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The extended second Painlevé hierarchy: Auto-Bäcklund transformations and special integrals

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Abstract

We return to our study of the extended second Painlevé hierarchy presented in a previous paper. For this hierarchy we give a new local auto-BT. We also give an extensive discussion of the iterative construction of solutions and special integrals using auto-BTs. Furthermore, we show that Lax pairs can be provided for special integrals. Even though this will, in fact, be the case quite generally, it seems that Lax pairs for special integrals have not been given previously. Amongst the equations for which we present Lax pairs are examples due to Cosgrove and, in classical Painlevé classification results, Chazy.

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1. Introduction

As is well-known, interest in the Painlevé equations was reignited by the discovery in [1] of a link between these equations and completely integrable partial differential equations (PDEs). However, even though the second Painlevé hierarchy, derived by similarity reduction using the Korteweg-de Vries (KdV) and modified Korteweg-de Vries (mKdV) hierarchies, followed soon

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after [1–3], with results on auto-Bäcklund transformations (auto-BTs) for all members of this hierarchy also being derived in [2], nearly another two decades were to pass before interest in such Painlevé hierarchies was to really take off.

The second Painlevé hierarchy is a sequence of ordinary differential equations (ODEs), one of each even order $2, 4, 6, \ldots$, the first member being the second Painlevé equation (P_{II}),

$$v_{xx} = 2v^3 + xv + \alpha, \tag{1.1}$$

where α is an arbitrary parameter, and where — as throughout this paper — we use subscripts, not only for PDEs but also for ODEs, to denote derivatives. This equation has the two auto-BTs,

$$v = y + \frac{\alpha - \tilde{\alpha}}{2y_x - 2y^2 - x}, \qquad \alpha = -\tilde{\alpha} + 1, \tag{1.2}$$

$$v = -y, \qquad \alpha = -\tilde{\alpha}, \tag{1.3}$$

which provide mappings from a solution y of P_{II} for parameter value $\tilde{\alpha}$, i.e., a solution of

$$y_{xx} = 2y^3 + xy + \tilde{\alpha}, \tag{1.4}$$

to a solution v of P_{II} for parameter value α, i.e., of (1.1). It was these results on auto-BTs that Airault [2] extended to every member of the P_{II} hierarchy. This hierarchy may be written

$$\psi[v]\tilde{\mathcal{R}}^{n-1}[v]v_x + 2h_1 xv - \alpha_n = 0, \qquad n = 1, 2, \ldots, \tag{1.5}$$

where $\psi[v] = \partial_x - 4v\partial_x^{-1}v$, $\tilde{\mathcal{R}}[v] = \partial_x \psi[v] = \partial_x\left(\partial_x - 4v\partial_x^{-1}v\right)$ is the recursion operator of the mKdV hierarchy, h_1 is a nonzero constant and α_n is an arbitrary constant. Without loss of generality we may take $h_1 = -1/2$, in which case (1.5) for $n = 1$ and $\alpha_1 = \alpha$ gives the second Painlevé equation (1.1). We usually refer to the hierarchy (1.5) employed by Airault as the standard P_{II} hierarchy. When additional terms corresponding to lower order mKdV flows are also included in the equations of the P_{II} hierarchy, i.e., when it has the form

$$\psi[v]\sum_{k=1}^{n} c_k \tilde{\mathcal{R}}^{k-1}[v]v_x + c_0 v + 2h_1 xv - \alpha_n = 0, \qquad n = 1, 2, \ldots, \tag{1.6}$$

where all c_k, $k = 0, 1, \ldots, n$, are constant (and h_1 is a nonzero constant and α_n is an arbitrary constant), we refer to it as the generalized P_{II} hierarchy; see [3] and [4]. The hierarchy under consideration in the current paper, which was first introduced in [5], is a still further extension of Airault's P_{II} hierarchy.

As remarked above, even though the paper [1] led to a huge amount of interest in the Painlevé equations, the question of Painlevé hierarchies seems not to have generated much interest until the rediscovery in [6] of the standard second Painlevé hierarchy, along with a standard first Painlevé (P_I) hierarchy, i.e., a hierarchy having as first member an ODE equivalent to the first Painlevé equation $y_{tt} = 6y^2 + t$. In fact, both of these results can also be found in [7] (but with the P_{II} hierarchy having zero additive constant). However, this last paper does not seem to have had much impact — outside string theory — on those interested in the Painlevé equations.

In [8], an alternative form of the auto-BTs of the P_{II} hierarchy was given. Also, the results in [9] on the relationship between nonisospectral scattering problems and linear problems for Painlevé equations were used and extended in [10], in order to derive Painlevé hierarchies and simultaneously their underlying linear problems, or Lax pairs. To date, a great number of papers have been published on Painlevé hierarchies, of different types, i.e., not only purely differential but also discrete and differential-delay. The interest in deriving auto-BTs for Painlevé hierarchies can be seen from the fact that, even in the case of the (standard/generalized) second Painlevé hierarchy, auto-BTs as formulated by Airault have since been rederived in [11–14]. Similarly, the interest in deriving Lax pairs for Painlevé hierarchies can be seen, again citing here results only for the second Painlevé hierarchy itself, from their derivation/rederivation in [3,7] and [11,15,16].

The authors of the current paper have continued in their use of nonisospectral scattering problems in order to derive Painlevé hierarchies, as they believe that it has a number of advantages over other approaches. The first of these advantages is the simplicity with which it allows the derivation of Lax pairs for ODEs obtained as reductions. Since the Lax pairs for modified hierarchies (nonisospectral or otherwise) can be obtained from those of the original (unmodified) hierarchies using standard, long-established techniques, then Lax pairs for ODE reductions of nonisospectral modified hierarchies are also readily obtained; for examples we refer, for instance, to [5]. A second advantage is the simplicity with which certain ODE reductions are obtained, when compared to other methods. Whilst the derivation of a scaling similarity reduction of a PDE hierarchy can be quite straightforward, that of other similarity reductions may not be. Indeed, the question of whether a (non-scaling) similarity reduction for the first member of an isospectral PDE hierarchy can be extended to all members of that hierarchy may not be so trivial. For example, the question of how to derive a first Painlevé hierarchy by extending the accelerating wave reduction of the KdV equation to every member of the KdV hierarchy only seems to have been resolved in [17]; the derivation given in [6] made use of the so-called “singular manifold equations” of the KdV hierarchy, a step that would not be so easy to repeat for other PDE hierarchies. Much easier is the derivation of a P_I hierarchy using the non-isospectral approach [10] (see also [5], where it appears, as a special case, as equation (3.19)), or, indeed, the closely related approach employed in [7]. For further remarks on the relationship between the use of nonisospectral scattering problems and similarity reductions in the derivation of Painlevé hierarchies, we refer to [18]. A more substantial discussion of the various techniques used to derive Painlevé hierarchies and their properties, including “integration via modification” [19,20] and a method of obtaining (both continuous and discrete) auto-BTs [21], can be found in the recent review [22]; see this review also for a description of results for matrix examples, e.g., [23–28], a topic of much current interest.

Recently, we have also explored a still further advantage to the use of nonisospectral scattering problems in order to obtain Painlevé hierarchies. For example, in the KdV case, we might consider the extended KdV hierarchy

$$u_t = \sum_{k=0}^{n} c_k \mathcal{R}^k[u] u_x + \sum_{k=0}^{p} h_k \mathcal{R}^k[u] 1, \tag{1.7}$$

where $\mathcal{R}[u] = \partial_x^2 + 4u + 2u_x \partial_x^{-1}$ is the recursion operator of the KdV hierarchy and where all $c_k = c_k(t)$ and $h_k = h_k(t)$ are functions of t. This nonisospectral hierarchy can be found in [29]; the addition of terms of the form $\mathcal{R}^k[u]1$ to the KdV hierarchy (as well as for $2+1$ generalizations thereof) was also considered in [10]. Whilst corresponding ODE reductions, and their Lax pairs, were indeed considered in [10], emphasis was given to the case where $\mathcal{R}^k[u]1$ is local and so ODEs containing terms of this form with $k \geq 2$ were not studied in detail. However, in [5], we showed that (1.7) can be used to derive the new hierarchy

$$\psi[v]\left(\sum_{k=1}^{n} c_k \tilde{\mathcal{R}}^{k-1}[v]v_x + 2\sum_{k=2}^{p} h_k \tilde{\mathcal{R}}^{k-2}[v](xv)_x\right) + c_0 v + 2h_1 xv - \alpha_n = 0,$$
$$n = 1, 2, \ldots, \tag{1.8}$$

where now all c_k, $k = 0, 1, \ldots, n$, and all h_k, $k = 1, 2, \ldots, p$, are assumed constant, and where α_n is an arbitrary constant; in the non-autonomous case where at least one of $h_1, h_2, \ldots, h_p$ is nonzero, we defined (1.8) to be an extended P_{II} hierarchy. This may be considered to be a natural extension of the generalized (and standard) second Painlevé hierarchies, which may be obtained by setting $h_2 = h_3 = \cdots = h_p = 0$ and assuming $h_1 \neq 0$. It can be written locally using suitable auxiliary dependent variables. Details of its derivation and some of its properties, as given in [5], are recalled in Section 2. We note that preliminary results for the particular case $n = 2$ and $p = 2$ of (1.8) were also presented in [30].

In other early papers of the current authors on Painlevé hierarchies, terms corresponding to $\mathcal{R}^k[u]1$ as discussed above were also considered; see, e.g., [31]. That is, the structure used in the derivation of (1.8) is quite general, and can be used within the context of other hierarchies; one example of a new extended Painlevé hierarchy, along with properties such as auto-BTs and basic special integrals, can be found in [32]. Further examples will be presented shortly. We expect the study of such extended Painlevé hierarchies to be of relevance, both for the information that may be obtained as to which classes of ODE might be of interest for Painlevé classification, as well as for its possible usefulness in allowing ODEs isolated in a particular classification process — or by any other techniques — to be identified. This dual motivation will also hold in the cases of discrete and differential-delay examples.

In the present paper, we return to our study of the extended second Painlevé hierarchy. In Section 2 we give a brief summary of basic facts related to this hierarchy: its relationship to corresponding extensions of the KdV and mKdV hierarchies; basic special integrals; Lax pairs; and auto-BTs. In Section 3 we consider its auto-BTs anew, giving a new local form of the auto-BT g corresponding to the discrete symmetry $(v, \alpha_n) \to (-v, -\alpha_n)$ of (1.8), and in addition give extensive results on the iteration of solutions using auto-BTs. In Section 4 we consider special integrals of our extended second Painlevé hierarchy. We first of all consider iteration using auto-BTs to generate sequences of special integrals beginning with a basic special integral. We then discuss the question of Lax pairs for special integrals. We believe that Lax pairs for the special integrals of Painlevé hierarchies, or, indeed, Painlevé equations, have not been given previously. In Section 5, we consider as an example the extended sixth order second Painlevé equation and a basic special integral thereof, corresponding to the case $n = 3$ and $p = 3$ of (1.8), as well as results for extended lower-order second Painlevé equations obtained by setting $c_3 = 0$ and also $c_3 = c_2 = 0$. Amongst the examples discussed are equations obtained using Painlevé classification by Chazy and Cosgrove, which appear as basic special integrals in the case $c_3 = 0$, corresponding to the choices $h_3 = h_2 = 0$, and $h_3 = 0$ but $h_2 \neq 0$, respectively. For each of these previously known equations, with the equation due to Chazy being known for well over a hundred years, as well as for other examples, we can now give Lax pairs. Section 6 is used for conclusions and a brief discussion of our results.

2. Preliminaries: the extended second Painlevé hierarchy

In this section we recall some basic facts and results with regard to the extended second Painlevé (P_{II}) hierarchy derived in [5]. In addition to its relation to the extended Korteweg-de Vries (KdV) and extended modified Korteweg-de Vries (mKdV) hierarchies, we also discuss here basic special integrals, Lax pairs and auto-Bäcklund transformations. We begin with a description of the extended KdV and mKdV hierarchies, these being nonisospectral generalizations of the KdV and mKdV hierarchies. We recall the KdV and mKdV recursion operators and Hamiltonian structures, and of course the Miura map, as well as Lax pairs for the given nonisospectral extensions. We refer to [33–40], and in addition to [29,9,10] and [5] for the results presented in this section. In subsequent sections we will give new results for the extended P_{II} hierarchy.

