Bäcklund transformations for a new extended Painlevé hierarchy

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Abstract

In a recent paper we introduced an extended second Painlevé hierarchy and studied its properties. The approach developed in order to derive these results is widely applicable. Here we use it to obtain a second example of an extended Painlevé hierarchy. We also give results on Bäcklund transformations, auto-Bäcklund transformations and other properties of this and related hierarchies of ordinary differential equations, as well as on the nesting of equations whereby we obtain relations between systems of different orders but of the same form.

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1 Introduction

Painlevé's original programme of classifiying ordinary differential equations (ODEs) with general solution free of movable branched singularities, i.e., having what today is referred to as the Painlevé property, foresaw dealing with classes of equations of increasing order. This classification process had as its underlying aim the discovery of new transcendental functions defined via differential eqautions; at second order this then led, as is well-known, to the discovery of the six Painlevé equations [1, 2, 3, 4]. However, since such a classification becomes increasingly difficult as the order of the class of equations studied increases, this programme began to stall somewhat, with only partial classifications of third and fourth order ODEs being carried out [5, 6, 7, 8, 9, 10].

The discovery of the connection between completely integrable partial differential equations (PDEs) and ODEs having the Painlevé property [11] led, through consideration of the Korteweg-de Vries (KdV) and modified KdV (mKdV) hierarchies, to the introduction of a hierarchy of ODEs having as first member the second Painlevé (P_{II}) equation, i.e., to a P_{II} hierarchy [11, 12]. This hierarchy, which we refer to as the standard P_{II} hierarchy in contrast to the generalized P_{II} hierarchy where terms derived from lower order mKdV flows are also included [13, 14], was shown in [12] to admit auto-Bäcklund transformations (auto-BTs) analogous to those known for the second Painlevé equation itself. This then meant that the way was open, at least in principle, to the derivation of higher order analogues of the Painlevé equations along with their properties.

However, no further work in this direction seems to have been undertaken until, some twenty years later, the standard P_{II} hierarchy was rederived and a first Painlevé (P_I) hierarchy also derived [15]. A wide variety of techniques have since been developed to derive Painlevé hierarchies and study their properties, although this is often more complicated than for the original P_{II} case. Examples include hierarchies of ODEs based on the first, second and fourth Painlevé equations, with more than one such hierarchy having been found in each of these cases; see, in addition to the papers cited previously, e.g., [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. Other examples of Painlevé hierarchies have been found which have ODEs of order higher than two or matrix ODEs as first members [27, 16, 28, 17, 18, 29, 25, 30, 31]. We note that the derivation of ODEs as members of Painlevé hierarchies — these being related to hierarchies of completely integrable PDEs — has proved to be of relevance in the Painlevé classification process; see, e.g., [32, 33, 34, 35].

The work of the current authors (and collaborators) on Painlevé hierarchies has exploited and extended the connection with nonisospectral scattering problems as described in [36]. In the course of this work, hierarchies of ODEs which included nonlocal terms were often presented (see, e.g., [16, 18, 19]). However, in the analyses of (most of) the properties of such hierarchies, only the local cases were considered. We have recently returned to the study of hierarchies of ODEs including such nonlocal terms, and in particular have obtained a new extended P_{II} hierarchy: in [37] we derived the hierarchy of ODEs

$$\psi[v]\left(\sum_{k=1}^{n} c_k\tilde{\mathcal{Q}}^{k-1}[v]v_x + 2\sum_{k=2}^{p} h_k\tilde{\mathcal{Q}}^{k-2}[v](xv)_x\right) + c_0 v + 2h_1 xv - \alpha_n = 0, \qquad n = 1, 2, \ldots, \tag{1.1}$$

— where all c_k, $k = 0, 1, \ldots, n$, and all h_k, $k = 1, 2, \ldots, p$, are constant, α_n is an arbitrary constant,

$$\psi[v] = \partial_x - 4v\partial_x^{-1}v \qquad \text{and} \qquad \tilde{\mathcal{Q}}[v] = \partial_x\psi[v] = \partial_x\left(\partial_x - 4v\partial_x^{-1}v\right) \tag{1.2}$$

is the recursion operator of the mKdV hierarchy — and we defined the non-autonomous case where at least one of $h_1, h_2, \ldots, h_p$ is nonzero to be an extended P_{II} hierarchy. We note that in the case where $h_2 = h_3 = \cdots = h_p = 0$ and $h_1 \neq 0$ we recover the generalized P_{II} hierarchy [13, 14], which then also includes of course as a further special case the standard P_{II} hierarchy [11, 12, 15]. In [37] we used a local form of the hierarchy (1.1) in order to derive, amongst other results, Bäcklund transformations (BTs) and auto-BTs for our extended P_{II} hierarchy. We also stated that, since the nonisospectral approach used in [37] is quite general, we expected to be able to derive extended versions of other continuous, discrete and differential-delay Painlevé hierarchies and equations. It is here that we begin this task.

In the present paper we define, and derive properties of, extensions of hierarchies of ODEs to be found in [27, 16, 28, 17, 29, 25, 30]. These new extensions include nonlocal terms and are related to completely integrable nonisospectral Kaup-Kupershmidt (KK) and Sawada-Kotera (SK) hierarchies, and their nonisospectral modified (mKK and mSK) hierarchies (these nonisospectral mKK and mSK hierarchies are equivalent, i.e., the nonisospectral KK and SK hierarchies have a common nonisospectral modified hierarchy, a result well-known in the isospectral case). We note that examples of hierarchies of ODEs related to nonisospectral versions of the KK and SK hierarchies and in which such nonlocal terms appear can be found in [16]. In the present paper we focus much of our interest on a new hierarchy of ODEs related to the afore-mentioned nonisospectral version of the common modified hierarchy. This new hierarchy of ODEs can be written in two different ways in forms which may be compared with (1.1). Similarly to (1.1) we have a set of constants $h_1, h_2, \ldots, h_p$, and in the case where least one of these constants $h_1, h_2, \ldots, h_p$ is nonzero we define this hierarchy to be an extended KK-SK Painlevé hierarchy. The layout of the paper is as follows. In Section 2 we consider nonisospectral KK and SK hierarchies along with their common nonisospectral modified hierarchy, giving also local forms for all of these nonisospectral hierarchies. We then derive, also in Section 2, our extended KK-SK Painlevé hierarchy, which we present in two different ways in forms analogous to (1.1), giving moreover in each case a corresponding local form. In Section 3 we use these two alternative local forms to derive results on BTs and auto-BTs as well as other aspects of our extended KK-SK Painlevé hierarchy, and also give results for other (again locally written) related ODE hierarchies. In Section 4 we discuss the nesting of equations, thus obtaining relations between systems of different orders but of the same form. Finally, in Section 5, we draw our conclusions.

2 Nonisospectral hierarchies and an extended Painlevé hierarchy

2.1 Nonisospectral hierarchies

In this section we consider nonisospectral KK and SK hierarchies along with their common nonisospectral modified hierarchy. Basic facts in relation to recursion operators, Hamiltonian structures, Miura maps and the equivalence of the modified hierarchies in the isospectral case, as well as non-isospectral extensions, can be found in [38, 39, 40, 41, 42, 43, 44, 45, 16]; it is the ideas and results given in these papers that we use below.

Let us begin with the nonisospectral KK hierarchy

$$u_t = \sum_{k=0}^{n} c_k \mathcal{T}^k[u] u_x + \sum_{k=0}^{q} a_k \mathcal{T}^k[u] \theta[u] H_1[u] + \sum_{k=1}^{p} h_k \mathcal{T}^{k-1}[u] \theta[u] x, \tag{2.1}$$

where

$$H_1[u] = u_{xx} + 4u^2, \tag{2.2}$$

and $\mathcal{T}[u] = \theta[u] K[u]$ with

$$\begin{aligned} \theta[u] &= \partial_x^3 + u\partial_x + \partial_x u, && (2.3)\\ K[u] &= \partial_x^{-1} \hat{K}[u] \partial_x^{-1} = \partial_x^{-1} \left[\partial_x^5 + 3\left(\partial_x u \partial_x^2 + \partial_x^2 u \partial_x\right) + 2\left(\partial_x^3 u + u\partial_x^3\right) + 8\left(\partial_x u^2 + u^2 \partial_x\right) \right] \partial_x^{-1} && (2.4) \end{aligned}$$

is the recursion operator of the KK hierarchy, and where all coefficients $c_k = c_k(t)$, $a_k = a_k(t)$ and $h_k = h_k(t)$ are functions of t. Denoting as usual

$$H_0[u] = 1, \qquad \text{so} \qquad u_x = \theta[u] H_0[u], \tag{2.5}$$

we see that by introducing auxiliary variables $\mathbf{w} = (w_1, w_2, \ldots, w_{p-1})$ and $\mathbf{z} = (z_1, z_2, \ldots, z_{p-1})$ we may write this hierarchy locally as

$$u_t = \sum_{k=0}^{n} c_k \mathcal{T}^k[u]\theta[u]H_0[u] + \sum_{k=0}^{q} a_k \mathcal{T}^k[u]\theta[u]H_1[u] + \sum_{k=2}^{p} h_k \theta[u] z_{k-1} + h_1 \theta[u] x \tag{2.6}$$

$$\left.\begin{array}{rcl} w_{1,x} &=& \theta[u]x \\ z_{1,x} &=& \hat{K}[u]w_1 \\ w_{2,x} &=& \theta[u]z_1 \\ z_{2,x} &=& \hat{K}[u]w_2 \\ &\vdots& \\ w_{p-1,x} &=& \theta[u]z_{p-2} \\ z_{p-1,x} &=& \hat{K}[u]w_{p-1} \end{array}\right\} \tag{2.7}$$

where we have used the identity $\mathcal{T}^{k-1}[u]\theta[u]x = \theta[u]z_{k-1}$, $k = 2, 3, \ldots, p$. Recalling the recursive definition of the sequence of variational derivatives of the Hamiltonian densities of the KK hierarchy,

$$H_{n+2}[u] = K[u]\theta[u]H_n[u], \qquad n = 0, 1, 2, \ldots, \tag{2.8}$$

we note that equation (2.6) can also be written

$$u_t = \theta[u]L[u, \mathbf{z}] \tag{2.9}$$

where

$$L[u, \mathbf{z}] = \sum_{k=0}^{n} c_k H_{2k}[u] + \sum_{k=0}^{q} a_k H_{2k+1}[u] + \sum_{k=2}^{p} h_k z_{k-1} + h_1 x. \tag{2.10}$$

