consider its use in defining matrix Schwarzian ODE and PDE hierarchies.

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