2.1. Extended KdV hierarchy

Let us recall some basic details of the structure of the extended KdV hierarchy (1.7) [29,10,5], i.e.,

$$u_t = \sum_{k=0}^{n} c_k \mathcal{R}^k[u]u_x + \sum_{k=0}^{p} h_k \mathcal{R}^k[u]1, \tag{2.1}$$

or

$$u_t = \sum_{k=0}^{n} c_k \mathcal{R}^k[u]u_x + \sum_{k=2}^{p} h_k \mathcal{R}^k[u]1 + h_1(4u + 2xu_x) + h_0, \tag{2.2}$$

which was used in [5] as the starting point for the construction of our extended P_{II} hierarchy. The members of this hierarchy consist of a sum of standard KdV flows and nonisospectral terms, the respective coefficients of which, i.e., $c_k = c_k(t)$ and $h_k = h_k(t)$, are functions of t; here for simplicity of notation we use t as the time variable for all the flows of the hierarchy. As is well-known, the KdV recursion operator

$$\mathcal{R}[u] = \partial_x^2 + 4u + 2u_x\partial_x^{-1} \tag{2.3}$$

is the quotient $\mathcal{R}[u] = \mathcal{B}_1[u]\mathcal{B}_0^{-1}[u]$ of the two KdV Hamiltonian operators

$$\mathcal{B}_1[u] = \partial_x^3 + 4u\partial_x + 2u_x \qquad \text{and} \qquad \mathcal{B}_0[u] = \partial_x. \tag{2.4}$$

The bi-Hamiltonian structure of the standard KdV hierarchy $u_t = \mathcal{R}^n[u]u_x$ is expressed via the identities

$$\mathcal{R}^n[u]u_x = \mathcal{B}_0[u]M_{n+1}[u] = \mathcal{B}_1[u]M_n[u], \qquad n = 0, 1, 2, \ldots, \tag{2.5}$$

wherein the quantities $M_n[u]$ (defined by $M_0[u] = 1/2$ and by the Lenard recursion relation given by the last equality in equation (2.5)), i.e., the sequence

$$M_0[u]=\frac{1}{2},\quad M_1[u]=u,\quad M_2[u]=u_{xx}+3u^2,$$
$$M_3[u]=u_{xxxx}+10uu_{xx}+5u_x^2+10u^3,\quad \dots \tag{2.6}$$

are the variational derivatives of a corresponding sequence of Hamiltonian densities ($M_n[u]=\delta\mathcal{H}_n$) given by

$$\mathcal{H}_0=\frac{1}{2}u,\quad \mathcal{H}_1=\frac{1}{2}u^2,\quad \mathcal{H}_2=u^3-\frac{1}{2}u_x^2,\quad \mathcal{H}_3=\frac{5}{2}u^4-5uu_x^2+\frac{1}{2}u_{xx}^2,\quad \dots \tag{2.7}$$

We note that for the purposes of the present paper we do not need the expressions for the Hamiltonian densities.

Defining also, as in [5], the series of quantities w_k, $k=0,1,2\dots$, via

$$\mathcal{B}_0[u]w_{k+1}=\mathcal{B}_1[u]w_k,\qquad w_0=x, \tag{2.8}$$

so that

$$\mathcal{B}_1[u]w_{k-1}=\mathcal{R}^k[u]1,\qquad k=1,2,3\dots, \tag{2.9}$$

we see that we may also write equation (2.1) as

$$u_t=\sum_{k=0}^{n}c_k\mathcal{R}^k[u]u_x+\sum_{k=2}^{p}h_k\mathcal{B}_1[u]w_{k-1}+h_1\mathcal{B}_1[u]x+h_0. \tag{2.10}$$

This then leads to a local form of (2.1), expressed as a system in the variables u and $\mathbf{w}=(w_1,\dots,w_{p-1})$:

$$u_t=\mathcal{B}_1[u]K[u,\mathbf{w}]+h_0 \tag{2.11}$$

and

$$\left.\begin{array}{rcl} w_{1,x} &=& \mathcal{B}_1[u]x\\ w_{2,x} &=& \mathcal{B}_1[u]w_1\\ &\vdots& \\ w_{p-1,x} &=& \mathcal{B}_1[u]w_{p-2}\end{array}\right\} \tag{2.12}$$

where

$$K[u,\mathbf{w}]=\sum_{k=0}^{n}c_kM_k[u]+\sum_{k=2}^{p}h_kw_{k-1}+h_1x. \tag{2.13}$$

The extended KdV hierarchy has the nonisospectral Lax pair

$$\Psi_{xx}=(\lambda-u)\Psi, \tag{2.14}$$
$$\Psi_t=2P\Psi_x-P_x\Psi, \tag{2.15}$$

where $\lambda = \lambda(t)$ satisfies

$$\lambda_t = \sum_{k=0}^{p} h_k (4\lambda)^k, \tag{2.16}$$

and

$$P = \sum_{k=0}^{n} c_k P_k + \sum_{k=1}^{p} h_k \tilde{P}_k \tag{2.17}$$

with

$$P_k = \sum_{i=0}^{k} (4\lambda)^i M_{k-i}[u], \quad k = 0, 1, 2, \dots, \qquad \text{and}$$
$$\tilde{P}_k = \sum_{i=0}^{k-1} (4\lambda)^i w_{k-1-i}, \qquad k = 1, 2, 3 \dots. \tag{2.18}$$

The Lax pair (2.14), (2.15) is equivalent to the system

$$\begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}_x = \mathcal{F} \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}, \qquad \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}_t = \mathcal{G} \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}, \tag{2.19}$$

where $\Theta_1 = \Psi$, $\Theta_2 = \Psi_x$,

$$\mathcal{F} = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}, \tag{2.20}$$

and

$$\mathcal{G} = \begin{pmatrix} -P_x & 2P \\ 2\lambda P - P_{xx} - 2Pu & P_x \end{pmatrix}. \tag{2.21}$$

2.2. *Extended mKdV hierarchy*

The KdV hierarchy is related to the mKdV hierarchy via the Miura map

$$u = M[v] = v_x - v^2, \tag{2.22}$$

under which the Hamiltonian operator $\mathcal{B}_1[u]$ factorizes as

$$\mathcal{B}_1[u] \Big|_{u=M[v]} = M'[v]\mathcal{B}[v](M'[v])^\dagger, \tag{2.23}$$

where

$$M'[v] = \partial_x - 2v \tag{2.24}$$

is the Fréchet derivative of $M[v]$,

$$(M'[v])^\dagger = -\partial_x - 2v \tag{2.25}$$

is the adjoint of this Fréchet derivative, and

$$\mathcal{B}[v] = -\partial_x \tag{2.26}$$

is the Hamiltonian operator of the mKdV hierarchy.

The Miura map (2.22), beginning with (2.1) for the choice $h_0 = 0$, then yields the extended mKdV hierarchy

$$v_t = \sum_{k=0}^{n} c_k \tilde{\mathcal{R}}^k[v]v_x + 2\sum_{k=1}^{p} h_k \tilde{\mathcal{R}}^{k-1}[v](xv)_x, \tag{2.27}$$

or

$$v_t = \sum_{k=0}^{n} c_k \tilde{\mathcal{R}}^k[v]v_x + 2\sum_{k=2}^{p} h_k \tilde{\mathcal{R}}^{k-1}[v](xv)_x + 2h_1(xv)_x, \tag{2.28}$$

where the mKdV recursion operator $\tilde{\mathcal{R}}[v]$ is given by

$$\tilde{\mathcal{R}}[v] = \partial_x(\partial_x + 2v)\partial_x^{-1}(\partial_x - 2v). \tag{2.29}$$

Similarly to the extended KdV hierarchy, the members of this hierarchy consist of a sum of standard mKdV flows and nonisospectral terms, again with respective coefficient functions $c_k = c_k(t)$ and $h_k = h_k(t)$. A local form of this extended mKdV hierarchy, written as a system in v and $\mathbf{w} = (w_1, \ldots, w_{p-1})$, is given by:

$$v_t = \mathcal{B}[v](M'[v])^\dagger K[M[v], \mathbf{w}] \tag{2.30}$$

and

$$\left.\begin{array}{rcl} w_{1,x} &=& \mathcal{B}_1[M[v]]x \\ w_{2,x} &=& \mathcal{B}_1[M[v]]w_1 \\ &\vdots& \\ w_{p-1,x} &=& \mathcal{B}_1[M[v]]w_{p-2} \end{array}\right\} \tag{2.31}$$

where $K[u, \mathbf{w}]$ is as given in (2.13).

A matrix Lax pair for the extended mKdV hierarchy is obtained via factorization of the Schrödinger operator in (2.14) under $u = v_x - v^2$,

$$(\partial_x - v)(\partial_x + v)\Psi = \lambda\Psi. \tag{2.32}$$

Setting $\Psi_1 = \Psi$ and $\Psi_2 = (\partial_x + v)\Psi_1$ we obtain the system

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}_x = \mathbf{F} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \qquad \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}_t = \mathbf{G} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \tag{2.33}$$

where

$$\mathbf{F} = \begin{pmatrix} -v & 1 \\ \lambda & v \end{pmatrix} \tag{2.34}$$

and

$$\mathbf{G} = \begin{pmatrix} -P_x - 2Pv & 2P \\ 2\lambda P - (P_x + 2Pv)_x + \mathcal{B}[v](M'[v])^\dagger K[M[v], \mathbf{w}] & P_x + 2Pv \end{pmatrix}. \tag{2.35}$$

An equivalent Lax pair for the extended mKdV hierarchy may be obtained via the transformation

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \mu & \mu \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \tag{2.36}$$

where $\lambda = \mu^2$, which then yields the spatial part

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}_x = \begin{pmatrix} \mu & v \\ v & -\mu \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \tag{2.37}$$

with the temporal part being easily written down.

Finally, we note that, if we assume $c_n \neq 0$ in the extended KdV/mKdV hierarchies, then we may set $c_n = 1$ (by redefining t). However, in order to simplify the presentation of examples, we prefer not to make this choice.

2.3. The extended second Painlevé hierarchy

The extended P_{II} hierarchy was defined in [5] as the integrated stationary extended mKdV hierarchy,

$$\mathcal{Q}_{n,p}[v, \alpha_n] \equiv \psi[v]\left(\sum_{k=1}^{n} c_k \tilde{\mathcal{R}}^{k-1}[v]v_x + 2\sum_{k=2}^{p} h_k \tilde{\mathcal{R}}^{k-2}[v](xv)_x\right) + c_0 v + 2h_1 xv - \alpha_n = 0,$$
$$n = 1, 2, \ldots, \tag{2.38}$$

wherein all c_k, $k = 0, 1, \ldots, n$, and all h_k, $k = 1, 2, \ldots, p$, are now constant, and where α_n is an arbitrary constant of integration, in the nonautonomous case where at least one of $h_1, h_2, \ldots, h_p$ is nonzero; here

$$\psi[v] = (\partial_x + 2v)\partial_x^{-1}(\partial_x - 2v) = \partial_x - 4v\partial_x^{-1}v, \tag{2.39}$$

so (see (2.29))

$$\tilde{\mathcal{R}}[v] = \partial_x \psi[v]. \tag{2.40}$$

This is the hierarchy (1.8). Equation (2.38) can also be written as a local system in v and $\mathbf{w} = (w_1, \ldots, w_{p-1})$:

$$(\partial_x + 2v)K[M[v], \mathbf{w}] - (h_1 + \alpha_n) = 0 \tag{2.41}$$

$$\left.\begin{array}{rcl} w_{1,x} & = & \mathcal{B}_1[M[v]]x \\ w_{2,x} & = & \mathcal{B}_1[M[v]]w_1 \\ & \vdots & \\ w_{p-1,x} & = & \mathcal{B}_1[M[v]]w_{p-2} \end{array}\right\} \tag{2.42}$$

where $K[u, \mathbf{w}]$ is as given in (2.13). In the special case where $h_2 = h_3 = \cdots = h_p = 0$ and $h_1 \neq 0$ the resulting scalar equation is just the generalized P_{II} hierarchy [3,4],

$$\psi[v] \sum_{k=1}^{n} c_k \tilde{\mathcal{R}}^{k-1}[v] v_x + c_0 v + 2h_1 x v - \alpha_n = 0, \qquad n = 1, 2, \ldots, \tag{2.43}$$

which in turn gives the standard P_{II} hierarchy [1–3] when $c_0 = c_1 = \cdots = c_{n-1} = 0$.

2.4. *Extended P_{II} hierarchy: basic special integrals*

We recall the basic special integrals of the extended second Painlevé hierarchy, as defined in [5].