Under the Miura map

$$u = M[v] = v_x - \frac{1}{2}v^2 \tag{2.11}$$

the Hamiltonian operator $\theta[u]$ factorizes as

$$\theta[u]\Bigg|_{u=M[v]} = M'[v]\mathcal{B}(M'[v])^{\dagger}, \tag{2.12}$$

where

$$M'[v] = \partial_x - v \tag{2.13}$$

is the Fréchet derivative of $M[v]$,

$$(M'[v])^{\dagger} = -\partial_x - v \tag{2.14}$$

is the adjoint of this Fréchet derivative, and

$$\mathcal{B} = -\partial_x \tag{2.15}$$

is the Hamiltonian operator of the mKK hierarchy. The modified version of the nonisospectral hierarchy (2.1) is then written

$$\begin{aligned} v_t &= \sum_{k=0}^{n} c_k \tilde{\mathcal{T}}^k[v]v_x + \sum_{k=0}^{q} a_k \tilde{\mathcal{T}}^k[v]\mathcal{B}(M'[v])^{\dagger} H_1[M[v]] + \sum_{k=1}^{p} h_k \tilde{\mathcal{T}}^{k-1}[v]\mathcal{B}(M'[v])^{\dagger} x \\ &= \sum_{k=0}^{n} c_k \tilde{\mathcal{T}}^k[v]v_x + \sum_{k=0}^{q} a_k \tilde{\mathcal{T}}^k[v]\mathcal{B}\tilde{H}_1[v] + \sum_{k=1}^{p} h_k \tilde{\mathcal{T}}^{k-1}[v](xv)_x \end{aligned} \tag{2.16}$$

where the recursion operator of the mKK hierarchy is given by

$$\begin{aligned}\tilde{\mathcal{T}}[v] &= \mathcal{B}(M'[v])^{\dagger}K[M[v]]M'[v] \\ &= \partial_x\left(\partial_x+v\right)\partial_x^{-1}\left(\partial_x-2v\right)\left(\partial_x-v\right)\partial_x\left(\partial_x+v\right)\left(\partial_x+2v\right)\partial_x^{-1}\left(\partial_x-v\right),\end{aligned} \tag{2.17}$$

and where we have set

$$\tilde{H}_1[v]=(M'[v])^{\dagger}H_1[M[v]]=-(v_{xxxx}+5v_xv_{xx}-5v^2v_{xx}-5vv_x^2+v^5). \tag{2.18}$$

This modified nonisospectral hierarchy can also be written locally as the system

$$v_t=\mathcal{B}(M'[v])^{\dagger}L[M[v],\mathbf{z}] \tag{2.19}$$

$$\left.\begin{aligned} w_{1,x} &= \theta[M[v]]x \\ z_{1,x} &= \hat{K}[M[v]]w_1 \\ w_{2,x} &= \theta[M[v]]z_1 \\ z_{2,x} &= \hat{K}[M[v]]w_2 \\ &\vdots \\ w_{p-1,x} &= \theta[M[v]]z_{p-2} \\ z_{p-1,x} &= \hat{K}[M[v]]w_{p-1}\end{aligned}\right\} \tag{2.20}$$

As is well-known in the isospectral case, this modified nonisospectral hierarchy can also be derived from a nonisospectral SK hierarchy. We thus begin with the nonisospectral SK hierarchy

$$U_t=\sum_{k=0}^{n}c_k\mathcal{R}^k[U]\theta[U]G_0[U]+\sum_{k=0}^{q}a_k\mathcal{R}^k[U]\theta[U]G_1[U]+\sum_{k=1}^{p}h_k\mathcal{R}^{k-1}[U]\theta[U]x, \tag{2.21}$$

where $\mathcal{R}[U]=\theta[U]J[U]$ with

$$\theta[U] = \partial_x^3+U\partial_x+\partial_xU, \tag{2.22}$$

$$J[U] = \partial_x^{-1}\hat{J}[U]\partial_x^{-1}=\partial_x^{-1}\left[\partial_x^5+\frac{1}{2}\left(\partial_x^3U+U\partial_x^3\right)+\frac{1}{8}\left(\partial_xU^2+U^2\partial_x\right)\right]\partial_x^{-1} \tag{2.23}$$

is the recursion operator of the SK hierarchy,

$$G_0[U]=1,\qquad \text{so}\qquad \theta[U]G_0[U]=U_x,\qquad \text{and}\qquad G_1[U]=U_{xx}+\frac{1}{4}U^2, \tag{2.24}$$

and the coefficients $c_k=c_k(t)$, $a_k=a_k(t)$ and $h_k=h_k(t)$ are the same functions of t as appear in (2.1). Recalling the recursive definition of the sequence of variational derivatives of the Hamiltonian densities of the SK hierarchy,

$$G_{n+2}[U]=J[U]\theta[U]G_n[U],\qquad n=0,1,2,\ldots, \tag{2.25}$$

we see that by introducing auxiliary variables $\mathbf{W}=(W_1,W_2,\ldots,W_{p-1})$ and $\mathbf{Z}=(Z_1,Z_2,\ldots,Z_{p-1})$ we may write this nonisospectral SK hierarchy locally as

$$U_t=\theta[U]N[U,\mathbf{Z}] \tag{2.26}$$

$$\left.\begin{aligned} W_{1,x} &= \theta[U]x \\ Z_{1,x} &= \hat{J}[U]W_1 \\ W_{2,x} &= \theta[U]Z_1 \\ Z_{2,x} &= \hat{J}[U]W_2 \\ &\vdots \\ W_{p-1,x} &= \theta[U]Z_{p-2} \\ Z_{p-1,x} &= \hat{J}[U]W_{p-1}\end{aligned}\right\} \tag{2.27}$$

where

$$N[U,\mathbf{Z}] = \sum_{k=0}^{n} c_k G_{2k}[U] + \sum_{k=0}^{q} a_k G_{2k+1}[U] + \sum_{k=2}^{p} h_k Z_{k-1} + h_1 x, \tag{2.28}$$

and where we have made use of the identity $\mathcal{R}^{k-1}[U]\theta[U]x = \theta[U]Z_{k-1}$, $k = 2, 3, \ldots, p$. Under the same Miura map as in the KK case but with dependent variables U and V,

$$U = M[V] = V_x - \frac{1}{2}V^2, \tag{2.29}$$

which then leads to the same factorization,

$$\theta[U]\Bigg|_{U=M[V]} = M'[V]\mathcal{B}(M'[V])^{\dagger}, \tag{2.30}$$

we obtain the modified version of the nonisospectral hierarchy (2.21) as

$$V_t = \sum_{k=0}^{n} c_k \tilde{\mathcal{R}}^k[V]V_x + \sum_{k=0}^{q} a_k \tilde{\mathcal{R}}^k[V]\mathcal{B}\tilde{G}_1[V] + \sum_{k=1}^{p} h_k \tilde{\mathcal{R}}^{k-1}[V](xV)_x \tag{2.31}$$

where the recursion operator of the mSK hierarchy is given by

$$\begin{aligned} \tilde{\mathcal{R}}[V] &= \mathcal{B}(M'[V])^{\dagger}J[M[V]]M'[V] \\ &= \partial_x\left(\partial_x + V\right)\partial_x^{-1}\left(\partial_x - V/2\right)\left(\partial_x + V/2\right)\partial_x\left(\partial_x - V/2\right)\left(\partial_x + V/2\right)\partial_x^{-1}\left(\partial_x - V\right), \end{aligned} \tag{2.32}$$

and where we have set

$$\tilde{G}_1[V] = (M'[V])^{\dagger}G_1[M[V]] = -\left(V_{xxxx} - \frac{5}{2}V_xV_{xx} - \frac{5}{4}V^2V_{xx} - \frac{5}{4}VV_x^2 + \frac{1}{16}V^5\right). \tag{2.33}$$

This modified nonisospectral hierarchy (2.31) can also be written locally as the system

$$V_t = \mathcal{B}(M'[V])^{\dagger}N[M[V],\mathbf{Z}] \tag{2.34}$$

$$\left.\begin{aligned} W_{1,x} &= \theta[M[V]]x \\ Z_{1,x} &= \hat{J}[M[V]]W_1 \\ W_{2,x} &= \theta[M[V]]Z_1 \\ Z_{2,x} &= \hat{J}[M[V]]W_2 \\ &\vdots \\ W_{p-1,x} &= \theta[M[V]]Z_{p-2} \\ Z_{p-1,x} &= \hat{J}[M[V]]W_{p-1} \end{aligned}\right\} \tag{2.35}$$

As in the isospectral case, the modified hierachies (2.16) and (2.31) are related via $V = -2v$. In order to see this we note that

$$-\frac{1}{2}\tilde{G}_1[-2v] = \tilde{H}_1[v], \tag{2.36}$$

and also that the identity

$$\left(\partial_x + \beta v\right)\partial_x^{-1}\left(\partial_x + \gamma v\right) = \left(\partial_x + \gamma v\right)\partial_x^{-1}\left(\partial_x + \beta v\right), \tag{2.37}$$

which holds for any two constants β and γ, implies that

$$\tilde{\mathcal{R}}[-2v] = \tilde{\mathcal{T}}[v]. \tag{2.38}$$

2.2 A new extended Painlevé hierarchy

The new extended Painlevé hierarchy which will form the focus of the current paper is a hierarchy of ODEs, subject to a condition given in Definition 2.1 below, which may be obtained — following the ideas in [36] and, applied to nonlocal examples, in [16] — as the (integrated) stationary flow of (2.16). Defining

$$\varphi[v] = (\partial_x + v)\,\partial_x^{-1}\,(\partial_x - 2v)\,(\partial_x - v)\,\partial_x\,(\partial_x + v)\,(\partial_x + 2v)\,\partial_x^{-1}\,(\partial_x - v)\,, \tag{2.39}$$

so that

$$\tilde{\mathcal{T}}[v] = \partial_x \varphi[v], \tag{2.40}$$

we see that the integrated stationary flow of (2.16) yields the hierarchy of ODEs

$$\varphi[v]\left(\sum_{k=1}^{n} c_k \tilde{\mathcal{T}}^{k-1}[v]v_x + \sum_{k=1}^{q} a_k \tilde{\mathcal{T}}^{k-1}[v]\mathcal{B}\tilde{H}_1[v] + \sum_{k=2}^{p} h_k \tilde{\mathcal{T}}^{k-2}[v](xv)_x\right) + c_0 v - a_0 \tilde{H}_1[v] + h_1 x v + \frac{1}{2}\alpha = 0, \tag{2.41}$$

where all coefficients c_k, a_k and h_k are now constant, and $\frac{1}{2}\alpha$ is an arbitrary constant of integration. We may alternatively obtain this hierarchy of ODEs as the integrated stationary flow of (2.31). Defining

$$\phi[V] = (\partial_x + V)\,\partial_x^{-1}\,(\partial_x - V/2)\,(\partial_x + V/2)\,\partial_x\,(\partial_x - V/2)\,(\partial_x + V/2)\,\partial_x^{-1}\,(\partial_x - V)\,, \tag{2.42}$$

so that

$$\tilde{\mathcal{R}}[V] = \partial_x \phi[V], \tag{2.43}$$

we see that the hierarchy (2.41), with the identification $V = -2v$, can also be written as

$$\phi[V]\left(\sum_{k=1}^{n} c_k \tilde{\mathcal{R}}^{k-1}[V]V_x + \sum_{k=1}^{q} a_k \tilde{\mathcal{R}}^{k-1}[V]\mathcal{B}\tilde{G}_1[V] + \sum_{k=2}^{p} h_k \tilde{\mathcal{R}}^{k-2}[V](xV)_x\right) + c_0 V - a_0 \tilde{G}_1[V] + h_1 x V - \alpha = 0. \tag{2.44}$$