Remark 2.4.1. [5] The basic special integrals of the extended P_{II} hierarchy are as given by the system

$$K[M[v], \mathbf{w}] = \sum_{k=0}^{n} c_k M_k[M[v]] + \sum_{k=2}^{p} h_k w_{k-1} + h_1 x = 0 \tag{2.44}$$

$$\left.\begin{array}{rcl} w_{1,x} & = & \mathcal{B}_1[M[v]]x \\ w_{2,x} & = & \mathcal{B}_1[M[v]]w_1 \\ & \vdots & \\ w_{p-1,x} & = & \mathcal{B}_1[M[v]]w_{p-2} \end{array}\right\} \tag{2.45}$$

It is clear that solutions of this system give solutions of (2.41), (2.42) for parameter value $\alpha_n = -h_1$.

Remark 2.4.2. [5] The system (2.44), (2.45) is equivalent to

$$K[u, \mathbf{w}] = \sum_{k=0}^{n} c_k M_k[u] + \sum_{k=2}^{p} h_k w_{k-1} + h_1 x = 0, \tag{2.46}$$

$$\left.\begin{array}{rcl} w_{1,x} & = & \mathcal{B}_1[u]x \\ w_{2,x} & = & \mathcal{B}_1[u]w_1 \\ & \vdots & \\ w_{p-1,x} & = & \mathcal{B}_1[u]w_{p-2} \end{array}\right\} \tag{2.47}$$

$$u = v_x - v^2, \tag{2.48}$$

Equation (2.48) can be linearized via $v = -\psi_x/\psi$ onto

$$\psi_{xx} + u\psi = 0. \tag{2.49}$$

Thus, given a solution u, $\mathbf{w}$ of (2.46), (2.47) we can obtain a solution v, $\mathbf{w}$ of (2.44), (2.45) by taking $v = -\psi_x/\psi$ where ψ satisfies the linear equation (2.49).

Remark 2.4.3. For the special case of the standard second Painlevé hierarchy, i.e., $h_2 = h_3 = \cdots = h_p = 0$ and $h_1 \neq 0$, and also $c_0 = c_1 = \cdots = c_{n-1} = 0$, it was observed in [6] that solutions for parameter value $\alpha_n = -h_1$ may be obtained from the first Painlevé hierarchy coupled to the Miura map. For our extended second Painlevé hierarchy, however, such a simple decoupling of the basic special integral is no longer possible.

2.5. *Extended P_{II} hierarchy: Lax pairs*

Let us recall the (two equivalent) Lax pairs given in [5] for our extended P_{II} hierarchy.

Theorem 2.5.1. *[5] A Lax pair for the hierarchy (2.41), (2.42) is given by*

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}_x = \mathbf{F} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \qquad \left(\sum_{k=1}^{p} h_k (4\lambda)^k\right) \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}_\lambda = \tilde{\mathbf{G}} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \tag{2.50}$$

where $\mathbf{F}$ *is given by (2.34),*

$$\tilde{\mathbf{G}} = \mathbf{G} + \begin{pmatrix} [(\partial_x + 2v)K[M[v], \mathbf{w}] - (h_1 + \alpha_n)] & 0 \\ 0 & -[(\partial_x + 2v)K[M[v], \mathbf{w}] - (h_1 + \alpha_n)] \end{pmatrix} \tag{2.51}$$

with $\mathbf{G}$ *given by (2.35), and where all c_k, $k = 0, 1, \ldots, n$, and all h_k, $k = 1, 2, \ldots, p$, are assumed constant. The compatibility condition of this Lax pair reads*

$$S^{(n,p)} \equiv \left(\sum_{k=1}^{p} h_k (4\lambda)^k\right) \mathbf{F}_\lambda - \tilde{\mathbf{G}}_x + [\mathbf{F}, \tilde{\mathbf{G}}] = 0. \tag{2.52}$$

Theorem 2.5.2. *[5] The hierarchy (2.41), (2.42) also has an equivalent Lax pair with x-part*

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}_x = \begin{pmatrix} \mu & v \\ v & -\mu \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \tag{2.53}$$

where $\lambda = \mu^2$ (obtained via the transformation (2.36) given in Section 2.2).

2.6. Extended P_{II} hierarchy: auto-Bäcklund transformations

Let us also recall some of the results on auto-BTs given in [5].

Theorem 2.6.1. *[5] (auto-BT* **f***) Consider a copy of the system (2.41), (2.42) written in terms of variables y and $\mathbf{z} = (z_1, \dots, z_{p-1})$, and with parameter $\tilde{\alpha}_n$, i.e.,*

$$(\partial_x + 2y)K[M[y], \mathbf{z}] - (h_1 + \tilde{\alpha}_n) = 0 \tag{2.54}$$

$$\left.\begin{array}{rcl} z_{1,x} &=& \mathcal{B}_1[M[y]]x \\ z_{2,x} &=& \mathcal{B}_1[M[y]]z_1 \\ &\vdots& \\ z_{p-1,x} &=& \mathcal{B}_1[M[y]]z_{p-2} \end{array}\right\} \tag{2.55}$$

Then solutions y, $\mathbf{z}$ of this system are mapped by the auto-BT

$$v = y + \frac{\alpha_n - \tilde{\alpha}_n}{2K[M[y], \mathbf{z}]}, \tag{2.56}$$

$$w_k = z_k, \qquad k = 1, \dots, p-1, \tag{2.57}$$

$$\alpha_n = -\tilde{\alpha}_n - 2h_1, \tag{2.58}$$

onto solutions v, $\mathbf{w}$ of the system (2.41), (2.42). This is our auto-BT f, and it is an involution.

Theorem 2.6.2. *[5] (auto-BT* **g***) Consider a copy of equation (2.38) written in terms of the variable y, and with parameter $\tilde{\alpha}_n$, i.e.,*

$$\mathcal{Q}_{n,p}[y, \tilde{\alpha}_n] = \psi[y]\left(\sum_{k=1}^{n} c_k \tilde{\mathcal{R}}^{k-1}[y]y_x + 2\sum_{k=2}^{p} h_k \tilde{\mathcal{R}}^{k-2}[y](xy)_x\right) + c_0 y + 2h_1 xy - \tilde{\alpha}_n = 0,$$
$$n = 1, 2, \dots. \tag{2.59}$$

Then solutions y of (2.59) are mapped onto solutions v of (2.38) by the auto-BT

$$v = -y, \qquad \alpha_n = -\tilde{\alpha}_n. \tag{2.60}$$

This is our auto-BT g for equation (2.38), and it is an involution.

3. A new local auto-Bäcklund transformation and iteration

We now give new results for the extended P_{II} hierarchy (2.41), (2.42). In Section 3.1 we give a new local form (i.e., written in terms of the variables v, $\mathbf{w}$ of the local system (2.41), (2.42)) of the auto-BT g for equation (2.38) given in Theorem 2.6.2. Unlike the local form given in [5], this is not defined by means of an alternative Miura map, but instead is given as a direct mapping between two different copies of the same hierarchy (2.41), (2.42). In Section 3.2 we give new results on the iteration of the auto-BTs of the hierarchy (2.41), (2.42). The results presented in Sections 3.1 and 3.2 will prove useful for our later discussion of special integrals.

3.1. A new local auto-Bäcklund transformation

Theorem 3.1.1. ***(new local form of auto-BT* g*)*** *A local form of the auto-BT g, providing a mapping from solutions* y, $\mathbf{z}$ *of the system (2.54), (2.55) to solutions* v, $\mathbf{w}$ *of the system (2.41), (2.42) is given by*

$$v = -y, \qquad \alpha_n = -\tilde{\alpha}_n, \tag{3.1}$$

together with

$$w_j = z_j - 2\partial_x(\partial_x + 2y)z_{j-1}, \qquad j = 1, 2, \dots, p-1 \tag{3.2}$$

(where $z_0 = x$*, and so for* $j = 1$ *this last equation reads* $w_1 = z_1 - 4(xy)_x$*).*

Proof. First of all we note that, since $\mathcal{B}_1[M[y]] = (\partial_x - 2y)\partial_x(\partial_x + 2y)$ (see the factorization (2.23)), then

$$\mathcal{B}_1[M[-y]] = (\partial_x + 2y)\partial_x(\partial_x - 2y). \tag{3.3}$$

In addition, from the definition of $\mathcal{B}_1[u]$ in (2.4) we see that

$$\mathcal{B}_1[M[-y]] = \mathcal{B}_1[M[y] - 2y_x] = \mathcal{B}_1[M[y]] - 8y_x\partial_x - 4y_{xx}. \tag{3.4}$$

Thus, under this auto-BT,

$$\begin{aligned} w_{1,x} - \mathcal{B}_1[M[v]]x &= (z_1 - 4(xy)_x)_x - \mathcal{B}_1[M[-y]]x \\ &= z_{1,x} - 4(xy)_{xx} - \big(\mathcal{B}_1[M[y]] - 8y_x\partial_x - 4y_{xx}\big)x \\ &= z_{1,x} - \mathcal{B}_1[M[y]]x, \end{aligned} \tag{3.5}$$

and, for $j \geq 2$ (with $z_0 = x$),

$$\begin{aligned} &w_{j,x} - \mathcal{B}_1[M[v]]w_{j-1} \\ &= \big(z_j - 2\partial_x(\partial_x + 2y)z_{j-1}\big)_x - \mathcal{B}_1[M[-y]]\big(z_{j-1} - 2\partial_x(\partial_x + 2y)z_{j-2}\big) \\ &= z_{j,x} - 2(\partial_x + 2y)\partial_x z_{j-1,x} - (8y_x\partial_x + 4y_{xx})z_{j-1} \\ &\quad - \big(\mathcal{B}_1[M[y]] - 8y_x\partial_x - 4y_{xx}\big)z_{j-1} + 2\mathcal{B}_1[M[-y]]\partial_x(\partial_x + 2y)z_{j-2} \\ &= z_{j,x} - 2(\partial_x + 2y)\partial_x z_{j-1,x} - \mathcal{B}_1[M[y]]z_{j-1} \\ &\quad + 2(\partial_x + 2y)\partial_x(\partial_x - 2y)\partial_x(\partial_x + 2y)z_{j-2} \\ &= z_{j,x} - \mathcal{B}_1[M[y]]z_{j-1} - 2(\partial_x + 2y)\partial_x\big(z_{j-1,x} - \mathcal{B}_1[M[y]]z_{j-2}\big). \end{aligned} \tag{3.6}$$

We now turn to equation (2.41). First we show by induction that $(\partial_x - 2y)M_k[M[-y]] = -(\partial_x + 2y)M_k[M[y]]$ for $k = 0, 1, \ldots$[1] This is easily seen to hold for $k = 0$, since $M_0[u] = 1/2$ and left- and right-hand sides of the claimed identity both give $-y$. Then, assuming now that this identity holds for $k = j$, the Lenard recursion relation given by the last equality in equation (2.5), along with the factorization (2.23), gives

$$M_{j+1}[M[y]] = \partial_x^{-1}(\partial_x - 2y)\partial_x(\partial_x + 2y)M_j[M[y]] \tag{3.7}$$

and so

$$\begin{aligned}(\partial_x - 2y)M_{j+1}[M[-y]] &= (\partial_x - 2y)\partial_x^{-1}(\partial_x + 2y)\partial_x(\partial_x - 2y)M_j[M[-y]] \\ &= -(\partial_x - 2y)\partial_x^{-1}(\partial_x + 2y)\partial_x(\partial_x + 2y)M_j[M[y]] \\ &= -(\partial_x + 2y)\partial_x^{-1}(\partial_x - 2y)\partial_x(\partial_x + 2y)M_j[M[y]],\end{aligned} \tag{3.8}$$

since $(\partial_x - 2y)\partial_x^{-1}(\partial_x + 2y) = (\partial_x + 2y)\partial_x^{-1}(\partial_x - 2y)$ (this corresponds to the invariance of $\psi[y]$ under $y \to -y$, see equation (2.39)). This last equation is just

$$(\partial_x - 2y)M_{j+1}[M[-y]] = -(\partial_x + 2y)M_{j+1}[M[y]], \tag{3.9}$$

and so we see that the identity holds for $k = j + 1$. Thus it holds for $k = 0, 1, \ldots$, as claimed. From the above we see that, under this auto-BT,

$$\begin{aligned}&(\partial_x + 2v)K[M[v], \mathbf{w}] - (h_1 + \alpha_n) \\ &= (\partial_x + 2v)\left(\sum_{k=0}^{n} c_k M_k[M[v]] + \sum_{k=2}^{p} h_k w_{k-1} + h_1 x\right) - (h_1 + \alpha_n) \\ &= (\partial_x - 2y)\sum_{k=0}^{n} c_k M_k[M[-y]] \\ &\quad +(\partial_x - 2y)\left(\sum_{k=2}^{p} h_k\big(z_{k-1} - 2\partial_x(\partial_x + 2y)z_{k-2}\big) + h_1 x\right) - (h_1 - \tilde{\alpha}_n) \\ &= -(\partial_x + 2y)\sum_{k=0}^{n} c_k M_k[M[y]] \\ &\quad +(\partial_x - 2y)\left(\sum_{k=2}^{p} h_k z_{k-1} + h_1 x\right) - 2\sum_{k=2}^{p} h_k(\partial_x - 2y)\partial_x(\partial_x + 2y)z_{k-2} \\ &\quad -(h_1 - \tilde{\alpha}_n) \\ &= -(\partial_x + 2y)\sum_{k=0}^{n} c_k M_k[M[y]]\end{aligned}$$

[1] We could alternatively appeal to the same invariance of the mKdV hierarchy, or the corresponding part of equation (2.59).

$$
\begin{aligned}
&+(\partial_x - 2y)\left(\sum_{k=2}^{p} h_k z_{k-1} + h_1 x\right) - 2\sum_{k=2}^{p} h_k \mathcal{B}_1[M[y]]z_{k-2} - (h_1 - \tilde{\alpha}_n)\\
&= -(\partial_x + 2y)\left(\sum_{k=0}^{n} c_k M_k[M[y]] + \sum_{k=2}^{p} h_k z_{k-1} + h_1 x\right)\\
&\quad +2\sum_{k=2}^{p} h_k\big(z_{k-1,x} - \mathcal{B}_1[M[y]]z_{k-2}\big) + (h_1 + \tilde{\alpha}_n)\\
&= -\Big((\partial_x + 2y)K[M[y],\mathbf{z}] - (h_1 + \tilde{\alpha}_n)\Big)\\
&\quad +2\sum_{k=2}^{p} h_k\big(z_{k-1,x} - \mathcal{B}_1[M[y]]z_{k-2}\big). \qquad (3.10)
\end{aligned}
$$

From the above results we see that, under the local form of the auto-BT g, if equations (2.54) and (2.55) are satisfied, then so are equations (2.41) and (2.42). □

Theorem 3.1.2. *The auto-BT given in Theorem 3.1.1 is an involution.*

Proof. In order to see this, let us consider a second iteration of the local form of the auto-BT g from the system (2.41), (2.42) to a solution of the same system written in terms of variables $\widehat{v}$ and $\widehat{\mathbf{w}} = (\widehat{w}_1, \dots, \widehat{w}_{p-1})$, and with parameter $\widehat{\alpha}_n$. We then obtain

$$
\widehat{v} = -v = y \qquad \text{and} \qquad \widehat{\alpha}_n = -\alpha_n = \tilde{\alpha}_n. \qquad (3.11)
$$

In addition,

$$
\widehat{w}_1 = w_1 - 4(xv)_x = z_1 - 4(xy)_x + 4(xy)_x = z_1, \qquad (3.12)
$$

and, for $j \geq 2$ (with $z_0 = x$),

$$
\begin{aligned}
\widehat{w}_j &= w_j - 2\partial_x(\partial_x + 2v)w_{j-1}\\
&= z_j - 2\partial_x(\partial_x + 2y)z_{j-1} - 2\partial_x(\partial_x - 2y)\big(z_{j-1} - 2\partial_x(\partial_x + 2y)z_{j-2}\big)\\
&= z_j - 4\partial_x\Big(z_{j-1,x} - (\partial_x - 2y)\partial_x(\partial_x + 2y)z_{j-2}\Big)\\
&= z_j - 4\partial_x\Big(z_{j-1,x} - \mathcal{B}_1[M[y]]z_{j-2}\Big) = z_j, \qquad (3.13)
\end{aligned}
$$

since the starting point for our iteration is a solution y, $\mathbf{z} = (z_1, \dots, z_{p-1})$ of the system (2.54), (2.55). □

3.2. Iteration of auto-Bäcklund transformations

Remark 3.2.1. (Group of auto-BTs) The group of auto-BTs of our extended second Painlevé hierarchy has generators f and g subject to the relations $f^2 = 1$ and $g^2 = 1$, i.e., it has the presentation

$$G = \langle f, g\ ;\ f^2 = g^2 = 1\rangle, \tag{3.14}$$

and is isomorphic, as in the case of the second Painlevé equation, to the affine Weyl group of type $A_1^{(1)}$.

Remark 3.2.2. (Composition of auto-BTs) In order to discuss the iteration of the auto-BTs f and g, let us consider the two composite auto-BTs $r = gf$ and $s = fg$, which we again express as mappings from solutions y, $\mathbf{z}$ of the system (2.54), (2.55) to solutions v, $\mathbf{w}$ of the system (2.41), (2.42):

$$r = gf: \qquad v = -y + \frac{\alpha_n + \tilde{\alpha}_n}{2K[M[y], \mathbf{z}]}, \tag{3.15}$$

$$w_k = z_k - 2\partial_x \left(\partial_x + 2y - \frac{\alpha_n + \tilde{\alpha}_n}{K[M[y], \mathbf{z}]}\right) z_{k-1}, \qquad k = 1, \ldots, p-1, \tag{3.16}$$

$$\alpha_n = \tilde{\alpha}_n + 2h_1, \tag{3.17}$$

and

$$s = fg: \qquad v = -y + \frac{\alpha_n + \tilde{\alpha}_n}{2K[M[-y], \mathbf{Z}]}, \qquad \mathbf{Z} = (z_1, \ldots, z_{p-1}) - 2\partial_x\left(\partial_x + 2y\right)(z_0, \ldots, z_{p-2}), \tag{3.18}$$

$$w_k = z_k - 2\partial_x\left(\partial_x + 2y\right) z_{k-1}, \quad k = 1, \ldots, p-1, \tag{3.19}$$

$$\alpha_n = \tilde{\alpha}_n - 2h_1 \tag{3.20}$$

(where $z_0 = x$). These transformations are inverse to each other: $rs = (gf)(fg) = gf^2g = g^2 = 1$. From the defining relations of the group G, it can then be seen that any composition of f and g can be written in one of the following forms:

$$f^\epsilon (gf)^q = f^\epsilon r^q, \qquad \epsilon \in \{0, 1\}, q \in \{0, 1, 2 \ldots\}; \tag{3.21}$$

$$g^\epsilon (fg)^q = g^\epsilon s^q, \qquad \epsilon \in \{0, 1\}, q \in \{0, 1, 2 \ldots\} \tag{3.22}$$

(each of which gives the identity transformation when $\epsilon = q = 0$).

The following will be of relevance when discussing (starting points for) the iteration of solutions:

Remark 3.2.3. The extended second Painlevé hierarchy (2.41), (2.42) has the solution $v = 0$ and $\mathbf{w}$ constant for parameter value $\alpha_n = 0$. This hierarchy also has solutions defined by the basic special integral (2.44), (2.45) for parameter value $\alpha_n = -h_1$. Note that in order for $v = 0$ and $\mathbf{w}$ constant to be a solution of (2.44), (2.45) we must have $h_1 = 0$ (so then $\alpha_n = 0$) and also that the constraint $\frac{1}{2}c_0 + \sum_{k=2}^{p} h_k w_{k-1} = 0$ holds.

Let us now assume $h_1 \neq 0$, in which case we may take $h_1 = -1/2$. We consider the iteration of solutions of (2.54), (2.55) beginning with seed solutions for initial parameter values $\tilde{\alpha}_n = \beta = 0$ and $\tilde{\alpha}_n = \beta = \frac{1}{2}$. The motivation for this lies in Remark 3.2.3 and the usual choices of initial parameter values for the iteration of solutions of the non-extended second Painlevé hierarchy, and in particular of the second Painlevé equation itself.

Remark 3.2.4. Take $h_1 = -1/2$. Then: the first composition (3.21) maps a solution of (2.54), (2.55) for initial parameter value $\tilde{\alpha}_n = \beta$ to a solution of (2.54), (2.55) for parameter value either $\tilde{\alpha}_n = \beta - q$ (if $\epsilon = 0$) or $\tilde{\alpha}_n = -\beta + q + 1$ (if $\epsilon = 1$); the second composition (3.22) maps a solution of (2.54), (2.55) for initial parameter value $\tilde{\alpha}_n = \beta$ to a solution of (2.54), (2.55) for parameter value either $\tilde{\alpha}_n = \beta + q$ (if $\epsilon = 0$) or $\tilde{\alpha}_n = -\beta - q$ (if $\epsilon = 1$).

Lemma 3.2.5. *Take $h_1 = -1/2$ and consider the iteration of solutions of (2.54), (2.55) beginning with a seed solution y_0, $\mathbf{z}_0$ for initial parameter value $\tilde{\alpha}_n = \beta = 0$. The composition (3.21) with $\epsilon = 1$ and $q = t - 1 \geq 0$ yields solutions $y_1 = fr^{t-1}y_0$, $\mathbf{z}_1 = fr^{t-1}\mathbf{z}_0$ of (2.54), (2.55) for integer parameter value $\tilde{\alpha}_n = t \geq 1$, and the composition (3.22) with $\epsilon = 0$ and $q = t \geq 1$ yields solutions $y_2 = s^t y_0$, $\mathbf{z}_2 = s^t \mathbf{z}_0$ of (2.54), (2.55) for integer parameter value $\tilde{\alpha}_n = t \geq 1$. Similarly, the composition (3.21) with $\epsilon = 0$ and $q = t \geq 0$ yields solutions $y_3 = r^t y_0$, $\mathbf{z}_3 = r^t \mathbf{z}_0$ of (2.54), (2.55) for integer parameter value $\tilde{\alpha}_n = -t \leq 0$, and the composition (3.22) with $\epsilon = 1$ and $q = t \geq 0$ yields solutions $y_4 = gs^t y_0$, $\mathbf{z}_4 = gs^t \mathbf{z}_0$ of (2.54), (2.55) for integer parameter value $\tilde{\alpha}_n = -t \leq 0$. The solutions y_1, $\mathbf{z}_1$ and y_2, $\mathbf{z}_2$ obtained for each positive integer parameter value $\tilde{\alpha}_n = t \geq 1$, and the solutions y_3, $\mathbf{z}_3$ and y_4, $\mathbf{z}_4$ obtained for each non-positive integer parameter value $\tilde{\alpha}_n = -t \leq 0$, satisfy:*

(a) $y_1 = y_2$ and $\mathbf{z}_1 = \mathbf{z}_2$ iff $y_0 = 0$ and $\mathbf{z}_0$ is constant;

(b) $y_3 = y_4$ and $\mathbf{z}_3 = \mathbf{z}_4$ iff $y_0 = 0$ and $\mathbf{z}_0$ is constant.

Proof. We begin by noting that

$y_1 = y_2$ and $\mathbf{z}_1 = \mathbf{z}_2$ iff $fr^{t-1}y_0 = s^t y_0$ and $fr^{t-1}\mathbf{z}_0 = s^t \mathbf{z}_0$ iff $y_0 = gy_0$ and $\mathbf{z}_0 = g\mathbf{z}_0$,

and

$y_3 = y_4$ and $\mathbf{z}_3 = \mathbf{z}_4$ iff $r^t y_0 = gs^t y_0$ and $r^t \mathbf{z}_0 = gs^t \mathbf{z}_0$ iff $y_0 = gy_0$ and $\mathbf{z}_0 = g\mathbf{z}_0$.