In the next section we will present results on BTs and auto-BTs for the members of this hierarchy. First of all, however, we note the two local formulations of this hierarchy, derived when written in the form (2.41) from (2.19), (2.20) with $L[u,\mathbf{z}]$ given by (2.10), and when written in the form (2.44) from (2.34), (2.35) with $N[U,\mathbf{Z}]$ given by (2.28). These formulations are respectively as the system

$$(\partial_x + v)\, L[M[v],\mathbf{z}] + \frac{1}{2}\alpha - h_1 = 0 \tag{2.45}$$

$$\left.\begin{array}{rcl} w_{1,x} &=& \theta[M[v]]x \\ z_{1,x} &=& \hat{K}[M[v]]w_1 \\ w_{2,x} &=& \theta[M[v]]z_1 \\ z_{2,x} &=& \hat{K}[M[v]]w_2 \\ &\vdots& \\ w_{p-1,x} &=& \theta[M[v]]z_{p-2} \\ z_{p-1,x} &=& \hat{K}[M[v]]w_{p-1} \end{array}\right\} \tag{2.46}$$

where

$$L[u,\mathbf{z}] = \sum_{k=0}^{n} c_k H_{2k}[u] + \sum_{k=0}^{q} a_k H_{2k+1}[u] + \sum_{k=2}^{p} h_k z_{k-1} + h_1 x, \tag{2.47}$$

and as the system

$$(\partial_x + V)\, N[M[V],\mathbf{Z}] - \alpha - h_1 = 0 \tag{2.48}$$

$$\left.\begin{array}{rcl} W_{1,x} & = & \theta[M[V]]x \\ Z_{1,x} & = & \hat{J}[M[V]]W_1 \\ W_{2,x} & = & \theta[M[V]]Z_1 \\ Z_{2,x} & = & \hat{J}[M[V]]W_2 \\ & \vdots & \\ W_{p-1,x} & = & \theta[M[V]]Z_{p-2} \\ Z_{p-1,x} & = & \hat{J}[M[V]]W_{p-1} \end{array}\right\} \tag{2.49}$$

where

$$N[U,\mathbf{Z}] = \sum_{k=0}^{n} c_k G_{2k}[U] + \sum_{k=0}^{q} a_k G_{2k+1}[U] + \sum_{k=2}^{p} h_k Z_{k-1} + h_1 x. \tag{2.50}$$

Definition 2.1 *In the non-autonomous case where at least one of $h_1, h_2, \ldots, h_p$ is nonzero we define the hierarchy (2.41), its equivalent formulation (2.44), and also the respective local versions of these hierarchies as given by the systems (2.45)—(2.47) and (2.48)—(2.50), as an extended KK-SK Painlevé hierarchy.*

Remark 2.2 *The above definition is analogous to that of the extended second Painlevé hierarchy given in [37]. The reason for focusing our discussion on this KK-SK Painlevé hierarchy instead of on the stationary (constant-coefficient) reduction of the nonisospectral KK hierarchy (2.7), (2.9), (2.10), or on the stationary (constant-coefficient) reduction of the nonisospectral SK hierarchy (2.26), (2.27), (2.28), is that these stationary reductions are in fact equivalent to (2.45)—(2.47) and (2.48)—(2.50) respectively (see Theorems 3.1 and 3.3). In the autonomous case where $h_1 = h_2 = \cdots = h_p = 0$, the equivalent hierarchies (2.41) and (2.44) are just the integrated stationary flows of the standard (isospectral) common modification of the KK and SK hierarchies. In the case where $h_2 = h_3 = \cdots = h_p = 0$ and $h_1 \neq 0$ the equivalent hierarchies (2.41) and (2.44) reduce to a hierarchy of ODEs the standard case of which is given in [27, 17, 28, 29, 30] and the generalised case, which includes terms derived from lower order flows of the corresponding common modification of the KK and SK hierarchies, in [25] (in its mKK formulation).*

3 Bäcklund transformations

In this section we will use in tandem the local formulations (2.45)—(2.47) and (2.48)—(2.50) of our extended KK-SK Painlevé hierarchy in order to present results on BTs, auto-BTs and other aspects of this new hierarchy. We also give results for other (again locally-formulated) related ODE hierarchies. The Theorems, Remarks and Definitions 3.1—3.7 and 3.12—3.16 generalize results, for individual equations or hierarchies, given in (or immediately deducible from) [27, 16, 17, 28, 29, 25, 30]. In these papers, except for [16, 25], the standard case was studied. Thus the great majority of the results given here are new not only for our new extended case but also for the generalized case. Furthermore, except for [25] (for which an SK/mSK version of the results therein can easily be given), the possibility that results may hold independently of the forms of $L[u,\mathbf{z}]$ and $N[U,\mathbf{Z}]$ was not remarked upon: below we note when this is the case. The Theorems and Remarks 3.8—3.11, in contrast, are particular to our new extended KK-SK Painlevé hierarchy since to be nontrivial we need at least one of $h_2, h_3, \ldots, h_p$ to be nonzero. Finally we remark that whereas some of the results given in this section become trivial in the autonomous case $h_1 = h_2 = \cdots = h_p = 0$, this is not true of all the results given here.

Theorem 3.1 *The BT*

$$(\partial_x + v)L[u,\mathbf{z}] + \frac{1}{2}\alpha - h_1 = 0 \tag{3.1}$$

$$\left.\begin{array}{rcl} w_{1,x} &=& \theta[u]x \\ z_{1,x} &=& \hat{K}[u]w_1 \\ w_{2,x} &=& \theta[u]z_1 \\ z_{2,x} &=& \hat{K}[u]w_2 \\ &\vdots& \\ w_{p-1,x} &=& \theta[u]z_{p-2} \\ z_{p-1,x} &=& \hat{K}[u]w_{p-1} \end{array}\right\} \tag{3.2}$$

$$u - M[v] = u - v_x + \frac{1}{2}v^2 = 0 \tag{3.3}$$

provides a mapping between the system (2.45), (2.46) in v, $\mathbf{w}$ *and* $\mathbf{z}$ *and the system*

$$L[u,\mathbf{z}](L[u,\mathbf{z}])_{xx} - \frac{1}{2}[(L[u,\mathbf{z}])_x]^2 + uL[u,\mathbf{z}]^2 + \frac{1}{2}\left(h_1 - \frac{1}{2}\alpha\right)^2 = 0 \tag{3.4}$$

$$\left.\begin{array}{rcl} w_{1,x} &=& \theta[u]x \\ z_{1,x} &=& \hat{K}[u]w_1 \\ w_{2,x} &=& \theta[u]z_1 \\ z_{2,x} &=& \hat{K}[u]w_2 \\ &\vdots& \\ w_{p-1,x} &=& \theta[u]z_{p-2} \\ z_{p-1,x} &=& \hat{K}[u]w_{p-1} \end{array}\right\} \tag{3.5}$$

in u, $\mathbf{w}$ *and* $\mathbf{z}$. *Moreover, the system (3.4), (3.5) provides a first integral for the stationary flow of the system (2.7), (2.9), i.e.,*

$$\theta[u]L[u,\mathbf{z}] = 0 \tag{3.6}$$

coupled with (3.5). We note that this result holds independently of the form of $L[u,\mathbf{z}]$, *and not just for the special case (2.47).*

Proof. The proof consists of straightforward calculations, which we leave to the reader. □

Remark 3.2 *We see from the above that solutions* u, $\mathbf{w}$, $\mathbf{z}$ *of*

$$L[u,\mathbf{z}] = 0 \tag{3.7}$$

$$\left.\begin{array}{rcl} w_{1,x} &=& \theta[u]x \\ z_{1,x} &=& \hat{K}[u]w_1 \\ w_{2,x} &=& \theta[u]z_1 \\ z_{2,x} &=& \hat{K}[u]w_2 \\ &\vdots& \\ w_{p-1,x} &=& \theta[u]z_{p-2} \\ z_{p-1,x} &=& \hat{K}[u]w_{p-1} \end{array}\right\} \tag{3.8}$$

are also solutions of (3.4), (3.5) for $\alpha = 2h_1$. *In addition, it is clear that solutions* u, $\mathbf{w}$, $\mathbf{z}$ *of (3.7), (3.8) are also solutions of (3.5), (3.6). Again, these results hold independently of the form of* $L[u,\mathbf{z}]$, *and not just as given by (2.47).*