Further,

solutions y_0, $\mathbf{z}_0 = (z_{1,0}, z_{2,0}, \ldots, z_{p-1,0})$ of (2.54), (2.55) for parameter $\tilde{\alpha}_n = 0$ satisfy $y_0 = gy_0$ and $\mathbf{z}_0 = g\mathbf{z}_0$

iff $y_0 = -y_0$ and $z_{j,0} = z_{j,0} - 2\partial_x(\partial_x + 2y_0)z_{j-1,0}$, $j = 1, 2, \ldots, p-1$ (where it is understood that $z_{0,0} = z_0 = x$)

iff $y_0 = 0$ and $\partial_x^2 z_{j-1,0} = 0$, $j = 1, 2, \ldots, p-1$ (note that here $j = 1$ just corresponds to $\partial_x^2 z_0 = \partial_x^2(x) = 0$)

iff $y_0 = 0$ and $\partial_x z_{j,0} = 0$, $j = 1, 2, \ldots, p-1$ (making use of (2.54), (2.55) with $y = 0$, $M[y] = 0$ and $\tilde{\alpha}_n = 0$). □

We thus obtain:

Theorem 3.2.6. *Take $h_1 = -1/2$. Then, given a seed solution y_0, $\mathbf{z}_0$ for initial parameter value $\tilde{\alpha}_n = \beta = 0$, combinations of the auto-BTs f and g yield, for each integer parameter value $\tilde{\alpha}_n$, either: exactly one solution of (2.54), (2.55), when $y_0 = 0$ and $\mathbf{z}_0$ is constant; or two distinct solutions of (2.54), (2.55), otherwise.*

Corollary 3.2.7. *Take $h_1 = -1/2$. Then, beginning with the seed solution $y_0 = 0$ and $\mathbf{z}_0$ constant for initial parameter value $\tilde{\alpha}_n = \beta = 0$, iteration of the auto-BTs f and g yields one rational solution of the extended P_{II} hierarchy (2.54), (2.55) for each integer parameter value $\tilde{\alpha}_n$.*

Remark 3.2.8. For the non-extended second Painlevé equation itself, this Corollary is a well-known result. We do not address here the question of whether there are any other rational solutions of the extended P_{II} hierarchy in the case $h_1 \neq 0$; for the non-extended second Painlevé equation, it is known there are not [41].

Lemma 3.2.9. *Take $h_1 = -1/2$ and consider the iteration of solutions of (2.54), (2.55), beginning with a seed solution y_0, $\mathbf{z}_0$ for initial parameter value $\tilde{\alpha}_n = \beta = \frac{1}{2}$. The composition (3.21) with $\epsilon = 1$ and $q = t \geq 0$ yields solutions $y_1 = fr^t y_0$, $\mathbf{z}_1 = fr^t \mathbf{z}_0$ of (2.54), (2.55) for half-odd-integer parameter value $\tilde{\alpha}_n = t + \frac{1}{2}$, and the composition (3.22) with $\epsilon = 0$ and $q = t \geq 0$ yields solutions $y_2 = s^t y_0$, $\mathbf{z}_2 = s^t \mathbf{z}_0$ of (2.54), (2.55) for half-odd-integer parameter value $\tilde{\alpha}_n = t + \frac{1}{2}$. Similarly, the composition (3.21) with $\epsilon = 0$ and $q = t + 1 \geq 1$ yields solutions $y_3 = r^{t+1} y_0$, $\mathbf{z}_3 = r^{t+1}\mathbf{z}_0$ of (2.54), (2.55) for half-odd-integer parameter value $\tilde{\alpha}_n = -t - \frac{1}{2}$, and the composition (3.22) with $\epsilon = 1$ and $q = t \geq 0$ yields solutions $y_4 = gs^t y_0$, $\mathbf{z}_4 = gs^t \mathbf{z}_0$ of (2.54), (2.55) for half-odd-integer parameter value $\tilde{\alpha}_n = -t - \frac{1}{2}$. The solutions y_1, $\mathbf{z}_1$ and y_2, $\mathbf{z}_2$ obtained for each positive half-odd-integer parameter value $\tilde{\alpha}_n = t + \frac{1}{2}$, $t \geq 0$, and the solutions y_3, $\mathbf{z}_3$ and y_4, $\mathbf{z}_4$ obtained for each negative half-odd-integer parameter value $\tilde{\alpha}_n = -t - \frac{1}{2}$, $t \geq 0$, satisfy:*

(a) $y_1 = y_2$ and $\mathbf{z}_1 = \mathbf{z}_2$;

(b) $y_3 = y_4$ and $\mathbf{z}_3 = \mathbf{z}_4$.

Proof. We begin by noting that

$y_1 = y_2$ and $\mathbf{z}_1 = \mathbf{z}_2$ iff $fr^t y_0 = s^t y_0$ and $fr^t \mathbf{z}_0 = s^t \mathbf{z}_0$ iff $fy_0 = y_0$ and $f\mathbf{z}_0 = \mathbf{z}_0$,

and

$y_3 = y_4$ and $\mathbf{z}_3 = \mathbf{z}_4$ iff $r^{t+1} y_0 = gs^t y_0$ and $r^{t+1}\mathbf{z}_0 = gs^t \mathbf{z}_0$ iff $fy_0 = y_0$ and $f\mathbf{z}_0 = \mathbf{z}_0$.

The condition $fy_0 = y_0$ is satisfied since if, in the auto-BT f, we have $\tilde{\alpha}_n = \frac{1}{2}$, then $\alpha_n = -\tilde{\alpha} - 2h_1 = -\frac{1}{2} + 1 = \frac{1}{2}$ and so $v = y$ (and we may define $v = y$ even in the case $K[M[y], \mathbf{z}] = 0$). The condition $f\mathbf{z}_0 = \mathbf{z}_0$ is satisfied since, in the auto-BT f, we have $w_k = z_k$, $k = 1, 2, \dots, p-1$. □

We thus obtain:

Theorem 3.2.10. *Take $h_1 = -1/2$. Then, given a seed solution y_0, $\mathbf{z}_0$ for initial parameter value $\tilde{\alpha}_n = \beta = \frac{1}{2}$ (where in the case $K[M[y_0], \mathbf{z}_0] = 0$ we may define $fy_0 = y_0$), combinations of the auto-BTs f and g yield, for each half-odd-integer parameter value $\tilde{\alpha}_n$, exactly one solution of (2.54), (2.55).*

Remark 3.2.11. Take $h_1 = -1/2$. In the previous theorem, the case where the seed solution y_0, $\mathbf{z}_0$ satisfies $K[M[y_0], \mathbf{z}_0] = 0$ corresponds to choosing this seed solution y_0, $\mathbf{z}_0$ to be a solution of the basic special integral

$$K[M[y],\mathbf{z}]=0 \qquad \text{and} \qquad z_{j,x}=\mathcal{B}_1[M[y]]z_{j-1}, \quad j=1,2,\ldots,p-1 \tag{3.23}$$

(where $z_0=x$) for initial parameter value $\tilde{\alpha}_n=\beta=\frac{1}{2}$. (For the non-extended second Painlevé equation itself, the sequence of solutions thus obtained are just the well-known Airy function solutions of this equation.)

We now turn to the case $h_1=0$; we must then have $p\geq 2$ and at least one of $h_2,h_3,\ldots,h_p$ nonzero. In this case $h_1=0$, each of the auto-BTs f and g maps from a solution of (2.54), (2.55) for parameter value $\tilde{\alpha}_n$ to a solution for parameter value $\alpha_n=-\tilde{\alpha}_n$. We have the following result:

Lemma 3.2.12. *Take $h_1=0$. Then, given a seed solution y_0, $\mathbf{z}_0$ of (2.54), (2.55) for parameter value $\tilde{\alpha}_n$:*

(a) in order for the results of applying the auto-BTs f and g to coincide, we must have $\tilde{\alpha}_n=0$;

(b) in the case $\tilde{\alpha}_n=0$, the results of applying the auto-BTs f and g coincide iff $y_0=0$ and $\mathbf{z}_0$ is constant.

Proof. (a) We take $h_1=0$. Assume $\tilde{\alpha}_n\neq 0$. Then $K[M[y_0],\mathbf{z}_0]\neq 0$, and $fy_0=gy_0$ implies $y_0=\frac{\tilde{\alpha}}{2K[M[y_0],\mathbf{z}_0]}\neq 0$; substituting $y=y_0$, $\mathbf{z}=\mathbf{z}_0$ and $K[M[y_0],\mathbf{z}_0]=\frac{\tilde{\alpha}}{2y_0}$ into equation (2.54) yields $\partial_x\left(\frac{\tilde{\alpha}}{2y_0}\right)=0$, and so $y_0=C$ for some nonzero constant C. But then $fz_{1,0}=gz_{1,0}$ gives $z_{1,0}=z_{1,0}-2\partial_x(\partial_x+2C)(x)$ [it is here that we assume that for $k\geq 2$ at least one $h_k\neq 0$], which implies $C=0$. This contradiction means that we must have $\tilde{\alpha}_n=0$.
(b) We take $h_1=0$ and $\tilde{\alpha}_n=0$. Then $fy_0=y_0$ since if, in the auto-BT f, we have $\tilde{\alpha}_n=0$, then $\alpha_n=0$ and so $v=y$ (and we may define $v=y$ even in the case $K[M[y],\mathbf{z}]=0$). The proof that $(fy_0=)\,y_0=gy_0$ and $(f\mathbf{z}_0=)\,\mathbf{z}_0=g\mathbf{z}_0$ iff $y_0=0$ and $\partial_x z_{j,0}=0$, $j=1,2,\ldots,p-1$, is as given as in the proof of Lemma 3.2.5, but now with $h_1=0$. □

Taking into account the proof of the above Lemma, we obtain:

Theorem 3.2.13. *Take $h_1=0$. Let y_0, $\mathbf{z}_0$ be a seed solution of (2.54), (2.55) for parameter value $\tilde{\alpha}_n=0$. Then the action of f on this seed solution corresponds to that of the identity transformation (by definition if $K[M[y_0],\mathbf{z}_0]=0$). The action of g on this seed solution provides, for parameter value $\tilde{\alpha}_n=0$, either: exactly one solution of (2.54), (2.55), when $y_0=0$ and $\mathbf{z}_0$ is constant; or two distinct solutions of (2.54), (2.55), otherwise. (In this case where $h_1=0$ and $\tilde{\alpha}_n=0$, if $K[M[y_0],\mathbf{z}_0]=0$ and also $y_0=0$ and $\mathbf{z}_0$ is constant, then we must have that the constraint $\frac{1}{2}c_0+\sum_{k=2}^{p}h_kz_{k-1,0}=0$ holds; see Remark 3.2.3.)*

4. Special integrals

4.1. Iteration of special integrals

Solutions v, $\mathbf{w}$ of the basic special integral (2.44), (2.45) provide solutions of the extended second Painlevé hierarchy (2.41), (2.42) for parameter value $\alpha_n=-h_1$. In addition to generating sequences of solutions from a known solution, the auto-BTs f and g may also be used to generate sequences of special integrals from a given basic special integral. It is to this iterative generation of special integrals to which we now turn. As in Section 3.2, we consider first of all the case $h_1\neq 0$, where we may take $h_1=-1/2$, and secondly the case $h_1=0$.

Set $h_1 = -1/2$. We begin with the basic special integral (2.44), (2.45), which gives rise to solutions v, $\mathbf{w}$ of (2.41), (2.42) for parameter value $\alpha_n = 1/2$. We now substitute for v, $\mathbf{w}$ in this special integral, using the auto-BT r given by (3.15)—(3.17), in terms of a solution y, $\mathbf{z}$ of (2.54), (2.55) for parameter value $\tilde{\alpha}_n = 3/2$, i.e., we set

$$v = -y + \frac{1}{K[M[y], \mathbf{z}]}; \quad w_k = z_k - 2\partial_x \left(\partial_x + 2y - \frac{2}{K[M[y], \mathbf{z}]} \right) z_{k-1}, \quad k = 1, \ldots, p-1, \tag{4.1}$$

and then work in the resulting expressions modulo equations (2.54), (2.55) with $h_1 = -1/2$ and $\tilde{\alpha}_n = 3/2$. In this way we obtain a special integral of our extended second Painlevé hierarchy for parameter value $\tilde{\alpha}_n = 3/2$.

We may repeat this procedure, and use the auto-BT r to substitute for y, $\mathbf{z}$ in the special integral we have obtained for parameter value $\tilde{\alpha}_n = 3/2$ in terms of a solution $\widehat{y}$, $\widehat{\mathbf{z}}$ of (2.54), (2.55) for corresponding parameter value $\widehat{\alpha}_n = 5/2$,

$$y = -\widehat{y} + \frac{2}{K[M[\widehat{y}], \widehat{\mathbf{z}}]}; \quad z_k = \widehat{z}_k - 2\partial_x \left(\partial_x + 2\widehat{y} - \frac{4}{K[M[\widehat{y}], \widehat{\mathbf{z}}]} \right) \widehat{z}_{k-1}, \quad k = 1, \ldots, p-1, \tag{4.2}$$

now working in the resulting expressions modulo equations (2.54), (2.55) in variables $\widehat{y}$, $\widehat{\mathbf{z}}$ with $h_1 = -1/2$ and parameter $\widehat{\alpha}_n = 5/2$. We thus obtain a special integral of our extended second Painlevé hierarchy for parameter value $\widehat{\alpha}_n = 5/2$.