Theorem 3.3 *The BT*

$$(\partial_x + V)N[U,\mathbf{Z}] - \alpha - h_1 = 0 \tag{3.9}$$

$$\left.\begin{array}{rcl} W_{1,x} &=& \theta[U]x \\ Z_{1,x} &=& \hat{J}[U]W_1 \\ W_{2,x} &=& \theta[U]Z_1 \\ Z_{2,x} &=& \hat{J}[U]W_2 \\ &\vdots& \\ W_{p-1,x} &=& \theta[U]Z_{p-2} \\ Z_{p-1,x} &=& \hat{J}[U]W_{p-1} \end{array}\right\} \tag{3.10}$$

$$U - M[V] = U - V_x + \frac{1}{2}V^2 = 0 \tag{3.11}$$

provides a mapping between the system (2.48), (2.49) in V, $\mathbf{W}$ *and* $\mathbf{Z}$ *and the system*

$$N[U,\mathbf{Z}](N[U,\mathbf{Z}])_{xx} - \frac{1}{2}[(N[U,\mathbf{Z}])_x]^2 + UN[U,\mathbf{Z}]^2 + \frac{1}{2}\left(h_1+\alpha\right)^2 = 0 \tag{3.12}$$

$$\left.\begin{array}{rcl} W_{1,x} &=& \theta[U]x \\ Z_{1,x} &=& \hat{J}[U]W_1 \\ W_{2,x} &=& \theta[U]Z_1 \\ Z_{2,x} &=& \hat{J}[U]W_2 \\ &\vdots& \\ W_{p-1,x} &=& \theta[U]Z_{p-2} \\ Z_{p-1,x} &=& \hat{J}[U]W_{p-1} \end{array}\right\} \tag{3.13}$$

in U, $\mathbf{W}$ *and* $\mathbf{Z}$*. Moreover, the system (3.12), (3.13) provides a first integral for the stationary flow of the system (2.26), (2.27), i.e.,*

$$\theta[U]N[U,\mathbf{Z}] = 0 \tag{3.14}$$

coupled with (3.13). We note that this result holds independently of the form of $N[U,\mathbf{Z}]$*, and not just for the special case (2.50).*

Proof. The proof consists of straightforward calculations, which we leave to the reader. □

Remark 3.4 *We see from the above that solutions* U, $\mathbf{W}$, $\mathbf{Z}$ *of*

$$N[U,\mathbf{Z}] = 0 \tag{3.15}$$

$$\left.\begin{array}{rcl} W_{1,x} &=& \theta[U]x \\ Z_{1,x} &=& \hat{J}[U]W_1 \\ W_{2,x} &=& \theta[U]Z_1 \\ Z_{2,x} &=& \hat{J}[U]W_2 \\ &\vdots& \\ W_{p-1,x} &=& \theta[U]Z_{p-2} \\ Z_{p-1,x} &=& \hat{J}[U]W_{p-1} \end{array}\right\} \tag{3.16}$$

are also solutions of (3.12), (3.13) for $\alpha = -h_1$*. In addition, it is clear that solutions* U, $\mathbf{W}$, $\mathbf{Z}$ *of (3.15), (3.16) are also solutions of (3.13), (3.14). Again, these results hold independently of the form of* $N[U,\mathbf{Z}]$*, and not just for (2.50).*

Remark 3.5 *By considering the Miura map $u = M[v]$ in the stationary case, we easily see that solutions v, $\mathbf{w}$, $\mathbf{z}$ of the system (2.45), (2.46) give solutions $u = v_x - \frac{1}{2}v^2$, $\mathbf{w}$, $\mathbf{z}$ of the system (3.5), (3.6). Similarly, consideration of the Miura map $U = M[V]$ in the stationary case tells us that solutions V, $\mathbf{W}$, $\mathbf{Z}$ of the system (2.48), (2.49) yield solutions $U = V_x - \frac{1}{2}V^2$, $\mathbf{W}$, $\mathbf{Z}$ of the system (3.13), (3.14). This is true independently of the forms of $L[u, \mathbf{z}]$ and $N[U, \mathbf{Z}]$, and not just for the special choices (2.47) and (2.50).*

Definition 3.6 *(***Basic special integrals***) We define*

$$L[M[v], \mathbf{z}] = 0 \tag{3.17}$$

$$\left.\begin{array}{rcl} w_{1,x} & = & \theta[M[v]]x \\ z_{1,x} & = & \hat{K}[M[v]]w_1 \\ w_{2,x} & = & \theta[M[v]]z_1 \\ z_{2,x} & = & \hat{K}[M[v]]w_2 \\ & \vdots & \\ w_{p-1,x} & = & \theta[M[v]]z_{p-2} \\ z_{p-1,x} & = & \hat{K}[M[v]]w_{p-1} \end{array}\right\} \tag{3.18}$$

as a basic special integral of the system (2.45), (2.46) for parameter value $\alpha = 2h_1$. We also define

$$N[M[V], \mathbf{Z}] = 0 \tag{3.19}$$

$$\left.\begin{array}{rcl} W_{1,x} & = & \theta[M[V]]x \\ Z_{1,x} & = & \hat{J}[M[V]]W_1 \\ W_{2,x} & = & \theta[M[V]]Z_1 \\ Z_{2,x} & = & \hat{J}[M[V]]W_2 \\ & \vdots & \\ W_{p-1,x} & = & \theta[M[V]]Z_{p-2} \\ Z_{p-1,x} & = & \hat{J}[M[V]]W_{p-1} \end{array}\right\} \tag{3.20}$$

as a basic special integral of the system (2.48), (2.49) for parameter value $\alpha = -h_1$. These definitions make sense independently of the form of $L[M[v], \mathbf{z}]$ and $N[M[V], \mathbf{Z}]$, and not just for the special cases corresponding to (2.47) and (2.50), i.e.

$$L[M[v], \mathbf{z}] = \sum_{k=0}^{n} c_k H_{2k}[M[v]] + \sum_{k=0}^{q} a_k H_{2k+1}[M[v]] + \sum_{k=2}^{p} h_k z_{k-1} + h_1 x \tag{3.21}$$

and

$$N[M[V], \mathbf{Z}] = \sum_{k=0}^{n} c_k G_{2k}[M[V]] + \sum_{k=0}^{q} a_k G_{2k+1}[M[V]] + \sum_{k=2}^{p} h_k Z_{k-1} + h_1 x \tag{3.22}$$

respectively.

Remark 3.7 *The basic special integral (3.17), (3.18) can be written as the system (3.7), (3.8) coupled to the Miura map $u = M[v] = v_x - \frac{1}{2}v^2$, this last being linearizable via $v = -2\Psi_x/\Psi$ onto $\Psi_{xx} + \frac{1}{2}u\Psi = 0$. The basic special integral (3.19), (3.20) can be written as the system (3.15), (3.16) coupled to the Miura map $U = M[V] = V_x - \frac{1}{2}V^2$, this last clearly also being linearizable via $V = -2\Phi_x/\Phi$ onto $\Phi_{xx} + \frac{1}{2}U\Phi = 0$. We recall that basic special integrals may provide a starting point for iteration using auto-BTs.*

Theorem 3.8 *Consider the system in y, $\mathbf{r} = (r_1, r_2, \ldots, r_{p-1})$ and $\mathbf{s} = (s_1, s_2, \ldots, s_{p-1})$ given by*

$$(\partial_x + y)\,\tilde{L}[M[y], \mathbf{s}] + \frac{1}{2}\tilde{\alpha} - h_1 = 0 \tag{3.23}$$

$$\left.\begin{array}{rcl} r_{1,x} &=& \theta[M[y]]x \\ s_{1,x} &=& \hat{K}[M[y]]r_1 \\ r_{2,x} &=& \theta[M[y]]s_1 \\ s_{2,x} &=& \hat{K}[M[y]]r_2 \\ &\vdots& \\ r_{p-1,x} &=& \theta[M[y]]s_{p-2} \\ s_{p-1,x} &=& \hat{K}[M[y]]r_{p-1} \end{array}\right\} \tag{3.24}$$

where

$$\begin{aligned} \tilde{L}[u, \mathbf{s}] &= \sum_{k=0}^{n} c_k H_{2k}[u] + \sum_{k=0}^{q} a_k H_{2k+1}[u] + \sum_{k=2}^{p} h_k \left(\sum_{j=1}^{k-1} 2d_j H_{2(k-j)-1}[u] + \sum_{j=1}^{k-1} e_j H_{2(k-j)-2}[u] \right) \\ &\quad + \sum_{k=2}^{p} h_k s_{k-1} + h_1 x \\ &= \sum_{k=0}^{n} c_k H_{2k}[u] + \sum_{k=0}^{q} a_k H_{2k+1}[u] + \sum_{k=0}^{p-2} \sigma_k H_{2k}[u] + \sum_{k=0}^{p-2} \rho_k H_{2k+1}[u] + \sum_{k=2}^{p} h_k s_{k-1} + h_1 x \end{aligned} \tag{3.25}$$

and (in addition to all coefficients c_k, a_k and h_k) $\tilde{\alpha}$ and all d_j and e_j are constant, and

$$\sigma_k = \sum_{j=1}^{p-1-k} h_{j+k+1} e_j, \qquad \rho_k = 2 \sum_{j=1}^{p-1-k} h_{j+k+1} d_j. \tag{3.26}$$

Then solutions y, $\mathbf{r}$, $\mathbf{s}$ of this system are mapped by the BT

$$v = y, \tag{3.27}$$

$$w_k = r_k + \sum_{j=1}^{k-1} 2d_j E_{2(k-j)-1}[M[y]] + \sum_{j=1}^{k-1} e_j E_{2(k-j)-2}[M[y]] + d_k H_0[M[y]], \qquad k = 1, \ldots, p-1, \tag{3.28}$$

$$z_k = s_k + \sum_{j=1}^{k} 2d_j H_{2(k-j)+1}[M[y]] + \sum_{j=1}^{k} e_j H_{2(k-j)}[M[y]], \qquad k = 1, \ldots, p-1, \tag{3.29}$$

$$\alpha = \tilde{\alpha}, \tag{3.30}$$

onto solutions v, $\mathbf{w}$, $\mathbf{z}$ of the system (2.45)—(2.47). We note that in (3.28) we have made use of the fact that the equations of the standard (isospectral) KK hierarchy are conservative in order to introduce $E_k[u]$, $k = 0, 1, 2, \ldots$, defined via

$$(E_k[u])_x = \theta[u] H_k[u], \qquad k = 0, 1, 2, \ldots \tag{3.31}$$

(so $E_0[u] = u$, $E_1[u] = u_{xxxx} + 10uu_{xx} + \frac{15}{2}u_x^2 + \frac{20}{3}u^3$, etc.). We also note that

$$(H_0[u])_x = 0, \qquad (H_1[u])_x = \frac{1}{2}\hat{K}[u] H_0[u], \qquad (H_{k+2}[u])_x = \hat{K}[u] E_k[u], \qquad k = 0, 1, 2, \ldots \tag{3.32}$$

(for the third of these identities see (2.8)).