Repeating this process gives a sequence of special integrals of the extended second Painlevé hierarchy (2.41), (2.42), one for each positive half-odd-integer parameter value $\alpha_n = 1/2, 3/2, 5/2, \ldots$. In this sequence of special integrals, substituting using the auto-BT g in terms of a solution for parameter value of opposite sign then yields a special integral for each negative half-odd-integer parameter value $\alpha_n = -1/2, -3/2, -5/2, \ldots$.

Taking into account the results of Secion 3.2, we see that one and only one special integral is obtained for each half-odd-integer parameter value:

Theorem 4.1.1. *Take $h_1 = -1/2$. Then, starting with the basic special integral (2.44), (2.45) of the extended second Painlevé hierarchy (2.41), (2.42) for parameter value $\alpha_n = 1/2$, combinations of the auto-BTs f and g provide exactly one special integral of (2.41), (2.42) for each half-odd-integer parameter value α_n.*

Proof. From the results of Section 3.2, we see that we may express the solution v, $\mathbf{w}$ of the basic special integral (2.44), (2.45) for parameter value $\alpha_n = 1/2$ in terms of solutions $y_1, \mathbf{z}_1$ and $y_2, \mathbf{z}_2$ of (2.54), (2.55) for half-odd integer parameter values $\tilde{\alpha}_n = t + 1/2$, $t \geq 0$, via the substitutions $v = s^t f y_1$, $\mathbf{w} = s^t f \mathbf{z}_1$ and $v = r^t y_2$, $\mathbf{w} = r^t \mathbf{z}_2$. However, since $v = f v$ and $\mathbf{w} = f \mathbf{w}$, we see that $v = f v = f r^t y_2 = s^t f y_2$ and $\mathbf{w} = f \mathbf{w} = f r^t \mathbf{z}_2 = s^t f \mathbf{z}_2$, i.e., the substitutions in terms of $y_1, \mathbf{z}_1$ and $y_2, \mathbf{z}_2$ are the same. Similarly, we may express the solution v, $\mathbf{w}$ of the basic special integral (2.44), (2.45) for parameter value $\alpha_n = 1/2$ in terms of solutions $y_3, \mathbf{z}_3$ and $y_4, \mathbf{z}_4$ of (2.54), (2.55) for half-odd integer parameter values $\tilde{\alpha}_n = -t - 1/2$, $t \geq 0$, via the substitutions $v = s^{t+1} y_3$, $\mathbf{w} = s^{t+1} \mathbf{z}_3$ and $v = r^t g y_4$, $\mathbf{w} = r^t g \mathbf{z}_4$. Again using the fact that $v = f v$ and $\mathbf{w} = f \mathbf{w}$, we see that $v = f v = f r^t g y_4 = s^{t+1} y_4$ and $\mathbf{w} = f \mathbf{w} = f r^t g \mathbf{z}_4 = s^{t+1} \mathbf{z}_4$, i.e., the substitutions in terms of $y_3, \mathbf{z}_3$ and $y_4, \mathbf{z}_4$ are also the same. □

We now consider the case $h_1 = 0$: we must then have $p \geq 2$ and at least one of $h_2, h_3, \ldots, h_p$ nonzero. The basic special integral (2.44), (2.45) then gives rise to solutions v, $\mathbf{w}$ of (2.41), (2.42) for parameter value $\alpha_n = 0$. Taking into account once again the results of Section 3.2, we obtain the following result:

Theorem 4.1.2. *Take $h_1 = 0$. Then, starting with the basic special integral (2.44), (2.45) of the extended second Painlevé hierarchy (2.41), (2.42) for parameter value $\alpha_n = 0$, the auto-BT f does not provide a new special integral for parameter value $\alpha_n = 0$, whereas the auto-BT g does do so. Any further use of the auto-BT g leads only to an oscillation between these two special integrals for parameter value $\alpha_n = 0$.*

Proof. From the results of Section 3.2, we see that if we express the solution v, $\mathbf{w}$ of the basic special integral (2.44), (2.45) for parameter value $\alpha_n = 0$ in terms of a solutions y, $\mathbf{z}$ (2.54), (2.55), also for parameter value $\tilde{\alpha}_n = 0$, via the substitution $v = fy$, $\mathbf{w} = f\mathbf{z}$, then since $v = fv$ and $\mathbf{w} = f\mathbf{w}$ we must have $v = y$ and $\mathbf{w} = \mathbf{z}$. On the other hand, if we express the solution v, $\mathbf{w}$ of the basic special integral (2.44), (2.45) for parameter value $\alpha_n = 0$ in terms of solutions y_1, $\mathbf{z}_1$ and y_2, $\mathbf{z}_2$ of (2.54), (2.55), both also for parameter value $\tilde{\alpha}_n = 0$, first via the substitution $v = gy_1$, $\mathbf{w} = g\mathbf{z}_1$ and subsequently $y_1 = gy_2$, $\mathbf{z}_1 = g\mathbf{z}_2$, it follows that $v = y_2$ and $\mathbf{w} = \mathbf{z}_2$. □

4.2. Lax pairs for special integrals

We now give Lax pairs for the special integrals of our extended second Painlevé hierarchy. Given these results, it is clear that this step will also be possible for the special integrals of other Painlevé equations and hierarchies, including discrete and differential-delay examples. We are unaware of Lax pairs for special integrals having been given previously.

We give here a direct derivation of Lax pairs for the basic special integrals of the extended second Painlevé hierarchy from the Lax pairs of the extended mKdV hierarchy. This is more straightforward than using a two-step process whereby first of all we reduce to the Lax pairs of the extended second Painlevé hierarchy given in Section 2.5, and then to Lax pairs for the special integrals (although the final result is, of course, the same). In Theorems 4.2.1 and 4.2.3 given below, we may set $h_1 = -1/2$ (if $h_1 \neq 0$), thus obtaining Lax pairs for a basic special integral for parameter value $\alpha_n = 1/2$, or $h_1 = 0$, giving Lax pairs for a basic special integral for parameter value $\alpha_n = 0$. The difference between these two cases is that, as explained in Section 4.1, in the former we may use the auto-BTs f and g to generate a doubly-infinite sequence of special integrals, whereas in the latter we may not.

Theorem 4.2.1. *A Lax pair for the basic special integral (2.44), (2.45) is given by*

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}_x = \mathbf{F} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \qquad \left(\sum_{k=1}^{p} h_k (4\lambda)^{k-1} \right) \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}_\lambda = \mathbf{H} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \tag{4.3}$$

where $\mathbf{F}$ *is given by (2.34),*

$$\mathbf{H} = \frac{1}{4\lambda} \left[\mathbf{G} + \begin{pmatrix} (\partial_x + 2v) K[M[v], \mathbf{w}] & -2K[M[v], \mathbf{w}] \\ 2\lambda K[M[v], \mathbf{w}] & -(\partial_x + 2v) K[M[v], \mathbf{w}] \end{pmatrix} \right] \tag{4.4}$$

with $\mathbf{G}$ *given by (2.35), and where all* c_k, $k = 0, 1, \ldots, n$, *and all* h_k, $k = 1, 2, \ldots, p$, *are assumed constant. The compatibility condition of this Lax pair reads*

$$T^{(n,p)} \equiv \left(\sum_{k=1}^{p} h_k (4\lambda)^{k-1} \right) \mathbf{F}_\lambda - \mathbf{H}_x + [\mathbf{F}, \mathbf{H}] = 0. \tag{4.5}$$

Proof. The only point which may not be entirely clear when passing from the mKdV Lax pair to the Lax pair for the basic special integral is how to use the equation $K[M[v], \mathbf{w}] = 0$ to eliminate derivatives of v of order $2n - 1$ and higher (we note $K[M[v], \mathbf{w}] = c_n v_{(2n-1)x} + \cdots$) from the matrix $\mathbf{G}$ given by (2.35). We observe that we may write the quantity P occurring in the matrix $\mathbf{G}$ as

$$P = \widehat{P} + K[M[v], \mathbf{w}], \quad \text{where} \quad \widehat{P} = \sum_{k=1}^{n} c_k \sum_{i=1}^{k} (4\lambda)^i M_{k-i}[M[v]] + \sum_{k=2}^{p} h_k \sum_{i=1}^{k-1} (4\lambda)^i w_{k-1-i} \tag{4.6}$$

is such that $\widehat{P} = 4\lambda c_n M_{n-1}[M[v]] + \cdots = 4\lambda c_n v_{(2n-3)x} + \cdots$. Considering the entries of the matrix

$$\mathbf{G} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & -g_{11} \end{pmatrix} \tag{4.7}$$

defined by (2.35), we thus see that

$$g_{11} = -(\partial_x + 2v)P = -(\partial_x + 2v)\widehat{P} - (\partial_x + 2v)K[M[v], \mathbf{w}] \tag{4.8}$$

$$g_{12} = 2P = 2\widehat{P} + 2K[M[v], \mathbf{w}] \tag{4.9}$$

$$\begin{aligned} g_{21} &= 2\lambda P - \partial_x(\partial_x + 2v)P + \partial_x(\partial_x + 2v)K[M[v], \mathbf{w}] \\ &= 2\lambda\widehat{P} + 2\lambda K[M[v], \mathbf{w}] - \partial_x(\partial_x + 2v)\widehat{P} \\ &= 2\lambda\widehat{P} + 2\lambda\Big(c_n v_{(2n-1)x} + \cdots\Big) - \partial_x(\partial_x + 2v)\Big(4\lambda c_n v_{(2n-3)x} + \cdots\Big) \\ &= -2\lambda c_n v_{(2n-1)x} + \cdots \end{aligned} \tag{4.10}$$

It then follows that the expression (4.4) for $\mathbf{H}$ is as required in order to use the equation $K[M[v], \mathbf{w}] = 0$ to eliminate derivatives of v of order $2n - 1$ and higher from $\mathbf{G}$ (and where, in addition, we divide by 4λ since $\widehat{P}$ has an overall factor of λ as do, as we see from (4.8)—(4.10), the entries of $\mathbf{G}$ after eliminating these derivatives). □

Remark 4.2.2. This represents an extension of the results of [9], where Lax pairs for Painlevé equations are obtained from those of PDEs having nonisospectral scattering problems via stationary reduction (see [10] for other reductions). Here a Lax pair is obtained for a lower order ODE system, the basic special integral, consistent with (i.e., defining solutions of), but not equivalent to, the stationary extended mKdV hierarchy.

Theorem 4.2.3. *The basic special integral (2.44), (2.45) also has an equivalent Lax pair with* x*-part*

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}_x = \begin{pmatrix} \mu & v \\ v & -\mu \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \tag{4.11}$$

where $\lambda = \mu^2$ (obtained via the transformation (2.36) given in Section 2.2).

Remark 4.2.4. Given that the basic special integral (2.44), (2.45) depends on v only via the combination of terms $v_x - v^2$, and so can be written as (2.46)—(2.48), we might seek to relate the Lax pair given in Theorem 4.2.1 to a Lax pair for the system (2.46), (2.47) since, if such a Lax pair could be found, the substitution $u = M[v] = v_x - v^2$ would lead to its compatibility condition being the basic special integral (2.44), (2.45). A Lax pair for the system (2.46), (2.47) was in fact presented in Theorem 3.2.1 of [5] (see also [10] for a related derivative equation), and is given, after making appropriate changes in notation, by

$$\begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}_x = \mathcal{F} \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}, \qquad \left(\sum_{k=1}^{p} h_k (4\lambda)^{k-1} \right) \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}_\lambda = \tilde{\mathcal{G}} \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}, \tag{4.12}$$

where $\mathcal{F}$ is as given by (2.20) and

$$\tilde{\mathcal{G}} = \begin{pmatrix} -\tilde{P}_x & 2\tilde{P} \\ 2\lambda \tilde{P} - \tilde{P}_{xx} - 2\tilde{P}u + K[u, \mathbf{w}] & \tilde{P}_x \end{pmatrix} \tag{4.13}$$

with

$$\tilde{P} = \sum_{k=1}^{n} c_k \sum_{i=1}^{k} (4\lambda)^{i-1} M_{k-i}[u] + \sum_{k=2}^{p} h_k \sum_{i=1}^{k-1} (4\lambda)^{i-1} w_{k-1-i}. \tag{4.14}$$

Setting $u = M[v] = v_x - v^2$ in (4.12)—(4.14) (which, as observed above, then gives the basic special integral (2.44), (2.45) as the compatibility condition of (4.12)), and making the gauge transformation

$$\begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \tag{4.15}$$

then yields — where it is useful to note that, after substituting $u = M[v]$, $\tilde{P} = \frac{1}{4\lambda} \widehat{P}$ — the Lax pair given in Theorem 4.2.1 for the basic special integral (2.44), (2.45). Thus we obtain the sought-after result. In some sense, (4.12) — which has as compatibility condition the system (2.46), (2.47) in u and $\mathbf{w}$ — may then also be considered to be a Lax pair for the basic special integral (2.44), (2.45). We recall that, given a solution of (2.46), (2.47), the solution v of (2.48) can then be found as $v = -\psi_x/\psi$ where ψ is the solution of (2.49).