Proof. Substitution of (3.27), (3.28) and (3.29) in (2.46), and using the relations

$$(H_0[M[y]])_x = 0, \quad (H_1[M[y]])_x = \frac{1}{2}\hat{K}[M[y]]H_0[M[y]], \quad (H_{k+2}[M[y]])_x = \hat{K}[M[y]]E_k[M[y]] \tag{3.33}$$

yields (3.24), whilst substitution of (3.27), (3.29) and (3.30) in (2.45) with $L[u, \mathbf{z}]$ given by (2.47) gives (3.23) with $\tilde{L}[u, \mathbf{s}]$ given by (3.25). It then follows that the BT (3.27)—(3.30) maps solutions y, $\mathbf{r}$, $\mathbf{s}$ of (3.23)—(3.25) onto solutions v, $\mathbf{w}$, $\mathbf{z}$ of (2.45)—(2.47). □

Theorem 3.9 *Consider the system in Y, $\mathbf{R} = (R_1, R_2, \ldots, R_{p-1})$ and $\mathbf{S} = (S_1, s_2, \ldots, S_{p-1})$ given by*

$$(\partial_x + Y)\, \tilde{N}[M[Y], \mathbf{S}] - \tilde{\alpha} - h_1 = 0 \tag{3.34}$$

$$\left.\begin{array}{rcl} R_{1,x} &=& \theta[M[Y]]x \\ S_{1,x} &=& \hat{J}[M[Y]]R_1 \\ R_{2,x} &=& \theta[M[Y]]S_1 \\ S_{2,x} &=& \hat{J}[M[Y]]R_2 \\ & \vdots & \\ R_{p-1,x} &=& \theta[M[Y]]S_{p-2} \\ S_{p-1,x} &=& \hat{J}[M[Y]]R_{p-1} \end{array}\right\} \tag{3.35}$$

where

$$\begin{aligned} \tilde{N}[U, \mathbf{S}] &= \sum_{k=0}^{n} c_k G_{2k}[U] + \sum_{k=0}^{q} a_k G_{2k+1}[U] + \sum_{k=2}^{p} h_k \left(\sum_{j=1}^{k-1} \frac{1}{2} f_j G_{2(k-j)-1}[U] + \sum_{j=1}^{k-1} g_j G_{2(k-j)-2}[U] \right) \\ &\quad + \sum_{k=2}^{p} h_k S_{k-1} + h_1 x \\ &= \sum_{k=0}^{n} c_k G_{2k}[U] + \sum_{k=0}^{q} a_k G_{2k+1}[U] + \sum_{k=0}^{p-2} \nu_k G_{2k}[U] + \sum_{k=0}^{p-2} \mu_k G_{2k+1}[U] + \sum_{k=2}^{p} h_k S_{k-1} + h_1 x \end{aligned} \tag{3.36}$$

and (in addition to all coefficients c_k, a_k and h_k) $\tilde{\alpha}$ and all f_j and g_j are constant, and

$$\nu_k = \sum_{j=1}^{p-1-k} h_{j+k+1} g_j, \qquad \mu_k = \frac{1}{2} \sum_{j=1}^{p-1-k} h_{j+k+1} f_j. \tag{3.37}$$

Then solutions Y, $\mathbf{R}$, $\mathbf{S}$ of this system are mapped by the BT

$$V = Y, \tag{3.38}$$

$$W_k = R_k + \sum_{j=1}^{k-1} \frac{1}{2} f_j F_{2(k-j)-1}[M[Y]] + \sum_{j=1}^{k-1} g_j F_{2(k-j)-2}[M[Y]] + f_k G_0[M[y]], \quad k = 1, \ldots, p-1, \tag{3.39}$$

$$Z_k = S_k + \sum_{j=1}^{k} \frac{1}{2} f_j G_{2(k-j)+1}[M[Y]] + \sum_{j=1}^{k} g_j G_{2(k-j)}[M[Y]], \qquad k = 1, \ldots, p-1, \tag{3.40}$$

$$\alpha = \tilde{\alpha}, \tag{3.41}$$

onto solutions V, $\mathbf{W}$, $\mathbf{Z}$ of the system (2.48)—(2.50). We note that in (3.39) we have made use of the fact that the equations of the standard (isospectral) SK hierarchy are conservative in order to introduce $F_k[U]$, $k = 0, 1, 2, \ldots$, defined via

$$(F_k[U])_x = \theta[U]G_k[U], \qquad k = 0, 1, 2, \ldots \tag{3.42}$$

(so $F_0[U] = U$, $F_1[U] = U_{xxxx} + \frac{5}{2}UU_{xx} + \frac{5}{12}U^3$, etc.). We also note that

$$(G_0[U])_x = 0, \qquad (G_1[U])_x = 2\hat{J}[U]G_0[U], \qquad (G_{k+2}[U])_x = \hat{J}[U]F_k[U], \qquad k = 0, 1, 2, \ldots \tag{3.43}$$

(for the third of these identities see (2.25).

Proof. Substitution of (3.38), (3.39) and (3.40) in (2.49), and using the relations

$$(G_0[M[Y]])_x = 0, \quad (G_1[M[Y]])_x = 2\hat{J}[M[Y]]G_0[M[Y]], \quad (G_{k+2}[M[Y]])_x = \hat{J}[M[Y]]F_k[M[Y]] \tag{3.44}$$

yields (3.35), whilst substitution of (3.38), (3.40) and (3.41) in (2.48) with $N[U, \mathbf{Z}]$ given by (2.50) gives (3.34) with $\tilde{N}[U, \mathbf{S}]$ given by (3.36). It then follows that the BT (3.38)—(3.41) maps solutions Y, $\mathbf{R}$, $\mathbf{S}$ of (3.34)—(3.36) onto solutions V, $\mathbf{W}$, $\mathbf{Z}$ of (2.48)—(2.50). □

Remark 3.10 *The BTs (3.27)—(3.30) and (3.38)—(3.41) are consequences of the structure of the systems (2.45)—(2.47) and (2.48)—(2.50) respectively. The inverses of these BTs are respectively*

$$y = v, \tag{3.45}$$

$$r_k = w_k - \sum_{j=1}^{k-1} 2d_j E_{2(k-j)-1}[M[v]] - \sum_{j=1}^{k-1} e_j E_{2(k-j)-2}[M[v]] - d_k H_0[M[v]], \qquad k = 1, \ldots, p-1, \tag{3.46}$$

$$s_k = z_k - \sum_{j=1}^{k} 2d_j H_{2(k-j)+1}[M[v]] - \sum_{j=1}^{k} e_j H_{2(k-j)}[M[v]], \qquad k = 1, \ldots, p-1, \tag{3.47}$$

$$\tilde{\alpha} = \alpha, \tag{3.48}$$

and

$$Y = V, \tag{3.49}$$

$$R_k = W_k - \sum_{j=1}^{k-1} \frac{1}{2} f_j F_{2(k-j)-1}[M[V]] - \sum_{j=1}^{k-1} g_j F_{2(k-j)-2}[M[V]] - f_k G_0[M[V]], \qquad k = 1, \ldots, p-1, \tag{3.50}$$

$$S_k = Z_k - \sum_{j=1}^{k} \frac{1}{2} f_j G_{2(k-j)+1}[M[V]] - \sum_{j=1}^{k} g_j G_{2(k-j)}[M[V]], \qquad k = 1, \ldots, p-1, \tag{3.51}$$

$$\tilde{\alpha} = \alpha, \tag{3.52}$$

The special cases of the BTs (3.27)—(3.30) and (3.38)—(3.41) where all d_j and e_j are zero, or where all f_j and g_j are zero, give the identity transformations of the systems (2.45)—(2.47) and (2.48)—(2.50).

Remark 3.11 *We define the conditions:*

$$\begin{aligned} A1: &\quad p-2 \leq n \quad \text{and} \quad p-2 \leq q; \\ A2a: &\quad p-2 > n, \quad \text{in which case we assume} \quad e_j = 0, \quad j = 1, 2, \ldots, p-n-2; \\ A2b: &\quad p-2 > q, \quad \text{in which case we assume} \quad d_j = 0, \quad j = 1, 2, \ldots, p-q-2. \end{aligned}$$

We note that condition A2a implies that $\sigma_k = 0$, $k = n+1, n+2, \ldots, p-2$, and that condition A2b implies that $\rho_k = 0$, $k = q+1, q+2, \ldots, p-2$. It then follows that the BT (3.27)—(3.30), in either of the two special cases where (i) condition A1 holds, or (ii) at least one of the conditions A2a or A2b holds, becomes an auto-BT, since we then have

$$\tilde{L}[u, \mathbf{s}] = \sum_{k=0}^{n} \gamma_k H_{2k}[u] + \sum_{k=0}^{q} \beta_k H_{2k+1}[u] + \sum_{k=2}^{p} h_k s_{k-1} + h_1 x, \quad \gamma_k = c_k + \sigma_k, \quad \beta_k = a_k + \rho_k \tag{3.53}$$

(where if $p-2 < n$ we take $\sigma_k = 0$, $k = p-1, \ldots, n$, and if $p-2 < q$ we take $\rho_k = 0$, $k = p-1, \ldots, q$). This auto-BT involves the changes in parameters $c_k = \gamma_k - \sigma_k$, $k = 0, \ldots, n$ and $a_k = \beta_k - \rho_k$, $k = 0, \ldots, q$.

Similarly we may define the conditions:

$$\begin{aligned}
B1: &\quad p-2 \leq n \quad \text{and} \quad p-2 \leq q;\\
B2a: &\quad p-2 > n, \quad \text{in which case we assume} \quad g_j = 0, \quad j = 1, 2, \ldots, p-n-2;\\
B2b: &\quad p-2 > q, \quad \text{in which case we assume} \quad f_j = 0, \quad j = 1, 2, \ldots, p-q-2.
\end{aligned}$$

We note that condition B2a implies that $\nu_k = 0$, $k = n+1, n+2, \ldots, p-2$, and that condition B2b implies that $\mu_k = 0$, $k = q+1, q+2, \ldots, p-2$. It then follows that the BT (3.38)—(3.41), in either of the two special cases where (i) condition B1 holds, or (ii) at least one of the conditions B2a or B2b holds, becomes an auto-BT, since we then have

$$\tilde{N}[U, \mathbf{S}] = \sum_{k=0}^{n} \delta_k G_{2k}[U] + \sum_{k=0}^{q} \epsilon_k G_{2k+1}[U] + \sum_{k=2}^{p} h_k S_{k-1} + h_1 x, \quad \delta_k = c_k + \nu_k, \quad \epsilon_k = a_k + \mu_k \tag{3.54}$$

(where if $p-2 < n$ we take $\nu_k = 0$, $k = p-1, \ldots, n$, and if $p-2 < q$ we take $\mu_k = 0$, $k = p-1, \ldots, q$). This auto-BT involves the changes in parameters $c_k = \delta_k - \nu_k$, $k = 0, \ldots, n$ and $a_k = \epsilon_k - \mu_k$, $k = 0, \ldots, q$.

The inverses of these auto-BTs derived as special cases of (3.27)—(3.30) and (3.38)—(3.41) are as given in Remark 3.10 along with the changes in parameters $\gamma_k = c_k + \sigma_k$, $k = 0, \ldots, n$ and $\beta_k = a_k + \rho_k$, $k = 0, \ldots, q$ in the first case, and $\delta_k = c_k + \nu_k$, $k = 0, \ldots, n$ and $\epsilon_k = a_k + \mu_k$, $k = 0, \ldots, q$ in the second case.

Theorem 3.12 *Solutions y, $\mathbf{r}$, $\mathbf{s}$ of the system*

$$(\partial_x + y)\, L[M[y], \mathbf{s}] + \frac{1}{2}\tilde{\alpha} - h_1 = 0 \tag{3.55}$$

$$\left.\begin{array}{rcl}
r_{1,x} &=& \theta[M[y]]x\\
s_{1,x} &=& \hat{K}[M[y]]r_1\\
r_{2,x} &=& \theta[M[y]]s_1\\
s_{2,x} &=& \hat{K}[M[y]]r_2\\
&\vdots&\\
r_{p-1,x} &=& \theta[M[y]]s_{p-2}\\
s_{p-1,x} &=& \hat{K}[M[y]]r_{p-1}
\end{array}\right\} \tag{3.56}$$

are mapped onto solutions v, $\mathbf{w}$, $\mathbf{z}$ of the system (2.45), (2.46) by the auto-BT

$$v = y - \frac{\alpha - \tilde{\alpha}}{2L[M[y], \mathbf{s}]}, \tag{3.57}$$

$$w_k = r_k, \qquad k = 1, \ldots, p-1, \tag{3.58}$$

$$z_k = s_k, \qquad k = 1, \ldots, p-1, \tag{3.59}$$

$$\alpha = -\tilde{\alpha} + 4h_1. \tag{3.60}$$

Proof. First of all we note that $M[v] = M[y]$:

$$\begin{aligned} M[v] = v_x - \frac{1}{2}v^2 &= y_x - \frac{1}{2}y^2 + \frac{\alpha - \tilde{\alpha}}{2L[M[y], \mathbf{s}]^2}\left((\partial_x + y)L[M[y], \mathbf{s}] - \frac{1}{4}(\alpha - \tilde{\alpha})\right) \\ &= y_x - \frac{1}{2}y^2 + \frac{\alpha - \tilde{\alpha}}{8L[M[y], \mathbf{s}]^2}(4h_1 - \alpha - \tilde{\alpha}) \\ &= y_x - \frac{1}{2}y^2 = M[y]. \end{aligned} \tag{3.61}$$

It then follows that solutions of (3.55) and (3.56) are mapped onto solutions of (2.46). Also,

$$\begin{aligned} (\partial_x + v)L[M[v], \mathbf{z}] + \frac{1}{2}\alpha - h_1 &= \left(\partial_x + y - \frac{\alpha - \tilde{\alpha}}{2L[M[y], \mathbf{s}]}\right) L[M[y], \mathbf{s}] + \frac{1}{2}\alpha - h_1 \\ &= (\partial_x + y)L[M[y], \mathbf{s}] + \frac{1}{2}\tilde{\alpha} - h_1, \end{aligned} \tag{3.62}$$

and so it then follows that solutions of equation (3.55) are mapped onto solutions of (2.45). □

Theorem 3.13 *Solutions Y, $\mathbf{R}$, $\mathbf{S}$ of the system*

$$(\partial_x + Y)\, N[M[Y], \mathbf{S}] - \tilde{\alpha} - h_1 = 0 \tag{3.63}$$

$$\left.\begin{aligned} R_{1,x} &= \theta[M[Y]]x \\ S_{1,x} &= \hat{J}[M[Y]]R_1 \\ R_{2,x} &= \theta[M[Y]]S_1 \\ S_{2,x} &= \hat{J}[M[Y]]R_2 \\ &\vdots \\ R_{p-1,x} &= \theta[M[Y]]S_{p-2} \\ S_{p-1,x} &= \hat{J}[M[Y]]R_{p-1} \end{aligned}\right\} \tag{3.64}$$

are mapped onto solutions V, $\mathbf{W}$, $\mathbf{Z}$ of the system (2.48), (2.49) by the auto-BT

$$V = Y + \frac{\alpha - \tilde{\alpha}}{N[M[Y], \mathbf{S}]}, \tag{3.65}$$

$$W_k = R_k, \qquad k = 1, \ldots, p-1, \tag{3.66}$$

$$Z_k = S_k, \qquad k = 1, \ldots, p-1, \tag{3.67}$$

$$\alpha = -\tilde{\alpha} - 2h_1. \tag{3.68}$$

Proof. First of all we note that $M[V] = M[Y]$:

$$\begin{aligned} M[V] = V_x - \frac{1}{2}V^2 &= Y_x - \frac{1}{2}Y^2 - \frac{\alpha - \tilde{\alpha}}{N[M[Y], \mathbf{S}]^2}\left((\partial_x + Y)N[M[Y], \mathbf{S}] + \frac{1}{2}(\alpha - \tilde{\alpha})\right) \\ &= Y_x - \frac{1}{2}Y^2 - \frac{\alpha - \tilde{\alpha}}{2N[M[Y], \mathbf{S}]^2}(2h_1 + \alpha + \tilde{\alpha}) \\ &= Y_x - \frac{1}{2}Y^2 = M[Y]. \end{aligned} \tag{3.69}$$

It then follows that solutions of (3.63) and (3.64) are mapped onto solutions of (2.49). Also,

$$\begin{aligned} (\partial_x + V)N[M[V], \mathbf{Z}] - \alpha - h_1 &= \left(\partial_x + Y + \frac{\alpha - \tilde{\alpha}}{N[M[Y], \mathbf{S}]}\right) N[M[Y], \mathbf{S}] - \alpha - h_1 \\ &= (\partial_x + Y)N[M[Y], \mathbf{S}] - \tilde{\alpha} - h_1, \end{aligned} \tag{3.70}$$

and so it then follows that solutions of equation (3.63) are mapped onto solutions of (2.48). □

Remark 3.14 *We note that the auto-BTs given in Theorems 3.12 and 3.13 hold independently of the forms of $L[u, \mathbf{z}]$ and $N[U, \mathbf{Z}]$, and not just for the special cases (2.47) and (2.50).*

Remark 3.15 *Since for the auto-BT (3.57)— (3.60) we have $M[v] = M[y]$, $\mathbf{w} = \mathbf{r}$ and $\mathbf{z} = \mathbf{s}$, its inverse is*

$$y = v - \frac{\tilde{\alpha} - \alpha}{2L[M[v], \mathbf{z}]}, \tag{3.71}$$

$$r_k = w_k, \qquad k = 1, \ldots, p-1, \tag{3.72}$$

$$s_k = z_k, \qquad k = 1, \ldots, p-1, \tag{3.73}$$

$$\tilde{\alpha} = -\alpha + 4h_1. \tag{3.74}$$

Thus we see that the auto-BT (3.57)— (3.60) is an involution. Similarly, the inverse of the auto-BT (3.65)— (3.68) is

$$Y = V + \frac{\tilde{\alpha} - \alpha}{N[M[V], \mathbf{Z}]}, \tag{3.75}$$

$$R_k = W_k, \qquad k = 1, \ldots, p-1, \tag{3.76}$$

$$S_k = Z_k, \qquad k = 1, \ldots, p-1, \tag{3.77}$$

$$\tilde{\alpha} = -\alpha - 2h_1, \tag{3.78}$$

and so we see that the auto-BT (3.65)— (3.68) is also an involution.

Remark 3.16 *Let us take $L[u, \mathbf{z}]$ as in (2.47) and $N[U, \mathbf{Z}]$ as in (2.50), so that (2.45)—(2.47) and (2.48)— (2.50) correspond under $V = -2v$ to local forms of the same hierarchy, and consider iterating the auto-BTs given in Theorems 3.12 and 3.13. Beginning with initial parameter value $\hat{\alpha}$, combining the shifts in parameter values (3.60) and (3.68) of these auto-BTs yields all parameter values of the form $\hat{\alpha} - 6m_1h_1$ and $-2h_1 - \hat{\alpha} - 6m_2h_1$ where m_1 and m_2 are integers. In the two special cases where either $\hat{\alpha} = -h_1 - 6h_1n_1$ for some integer n_1, or $\hat{\alpha} = 2h_1 - 6h_1n_2$ for some integer n_2, the set of iterated parameter values is evenly spaced (being either all parameter values of the form $-h_1 - 6M_1h_1$ where M_1 is integer, or all parameter values of the form $2h_1 - 6M_2h_1$ where M_2 is integer, respectively). We note that the particular cases $\hat{\alpha} = -h_1$ and $\hat{\alpha} = 2h_1$ are the parameter values for which the basic special integrals in Definition 3.6 are defined.*

4 Nested equations

In this section we give results on the nesting of equations analogous to those presented in [37] for the extended P_{II} and related hierarchies. We thus obtain relations between systems of different orders but of the same form. We note (see Remark 4.11) that it is impossible to obtain lower order nested systems of the standard or generalized cases of our hierarchies, and in this sense, in the non-autonomous case, our results are particular to our extended hierarchies.