Remark 4.2.5. We could also provide Lax pairs for the iterated special integrals (of which there are, for $h_1 = -1/2$, one for each parameter value $\alpha_n = \pm 1/2, \pm 3/2, \pm 5/2, \ldots$, and, for $h_1 = 0$, two for parameter value $\alpha_n = 0$) but, since these special integrals are all Bäcklund-equivalent to the basic special integrals (2.44), (2.45) (for $h_1 = -1/2$, for $\alpha_n = 1/2$, and for $h_1 = 0$, for $\alpha_n = 0$), giving such Lax pairs is not, in fact, necessary. (This is fortunate since, even with the use of symbolic manipulation programmes, the Lax pairs for the iterated special integrals for parameter values $\alpha_n = \pm 3/2, \pm 5/2, \pm 7/2, \ldots$ are difficult to handle.)

Remark 4.2.6. We have assumed in this paper that at least one of $h_1, h_2, \dots h_p$ be nonzero, since our interest is in obtaining results for a genuine analogue of the second Painlevé hierarchy. We note, however, that we may also use Theorems 4.2.1 and 4.2.3 to give Lax pairs for the system (2.44), (2.45) in the autonomous case $h_1 = h_2 = \cdots h_p = 0$ (this system is thus obtained, for example, from the condition $\mathbf{H}_x = [\mathbf{F}, \mathbf{H}]$). We then have $\alpha_n = 0$ and the only iteration is via g, leading to an oscillation between two special integrals. These two equivalent special integrals, of the integrated stationary mKdV hierarchy for zero integration constant, then also define subsets of solutions of the stationary mKdV hierarchy, of order $2n + 1$; we believe that these special integrals of order $2n - 1$ together with their Lax pairs represent new results even for this hierarchy. We recall that such linear problems may be used to obtain constants of motion of the corresponding equations.

5. Examples of special integrals: n = 3 and p = 3

In order to illustrate our results on Lax pairs for (basic) special integrals, we consider the case $n = 3$ and $p = 3$. In light of Remark 4.2.6, in this section we allow also the autonomous case where all $h_k = 0$, $k = 1, 2, \dots, p$. We then have

$$K[u, \mathbf{w}] = c_3 M_3[u] + c_2 M_2[u] + c_1 M_1[u] + c_0 M_0[u] + h_3 w_2 + h_2 w_1 + h_1 x, \tag{5.1}$$

where $\mathbf{w} = (w_1, w_2)$ and the quantities $M_k[u]$, $k = 0, 1, 2, 3$, are as given in (2.6). The extended sixth order second Painlevé equation, i.e.,

$$(\partial_x + 2v) K[M[v], \mathbf{w}] - (h_1 + \alpha) = 0 \tag{5.2}$$

$$\left.\begin{array}{rcl} w_{1,x} &=& \mathcal{B}_1[M[v]]x \\ w_{2,x} &=& \mathcal{B}_1[M[v]]w_1 \end{array}\right\} \tag{5.3}$$

where $M[v] = v_x - v^2$, and where we now denote the parameter by α rather than α_3, may be written explicitly as

$$\begin{aligned} & c_3(v_{xxxxxx} - 14v^2 v_{xxxx} - 56 v v_x v_{xxx} - 42 v v_{xx}^2 \\ & -70 v_x^2 v_{xx} + 70 v^4 v_{xx} + 140 v^3 v_x^2 - 20 v^7) + c_2(v_{xxxx} \\ & -10 v^2 v_{xx} - 10 v v_x^2 + 6 v^5) + c_1(v_{xx} - 2v^3) + c_0 v \\ & +(\partial_x + 2v)(h_3 w_2 + h_2 w_1) + 2h_1 x v - \alpha = 0 \end{aligned} \tag{5.4}$$

$$\left.\begin{array}{rcl} w_{1,x} &=& 4(v_x - v^2) + 2x(v_{xx} - 2vv_x) \\ w_{2,x} &=& w_{1,xxx} + 4(v_x - v^2) w_{1,x} + 2(v_{xx} - 2vv_x) w_1 \end{array}\right\} \tag{5.5}$$

Equations (2.44), (2.45) for $n = 3$ and $p = 3$ then give the corresponding extended fifth order basic special integral

$$\begin{aligned} & c_3(v_{xxxxx} - 2v v_{xxxx} + 2v_x v_{xxx} - 10 v^2 v_{xxx} - v_{xx}^2 \\ & -40 v v_x v_{xx} + 20 v^3 v_{xx} - 10 v_x^3 + 10 v^2 v_x^2 + 30 v^4 v_x \\ & -10 v^6) + c_2(v_{xxx} - 2v v_{xx} + v_x^2 - 6 v^2 v_x + 3v^4) \end{aligned}$$

$$+c_1(v_x - v^2) + \frac{1}{2}c_0 + h_3 w_2 + h_2 w_1 + h_1 x = 0 \tag{5.6}$$

$$\left.\begin{array}{rcl} w_{1,x} &=& 4(v_x - v^2) + 2x(v_{xx} - 2vv_x) \\ w_{2,x} &=& w_{1,xxx} + 4(v_x - v^2)w_{1,x} + 2(v_{xx} - 2vv_x)w_1 \end{array}\right\} \tag{5.7}$$

of the system (5.4), (5.5) for parameter value $\alpha = -h_1$. This basic special integral has the Lax pair, as given by Theorem 4.2.1,

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}_x = \mathbf{F}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \qquad \left(16\lambda^2 h_3 + 4\lambda h_2 + h_1\right)\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}_\lambda = \mathbf{H}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \tag{5.8}$$

where

$$\mathbf{F} = \begin{pmatrix} -v & 1 \\ \lambda & v \end{pmatrix} \quad \text{and} \quad \mathbf{H} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & -h_{11} \end{pmatrix} \tag{5.9}$$

with

$$\begin{aligned} h_{11} = &-16\lambda^2 c_3 v - 4\lambda[c_3(v_{xx} - 2v^3) + c_2 v + h_3(2xv + 1)] \\ &-[c_3(v_{xxxx} - 10v^2 v_{xx} - 10vv_x^2 + 6v^5) \\ &+c_2(v_{xx} - 2v^3) + c_1 v + h_3(w_{1,x} + 2vw_1) + h_2(2xv + 1)], \end{aligned} \tag{5.10}$$

$$\begin{aligned} h_{12} = &\,16\lambda^2 c_3 + 4\lambda[2c_3(v_x - v^2) + c_2 + 2h_3 x] \\ &+[2c_3(v_{xxx} - 2vv_{xx} + v_x^2 - 6v^2 v_x + 3v^4) \\ &+2c_2(v_x - v^2) + c_1 + 2h_3 w_1 + 2h_2 x], \end{aligned} \tag{5.11}$$

$$\begin{aligned} h_{21} = &\,16\lambda^3 c_3 - 4\lambda^2[2c_3(v_x + v^2) - c_2 - 2h_3 x] \\ &+\lambda[2c_3(-v_{xxx} - 2vv_{xx} + v_x^2 + 6v^2 v_x + 3v^4) \\ &-2c_2(v_x + v^2) + c_1 + 2h_3(w_1 - 4xv_x - 4v) + 2h_2 x] \\ &-[c_3(2vv_{xxxx} - 2v_x v_{xxx} + v_{xx}^2 - 20v^3 v_{xx} \\ &-10v^2 v_x^2 + 10v^6) - c_2(v_x^2 - 2vv_{xx} + 3v^4) + c_1 v^2 \\ &-(c_0/2) - h_3(w_2 - w_{1,xx} - 2w_1 v_x - 2vw_{1,x}) \\ &-h_2(w_1 - 2xv_x - 2v) - h_1 x]. \end{aligned} \tag{5.12}$$

The compatibility condition (4.5) of this Lax pair, i.e.,

$$T^{(3,3)} \equiv \left(16\lambda^2 h_3 + 4\lambda h_2 + h_1\right)\mathbf{F}_\lambda - \mathbf{H}_x + [\mathbf{F}, \mathbf{H}] = 0, \tag{5.13}$$

and likewise the compatibility condition of the equivalent Lax pair as given by Theorem 4.2.3, is the system (5.6), (5.7). For the various subcases of (5.6), (5.7) considered below, we may then, in fact, give either of the çcorresponding reduced Lax pairs.

Let us assume $c_3 \neq 0$. In the case $h_2 = h_3 = 0$ equation (5.6) gives a fifth order polynomial ODE[2]; for this equation, for $h_1 \neq 0$ but also for the autonomous case $h_1 = 0$, we have thus given a Lax pair. For $h_3 = 0$ and $h_2 \neq 0$, elimination of w_1 from the system consisting of (5.6) with $h_3 = 0$ and the first equation of (5.7) yields a sixth order polynomial ODE in v, for which we have also given a Lax pair. For the case $h_3 \neq 0$, elimination of w_1 and w_2 from the system (5.6), (5.7) gives rise to a seventh order non-polynomial ODE in v, for which once again we have given a Lax pair. (For the basic special integral (2.44), (2.45), in the general case, details of the process of elimination of the variables $w_1, w_2, \ldots, w_{p-1}$ in order to obtain an ODE in v can be found in [5].)

We note that setting $c_3 = 0$ (and also assuming $c_2 \neq 0$) gives the extended third order basic special integral

$$\begin{aligned} &c_2(v_{xxx} - 2vv_{xx} + v_x^2 - 6v^2v_x + 3v^4) + c_1(v_x - v^2) \\ &+ \frac{1}{2}c_0 + h_3w_2 + h_2w_1 + h_1x = 0 \end{aligned} \tag{5.14}$$

$$\left.\begin{aligned} w_{1,x} &= 4(v_x - v^2) + 2x(v_{xx} - 2vv_x) \\ w_{2,x} &= w_{1,xxx} + 4(v_x - v^2)w_{1,x} + 2(v_{xx} - 2vv_x)w_1 \end{aligned}\right\} \tag{5.15}$$

of the corresponding extended fourth order second Painlevé equation for parameter value $\alpha = -h_1$. Remarks similar to those made above can also be made here. For $h_3 = 0$ and $h_2 \neq 0$, elimination of w_1 yields the following fourth order polynomial ODE in v:

$$\begin{aligned} &c_2(v_{xxx} - 2vv_{xx} + v_x^2 - 6v^2v_x + 3v^4)_x + c_1(v_x - v^2)_x \\ &+ h_2[4(v_x - v^2) + 2x(v_{xx} - 2vv_x)] + h_1 = 0. \end{aligned} \tag{5.16}$$

As observed in [5], this turns out to be Cosgrove's equation F-XI for $N = 3$, which was obtained using Painlevé classification in [42] (see equations (6.67), (6.68) and (6.87) therein). So for this equation of Cosgrove we now have a Lax pair as given by equation (5.8) with $h_3 = 0$, and $\mathbf{F}$ and $\mathbf{H}$ as given by (5.9), but now where

$$h_{11} = -4\lambda c_2 v - [c_2(v_{xx} - 2v^3) + c_1 v + h_2(2xv + 1)], \tag{5.17}$$

$$h_{12} = 4\lambda c_2 + [2c_2(v_x - v^2) + c_1 + 2h_2 x], \tag{5.18}$$

$$\begin{aligned} h_{21} = &4\lambda^2 c_2 - \lambda[2c_2(v_x + v^2) - c_1 - 2h_2 x] \\ &-[c_2(v_{xx} - 2v^3)_x + c_1 v_x + 2h_2(xv)_x]. \end{aligned} \tag{5.19}$$

The compatibility condition $(4\lambda h_2 + h_1)\mathbf{F}_\lambda - \mathbf{H}_x + [\mathbf{F}, \mathbf{H}] = 0$ of this Lax pair, and likewise the compatibility condition of the equivalent Lax pair as given by Theorem 4.2.3, yields equation (5.16). For $h_3 \neq 0$, elimination of w_1 and w_2 gives a fifth order non-polynomial ODE in v; for this equation we have thus also given a Lax pair.