Theorem 4.1 *Let u, $\bar{\mathbf{w}} = (w_1, w_2, \ldots, w_{p-2})$, $\bar{\mathbf{z}} = (z_1, z_2, \ldots, z_{p-2})$ be a solution of the system*

$$\bar{L}[u, \bar{\mathbf{z}}] \equiv \sum_{k=1}^{n} c_k H_{2k-2}[u] + \sum_{k=1}^{q} a_k H_{2k-1}[u] + \sum_{k=3}^{p} h_k z_{k-2} + h_2 x = 0 \tag{4.1}$$

$$\left.\begin{array}{rcl} w_{1,x} &=& \theta[u]x \\ z_{1,x} &=& \hat{K}[u]w_1 \\ w_{2,x} &=& \theta[u]z_1 \\ z_{2,x} &=& \hat{K}[u]w_2 \\ &\vdots& \\ w_{p-2,x} &=& \theta[u]z_{p-3} \\ z_{p-2,x} &=& \hat{K}[u]w_{p-2} \end{array}\right\} \tag{4.2}$$

If we introduce functions w_{p-1} and z_{p-1} which satisfy the additional equations

$$w_{p-1,x} = \theta[u]z_{p-2}, \qquad z_{p-1,x} = \hat{K}[u]w_{p-1}, \tag{4.3}$$

then we may choose c_0 and a_0 such that u, $\mathbf{w} = (w_1, w_2, \ldots, w_{p-1})$ and $\mathbf{z} = (z_1, z_2, \ldots, z_{p-1})$ satisfy (3.7), (3.8) with $L[u, \mathbf{z}]$ given by (2.47) in the case where $h_1 = 0$, i.e., we may choose c_0 and a_0 such that u, $\mathbf{w}$ and $\mathbf{z}$ satisfy the system

$$L_0[u, \mathbf{z}] = \sum_{k=0}^{n} c_k H_{2k}[u] + \sum_{k=0}^{q} a_k H_{2k+1}[u] + \sum_{k=2}^{p} h_k z_{k-1} = 0 \tag{4.4}$$

$$\left.\begin{array}{rcl} w_{1,x} &=& \theta[u]x \\ z_{1,x} &=& \hat{K}[u]w_1 \\ w_{2,x} &=& \theta[u]z_1 \\ z_{2,x} &=& \hat{K}[u]w_2 \\ &\vdots& \\ w_{p-1,x} &=& \theta[u]z_{p-2} \\ z_{p-1,x} &=& \hat{K}[u]w_{p-1} \end{array}\right\} \tag{4.5}$$

Proof. That u, $\mathbf{w}$ and $\mathbf{z}$ satisfy (4.5) is immediate. We define

$$\Gamma[u, \mathbf{w}] \equiv \sum_{k=1}^{n} c_k E_{2k-2}[u] + \sum_{k=1}^{q} a_k E_{2k-1}[u] + \frac{1}{2}a_0 + \sum_{k=3}^{p} h_k w_{k-1} + h_2 w_1, \tag{4.6}$$

where $E_k[u]$, $k = 0, 1, 2, \ldots$, are as defined in (3.31), and where we recall that (3.32) also holds. Then, since (4.5) holds, we have

$$\begin{aligned} (L_0[u, \mathbf{z}] - c_0)_x &= \partial_x \left(\sum_{k=1}^{n} c_k H_{2k}[u] + \sum_{k=0}^{q} a_k H_{2k+1}[u] + \sum_{k=2}^{p} h_k z_{k-1} \right) \\ &= \hat{K}[u] \left(\sum_{k=1}^{n} c_k E_{2k-2}[u] + \sum_{k=1}^{q} a_k E_{2k-1}[u] + \frac{1}{2}a_0 + \sum_{k=2}^{p} h_k w_{k-1} \right) \\ &= \hat{K}[u]\Gamma[u, \mathbf{w}] \end{aligned} \tag{4.7}$$

and also

$$\begin{aligned} \left(\Gamma[u, \mathbf{w}] - \frac{1}{2}a_0\right)_x &= \partial_x \left(\sum_{k=1}^{n} c_k E_{2k-2}[u] + \sum_{k=1}^{q} a_k E_{2k-1}[u] + \sum_{k=3}^{p} h_k w_{k-1} + h_2 w_1 \right) \\ &= \theta[u] \left(\sum_{k=1}^{n} c_k H_{2k-2}[u] + \sum_{k=1}^{q} a_k H_{2k-1}[u] + \sum_{k=3}^{p} h_k z_{k-2} + h_2 x \right) \\ &= \theta[u]\bar{L}[u, \bar{\mathbf{z}}]. \end{aligned} \tag{4.8}$$

From (4.8) we see that if $\bar{L}[u, \bar{\mathbf{z}}] = 0$, i.e., if (4.1) holds, then we may choose a_0 such that $\Gamma[u, \mathbf{w}] = 0$. For this choice of a_0, we see from (4.7) that we may then choose c_0 such that $L_0[u, \mathbf{z}] = 0$, i.e., such that (4.4) holds. □

Definition 4.2 *We refer to (4.1), (4.2) as a lower order nested system of (3.7), (3.8) with $L[u, \mathbf{z}]$ given by (2.47). We note that we are assuming that c_0, a_0 and h_1 are parameters whose values we are free to set.*

Theorem 4.3 *Let $\bar{v}$, $\bar{\mathbf{w}} = (w_1, w_2, \ldots, w_{p-2})$, $\bar{\mathbf{z}} = (z_1, z_2, \ldots, z_{p-2})$ be a solution of the system*

$$(\partial_x + \bar{v})\bar{L}[M[\bar{v}], \bar{\mathbf{z}}] + \frac{1}{2}\bar{\alpha} - h_2 = 0 \tag{4.9}$$

$$\left.\begin{array}{rcl} w_{1,x} & = & \theta[M[\bar{v}]]x \\ z_{1,x} & = & \hat{K}[M[\bar{v}]]w_1 \\ w_{2,x} & = & \theta[M[\bar{v}]]z_1 \\ z_{2,x} & = & \hat{K}[M[\bar{v}]]w_2 \\ & \vdots & \\ w_{p-2,x} & = & \theta[M[\bar{v}]]z_{p-3} \\ z_{p-2,x} & = & \hat{K}[M[\bar{v}]]w_{p-2} \end{array}\right\} \tag{4.10}$$

where $\bar{\alpha}$ is an arbitrary constant and, as in Theorem 4.1,

$$\bar{L}[u, \bar{\mathbf{z}}] = \sum_{k=1}^{n} c_k H_{2k-2}[u] + \sum_{k=1}^{q} a_k H_{2k-1}[u] + \sum_{k=3}^{p} h_k z_{k-2} + h_2 x. \tag{4.11}$$

If we introduce functions w_{p-1} and z_{p-1} which satisfy the additional equations

$$w_{p-1,x} = \theta[M[\bar{v}]]z_{p-2}, \qquad z_{p-1,x} = \hat{K}[M[\bar{v}]]w_{p-1}, \tag{4.12}$$

and choose v such that $M[v] = M[\bar{v}]$, then we may choose c_0 and a_0 such that v, $\mathbf{w} = (w_1, w_2, \ldots, w_{p-1})$ and $\mathbf{z} = (z_1, z_2, \ldots, z_{p-1})$ satisfy, for parameter value $\alpha = 0$, the system (2.45)—(2.47) in the case where $h_1 = 0$. That is, we may choose c_0 and a_0 such that v, $\mathbf{w}$ and $\mathbf{z}$ satisfy, for parameter value $\alpha = 0$, the system

$$(\partial_x + v)\, L_0[M[v], \mathbf{z}] + \frac{1}{2}\alpha = 0 \tag{4.13}$$

$$\left.\begin{array}{rcl} w_{1,x} & = & \theta[M[v]]x \\ z_{1,x} & = & \hat{K}[M[v]]w_1 \\ w_{2,x} & = & \theta[M[v]]z_1 \\ z_{2,x} & = & \hat{K}[M[v]]w_2 \\ & \vdots & \\ w_{p-1,x} & = & \theta[M[v]]z_{p-2} \\ z_{p-1,x} & = & \hat{K}[M[v]]w_{p-1} \end{array}\right\} \tag{4.14}$$

where, as in Theorem 4.1,

$$L_0[u, \mathbf{z}] = \sum_{k=0}^{n} c_k H_{2k}[u] + \sum_{k=0}^{q} a_k H_{2k+1}[u] + \sum_{k=2}^{p} h_k z_{k-1}. \tag{4.15}$$

Proof. Setting $u = M[\bar{v}]$, we obtain immediately that (4.5) holds and also, with $\Gamma[u, \mathbf{w}]$ as defined in (4.6), that (4.7) and (4.8) hold. Since equation (4.8) gives

$$\left(\Gamma[u, \mathbf{w}] - \frac{1}{2}a_0\right)_x = (\partial_x - \bar{v})\partial_x \left[(\partial_x + \bar{v})\bar{L}[M[\bar{v}], \bar{\mathbf{z}}] + \frac{1}{2}\bar{\alpha} - h_2\right] \tag{4.16}$$

for any arbitrary constant $\bar{\alpha}$, it then follows, similarly to the proof of Theorem 4.1, that since (4.9) holds then we may choose c_0 and a_0 such that $L_0[u, \mathbf{z}] = 0$, i.e., such that (4.4) is satisfied. Thus equations (4.4) and (4.5) are satisfied: with $u = M[v]$ these equations then define a basic special integral, for parameter value $\alpha = 0$, of the system (4.13)—(4.15) (see Definition 3.6). This basic special integral consists of the equation

$$L_0[M[v], \mathbf{z}] = 0 \tag{4.17}$$

along with (4.14). Solutions v, $\mathbf{w}$ and $\mathbf{z}$ of this basic special integral then give, for parameter value $\alpha = 0$, solutions of the system (4.13)—(4.15). (We note that solutions of this basic special integral can be obtained from solutions u, $\mathbf{w}$, $\mathbf{z}$ of (4.4), (4.5) by taking $v = -2\Psi_x/\Psi$ where $\Psi_{xx} + \frac{1}{2}u\Psi = 0$; see Remark 3.7). □

Definition 4.4 *We refer to (4.9)—(4.11) as a lower order nested system of (2.45)—(2.47). Again we note that we are assuming that c_0, a_0 and h_1 are parameters whose values we are free to set.*

We now turn to the SK/mSK version of the above results, and then make some remarks common to both the KK/mKK and SK/mSK versions of our results on nested equations.