Let us now consider the case $h_3 = h_2 = 0$, for which the above special integral reduces to the equation

[2] That is, the highest order derivative of v may be expressed as a polynomial in v and lower-order derivatives thereof.

$$c_2(v_{xxx} - 2vv_{xx} + v_x^2 - 6v^2v_x + 3v^4) + c_1(v_x - v^2) + \frac{1}{2}c_0 + h_1x = 0 \tag{5.20}$$

(given with $h_1 = -1/2$ in the special case $c_1 = c_0 = 0$ in [12] as a special integral of the corresponding standard fourth order second Painlevé equation). Equation (5.20) is equivalent (see, e.g., [5]) to an ODE found by Chazy [43] in a classification of third order equations with the Painlevé property (Chazy class XI with $k = 3$). We see here that it has a Lax pair as given by equation (5.8) with $h_3 = h_2 = 0$, and $\mathbf{F}$ and $\mathbf{H}$ as given by (5.9), with h_{11}, h_{12} and h_{21} obtained from the corresponding special case of (5.10)—(5.12) as

$$h_{11} = -4\lambda c_2v - [c_2(v_{xx} - 2v^3) + c_1v], \tag{5.21}$$

$$h_{12} = 4\lambda c_2 + [2c_2(v_x - v^2) + c_1], \tag{5.22}$$

$$\begin{aligned} h_{21} = &\, 4\lambda^2c_2 - \lambda[2c_2(v_x + v^2) - c_1] \\ &+ [c_2(v_x^2 - 2vv_{xx} + 3v^4) - c_1v^2 + (c_0/2) + h_1x]. \end{aligned} \tag{5.23}$$

We believe that a Lax pair for this equation of Chazy has not been given previously. For $h_1 \neq 0$, and also $h_1 = 0$, the condition $h_1\mathbf{F}_\lambda - \mathbf{H}_x + [\mathbf{F}, \mathbf{H}] = 0$ yields (5.20), as will the corresponding condition given by Theorem 4.2.3. In the case $h_1 = 0$, the linear problem provides the constant of motion $c_2(v_{xx}^2 - 4vv_xv_{xx} + 2v_x^3 - 2v^2v_x^2 + 6v^4v_x - 2v^6) + c_1(v_x - v^2)^2 + c_0(v_x - v^2)$, which we recognise as $c_2(u_x^2 + 2u^3) + c_1u^2 + c_0u$ with $u = v_x - v^2$; see later.

For the case $c_3 = c_2 = 0$ (and $c_1 \neq 0$) of (5.6), (5.7), i.e., the extended first order basic special integral

$$c_1(v_x - v^2) + \frac{1}{2}c_0 + h_3w_2 + h_2w_1 + h_1x = 0 \tag{5.24}$$

$$\left.\begin{aligned} w_{1,x} &= 4(v_x - v^2) + 2x(v_{xx} - 2vv_x) \\ w_{2,x} &= w_{1,xxx} + 4(v_x - v^2)w_{1,x} + 2(v_{xx} - 2vv_x)w_1 \end{aligned}\right\} \tag{5.25}$$

of the extended second Painlevé equation for parameter value $\alpha = -h_1$, we can again give Lax pairs for any choice of h_k, $k = 1, 2, 3$. We note that for $h_3 = 0$, solutions of the resulting ODEs, which include the well-known Airy case for $h_2 = 0$ and $h_1 \neq 0$, are discussed in [5]; for these ODEs we have thus now given Lax pairs. For $h_3 \neq 0$, elimination of w_1 and w_2 from the system (5.24), (5.25) yields a fifth order non-polynomial ODE in v, for which we have thus also given a Lax pair.

Finally, returning to the basic special integral (5.6), (5.7), we recall that this may be written

$$K[u, \mathbf{w}] = 0, \qquad w_{1,x} = \mathcal{B}_1[u]x, \qquad w_{2,x} = \mathcal{B}_1[u]w_1, \qquad u = v_x - v^2, \tag{5.26}$$

or, explicitly, as

$$\begin{aligned} &c_3(u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3) + c_2(u_{xx} + 3u^2) \\ &+ c_1u + \frac{1}{2}c_0 + h_3w_2 + h_2w_1 + h_1x = 0, \end{aligned} \tag{5.27}$$

$$\left.\begin{aligned} w_{1,x} &= 4u + 2xu_x \\ w_{2,x} &= w_{1,xxx} + 4uw_{1,x} + 2u_xw_1 \end{aligned}\right\} \tag{5.28}$$

$$u = v_x - v^2. \tag{5.29}$$

The system (5.27), (5.28) has the Lax pair (4.12) as given in Remark 4.2.4, with $\mathcal{F}$ given by (2.20) and $\tilde{\mathcal{G}}$ by (4.13), where

$$\tilde{P} = c_3(8\lambda^2 + 4\lambda u + (u_{xx} + 3u^2)) + c_2(u + 2\lambda) + \frac{1}{2}c_1 + h_3(4\lambda x + w_1) + h_2 x, \tag{5.30}$$

i.e.,

$$\tilde{\mathcal{G}} = \begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{21} & -\tilde{g}_{11} \end{pmatrix} \tag{5.31}$$

with

$$\tilde{g}_{11} = -4\lambda[c_3 u_x + h_3] - [c_3(u_{xxx} + 6uu_x) + c_2 u_x + h_3 w_{1,x} + h_2], \tag{5.32}$$

$$\begin{aligned} \tilde{g}_{12} = {} & 16\lambda^2 c_3 + 4\lambda[2c_3 u + c_2 + 2h_3 x] \\ & + 2[c_3(u_{xx} + 3u^2) + c_2 u + (c_1/2) + h_3 w_1 + h_2 x], \end{aligned} \tag{5.33}$$

$$\begin{aligned} \tilde{g}_{21} = {} & 16\lambda^3 c_3 - 4\lambda^2(2c_3 u - c_2 - 2h_3 x) \\ & - 2\lambda[c_3(u_{xx} + u^2) + c_2 u - (c_1/2) + h_3(4xu - w_1) - h_2 x] \\ & + [c_3(2uu_{xx} - u_x^2 + 4u^3) + c_2 u^2 + (c_0/2) \\ & + h_3(w_2 - w_{1,xx} - 2uw_1) + h_2(w_1 - 2xu) + h_1 x]. \end{aligned} \tag{5.34}$$

As noted in Remark 4.2.4, substituting $u = v_x - v^2$ into this Lax pair for (5.27), (5.28) allows it to be mapped onto the Lax pair (5.8)—(5.12) for the basic special integral (5.6), (5.7). In some sense, we may thus regard the above Lax pair for the system (5.27), (5.28) in u, w_1, and w_2 also as a Lax pair for the basic special integral (5.6), (5.7). Within this context, let us recall, moreover, that given a solution u, w_1, w_2 of (5.27), (5.28), the solution v of (5.29) can then be found as $v = -\psi_x/\psi$ where ψ is the solution of $\psi_{xx} + u\psi = 0$. Finally, note that for $h_3 = h_2 = h_1 = 0$, (5.27) is a well-known completely integrable Hamiltonian system [44]; for the further special case $c_3 = 0$ (and $c_2 \neq 0$), the linear problem provides the trivial constant of motion (the Hamiltonian) $c_2(u_x^2 + 2u^3) + c_1 u^2 + c_0 u$, the same as for the autonomous case $h_1 = 0$ of Chazy's equation (5.20).

6. Conclusions and discussion

In this paper we have continued our study of the extended second Painlevé hierarchy introduced in [5]. We have obtained new results with respect to its auto-BTs and special integrals, in particular:

- a new local form of the auto-BT g;
- an iterative construction of solutions and special integrals using auto-BTs;
- Lax pairs for special integrals.

Let us recall once again the interest in deriving auto-BTs and Lax pairs for the case of the non-extended second Painlevé hierarchy, and the use of the former in constructing solutions and special integrals, as evidenced by the work in [2,3,7] and [11–16].

The above results will prove fundamental for our planned future studies of the properties and solutions of our extended second Painlevé hierarchy. However, here we would like to comment on the third of the above items, for which here our extended second Painlevé hierarchy serves more as an example of a general process: that of providing Lax pairs for special integrals. It is clear that this will also be possible for the special integrals of other Painlevé equations and hierarchies, including discrete and differential-delay examples. We are unaware of Lax pairs for special integrals having been given previously in the literature.

We recall, as remarked in the Introduction and also in [5], that our motivation for studying the extended second Painlevé hierarchy (and other extended Painlevé hierarchies) lies in the information that it could give with regard to which classes of ODE might be of interest for Painlevé classification, as well as in its possible usefulness in the identification of ODEs obtained from such a classification or any other process. These two motives are valid also for the special integrals of this hierarchy. By providing Lax pairs for special integrals, we give further information about any equations — i.e., equations found by (future or past) Painlevé classification, or by any other techniques — to which they correspond. For example, our results have allowed us to give Lax pairs for equations originally found by Cosgrove and Chazy, the latter being an equation known for well over a century. It is also interesting to note that Cosgrove's equation F-XI for $N = 3$ arises within the context of the systems studied in this paper as a genuinely extended example, since it occurs in the case $h_2 \neq 0$.

The case of the above-mentioned equation of Cosgrove serves to illustrate how useful the explicit form of Lax pairs is in the relationship between our extended hierarchy and Painlevé classification problems. Bearing in mind our discussion of gauge transformations, or more simply by observation, we find that in the case $h_1 = 0$ its Lax pair can be made to yield that of the second Painlevé equation: the restriction $c_2(v_{xx} - 2v^3) + c1v + 2h_2xv - \gamma = 0$ is consistent, and (5.17)—(5.19) then yield $h_{11} = -4\lambda c_2 v - h_2 - \gamma$, $h_{12} = 4\lambda c_2 + [2c_2(v_x - v^2) + c_1 + 2h_2 x]$ and $h_{21} = 4\lambda^2 c_2 - \lambda[2c_2(v_x + v^2) - c_1 - 2h_2 x]$, with compatibility condition $4\lambda h_2 \mathbf{F}_\lambda - \mathbf{H}_x + [\mathbf{F}, \mathbf{H}] = 0$ giving precisely this second Painlevé equation with arbitrary parameter γ. (Note also that the corresponding direct mapping of solutions v of the second Painlevé equation to solutions v of Cosgrove's equation with $h_1 = 0$ has not been given previously.) Similar results can be expected at higher orders. The use of the explicit form of Lax pairs to derive such connections between equations of our extended hierarchy and equations obtained from Painlevé classification problems (and generalizations thereof, as well as equations obtained by other means) — along also with the corresponding mappings between solutions — is a topic we aim to explore in future papers.

Our results on auto-BTs and the iterative construction of solutions also leads naturally to questions to be explored in future studies, where the results obtained here will play an important role. For example, we have shown that, in the case $h_1 \neq 0$, the iteration of auto-BTs may be used to construct a rational solution of our extended hierarchy for each integer value of the parameter α_n. Once we have a means of deriving these rational solutions, their properties may then be explored. It is known, for example, that the rational solutions of the usual second Painlevé hierarchy can be expressed in terms of a family of monic polynomials [45–47]; it would be interesting to see if analogues of these results can be obtained for our extended hierarchy. If that is so, then properties such as the locations of their zeros may be studied, again similarly to the non-extended case [45].

An interesting question that arises from the results presented in this paper is that of how we might know that a basic special integral can be expressed using a (perhaps higher order/extended) Painlevé equation, which is in turn connected to the question of the relationships between the Lax pairs of special integrals and those of such Painlevé equations. This is, in fact, quite a novel question, as it arises within the context of higher order Painlevé equations/hierarchies, but not for the original six Painlevé equations themselves. For the examples considered here, this question is related to the existence of the Miura map for related over-arching PDE hierarchies and so, in this sense, points to an advantage of the approach adopted by the current authors to the derivation of Painlevé hierarchies and their properties. Our use of nonisospectral PDE hierarchies and their modifications leads naturally, under reduction to ODEs, not only to Lax pairs but also to expressions for special integrals in terms of (Miura maps and) related Painlevé hierarchies and, as we have seen here, Lax pairs for these systems (i.e., special integrals and related Painlevé hierarchies), and transformations between these Lax pairs, will then also follow. Similar comments can be made in the discrete and differential-delay cases. Examples for future study may be found in our previous papers; see also the discussion in [48]. We will return to these and other examples, as well as to further extended Painlevé hierarchies and related questions, in forthcoming papers.

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Data availability

No data was used for the research described in the article.

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