Theorem 4.5 *Let U, $\bar{\mathbf{W}} = (W_1, W_2, \ldots, W_{p-2})$, $\bar{\mathbf{Z}} = (Z_1, Z_2, \ldots, Z_{p-2})$ be a solution of the system*

$$\bar{N}[U, \bar{\mathbf{Z}}] \equiv \sum_{k=1}^{n} c_k G_{2k-2}[U] + \sum_{k=1}^{q} a_k G_{2k-1}[U] + \sum_{k=3}^{p} h_k Z_{k-2} + h_2 x = 0 \tag{4.18}$$

$$\left.\begin{array}{rcl} W_{1,x} &=& \theta[U]x \\ Z_{1,x} &=& \hat{J}[U]W_1 \\ W_{2,x} &=& \theta[U]Z_1 \\ Z_{2,x} &=& \hat{J}[U]W_2 \\ &\vdots& \\ W_{p-2,x} &=& \theta[U]Z_{p-3} \\ Z_{p-2,x} &=& \hat{J}[U]W_{p-2} \end{array}\right\} \tag{4.19}$$

If we introduce functions W_{p-1} and Z_{p-1} which satisfy the additional equations

$$W_{p-1,x} = \theta[U]Z_{p-2}, \qquad Z_{p-1,x} = \hat{J}[U]W_{p-1}, \tag{4.20}$$

then we may choose c_0 and a_0 such that U, $\mathbf{W} = (W_1, W_2, \ldots, W_{p-1})$ and $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_{p-1})$ satisfy (3.15), (3.16) with $N[U, \mathbf{Z}]$ given by (2.50) in the case where $h_1 = 0$, i.e., we may choose c_0 and a_0 such that U, $\mathbf{W}$ and $\mathbf{Z}$ satisfy the system

$$N_0[U, \mathbf{Z}] = \sum_{k=0}^{n} c_k G_{2k}[U] + \sum_{k=0}^{q} a_k G_{2k+1}[U] + \sum_{k=2}^{p} h_k Z_{k-1} = 0 \tag{4.21}$$

$$\left.\begin{array}{rcl} W_{1,x} &=& \theta[U]x \\ Z_{1,x} &=& \hat{J}[U]W_1 \\ W_{2,x} &=& \theta[U]Z_1 \\ Z_{2,x} &=& \hat{J}[U]W_2 \\ &\vdots& \\ W_{p-1,x} &=& \theta[U]Z_{p-2} \\ Z_{p-1,x} &=& \hat{J}[U]W_{p-1} \end{array}\right\} \tag{4.22}$$

Proof. That U, $\mathbf{W}$ and $\mathbf{Z}$ satisfy (4.22) is immediate. We define

$$\Delta[U,\mathbf{W}] \equiv \sum_{k=1}^{n} c_k F_{2k-2}[U] + \sum_{k=1}^{q} a_k F_{2k-1}[U] + 2a_0 + \sum_{k=3}^{p} h_k W_{k-1} + h_2 W_1, \tag{4.23}$$

where $F_k[u]$, $k=0,1,2,\ldots$, are as defined in (3.42), and where we recall that (3.43) also holds. Then, since (4.22) holds, we have

$$\begin{aligned} (N_0[U,\mathbf{Z}]-c_0)_x &= \partial_x\left(\sum_{k=1}^{n} c_k G_{2k}[U] + \sum_{k=0}^{q} a_k G_{2k+1}[U] + \sum_{k=2}^{p} h_k Z_{k-1}\right) \\ &= \hat{J}[U]\left(\sum_{k=1}^{n} c_k F_{2k-2}[U] + \sum_{k=1}^{q} a_k F_{2k-1}[U] + 2a_0 + \sum_{k=2}^{p} h_k W_{k-1}\right) \\ &= \hat{J}[U]\Delta[U,\mathbf{W}] \end{aligned} \tag{4.24}$$

and also

$$\begin{aligned} (\Delta[U,\mathbf{W}]-2a_0)_x &= \partial_x\left(\sum_{k=1}^{n} c_k F_{2k-2}[U] + \sum_{k=1}^{q} a_k F_{2k-1}[U] + \sum_{k=3}^{p} h_k W_{k-1} + h_2 W_1\right) \\ &= \theta[U]\left(\sum_{k=1}^{n} c_k G_{2k-2}[U] + \sum_{k=1}^{q} a_k G_{2k-1}[U] + \sum_{k=3}^{p} h_k Z_{k-2} + h_2 x\right) \\ &= \theta[U]\bar{N}[U,\bar{\mathbf{Z}}] \end{aligned} \tag{4.25}$$

From (4.25) we see that if $N[U,\bar{\mathbf{Z}}]=0$, i.e., if (4.18) holds, then we may choose a_0 such that $\Delta[U,\mathbf{W}]=0$. For this choice of a_0, we see from (4.24) that we may then choose c_0 such that $N_0[U,\mathbf{Z}]=0$, i.e., such that (4.21) holds. □

Definition 4.6 *We refer to (4.18), (4.19) as a lower order nested system of (3.15), (3.16) with* $N[U,\mathbf{Z}]$ *given by (2.50). We note that we are assuming that* c_0*,* a_0 *and* h_1 *are parameters whose values we are free to set.*

Theorem 4.7 *Let* $\bar{V}$*,* $\bar{\mathbf{W}}=(W_1,W_2,\ldots,W_{p-2})$*,* $\bar{\mathbf{Z}}=(Z_1,Z_2,\ldots,Z_{p-2})$ *be a solution of the system*

$$(\partial_x+\bar{V})\bar{N}[M[\bar{V}],\bar{\mathbf{Z}}]-\bar{\alpha}-h_2=0 \tag{4.26}$$

$$\left.\begin{array}{rcl} W_{1,x} &=& \theta[M[\bar{V}]]x \\ Z_{1,x} &=& \hat{J}[M[\bar{V}]]W_1 \\ W_{2,x} &=& \theta[M[\bar{V}]]Z_1 \\ Z_{2,x} &=& \hat{J}[M[\bar{V}]]W_2 \\ &\vdots& \\ W_{p-2,x} &=& \theta[M[\bar{V}]]Z_{p-3} \\ Z_{p-2,x} &=& \hat{J}[M[\bar{V}]]W_{p-2} \end{array}\right\} \tag{4.27}$$

where $\bar{\alpha}$ is an arbitrary constant and, as in Theorem 4.5,

$$\bar{N}[U,\bar{\mathbf{Z}}] = \sum_{k=1}^{n} c_k G_{2k-2}[U] + \sum_{k=1}^{q} a_k G_{2k-1}[U] + \sum_{k=3}^{p} h_k Z_{k-2} + h_2 x. \tag{4.28}$$

If we introduce functions W_{p-1} and Z_{p-1} which satisfy the additional equations

$$W_{p-1,x} = \theta[M[\bar{V}]]Z_{p-2}, \qquad Z_{p-1,x} = \hat{J}[M[\bar{V}]]W_{p-1}, \tag{4.29}$$

and choose V such that $M[V] = M[\bar{V}]$, then we may choose c_0 and a_0 such that V, $\mathbf{W} = (W_1, W_2, \ldots, W_{p-1})$ and $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_{p-1})$ satisfy, for parameter value $\alpha = 0$, the system (2.48)—(2.50) in the case where $h_1 = 0$. That is, we may choose c_0 and a_0 such that V, $\mathbf{W}$ and $\mathbf{Z}$ satisfy, for parameter value $\alpha = 0$, the system

$$(\partial_x + V)\, N_0[M[V],\mathbf{Z}] - \alpha = 0 \tag{4.30}$$

$$\left.\begin{array}{rcl} W_{1,x} &=& \theta[M[V]]x \\ Z_{1,x} &=& \hat{J}[M[V]]W_1 \\ W_{2,x} &=& \theta[M[V]]Z_1 \\ Z_{2,x} &=& \hat{J}[M[V]]W_2 \\ &\vdots& \\ W_{p-1,x} &=& \theta[M[V]]Z_{p-2} \\ Z_{p-1,x} &=& \hat{J}[M[V]]W_{p-1} \end{array}\right\} \tag{4.31}$$

where, as in Theorem 4.5,

$$N_0[U,\mathbf{Z}] = \sum_{k=0}^{n} c_k G_{2k}[U] + \sum_{k=0}^{q} a_k G_{2k+1}[U] + \sum_{k=2}^{p} h_k Z_{k-1}. \tag{4.32}$$

Proof. Setting $U = M[\bar{V}]$, we obtain immediately that (4.22) holds and also, with $\Delta[U,\mathbf{W}]$ as defined in (4.23), that (4.24) and (4.25) hold. Since equation (4.25) gives

$$(\Delta[U,\mathbf{W}] - 2a_0)_x = (\partial_x - \bar{V})\partial_x\Big[(\partial_x + \bar{V})\bar{N}[M[\bar{V}],\bar{\mathbf{Z}}] - \bar{\alpha} - h_2\Big] \tag{4.33}$$

for any arbitrary constant $\bar{\alpha}$, it then follows, similarly to the proof of Theorem 4.5, that since (4.26) holds then we may choose c_0 and a_0 such that $N_0[U,\mathbf{Z}] = 0$, i.e., such that (4.21) is satisfied. Thus equations (4.21) and (4.22) are satisfied: with $U = M[V]$ these equations then define a basic special integral, for parameter value $\alpha = 0$, of the system (4.30)—(4.32) (see Definition 3.6). This basic special integral consists of the equation

$$N_0[M[V],\mathbf{Z}] = 0 \tag{4.34}$$

along with (4.31). Solutions V, $\mathbf{W}$ and $\mathbf{Z}$ of this basic special integral then give, for parameter value $\alpha = 0$, solutions of the system (4.30)—(4.32). (We note that solutions of this basic special integral can be obtained from solutions U, $\mathbf{W}$, $\mathbf{Z}$ of (4.21), (4.22) by taking $V = -2\Phi_x/\Phi$ where $\Phi_{xx} + \frac{1}{2}U\Phi = 0$; see Remark 3.7). □

Definition 4.8 *We refer to (4.26)—(4.28) as a lower order nested system of (2.48)—(2.50). Again we note that we are assuming that c_0, a_0 and h_1 are parameters whose values we are free to set.*

Remark 4.9 *We note that in Theorem 4.3, the solutions v and $\bar{v}$ need not coincide (we only require $M[v] = M[\bar{v}]$). Similarly for the solutions V and $\bar{V}$ in Theorem 4.7.*

Remark 4.10 *Each of the Theorems 4.1, 4.3, 4.5 and 4.7 may be applied repeatedly, in order to obtain successive lower order nested systems. (Second lower order nested systems, for example, would be for* $h_2 = 0$ *and a choice of* c_1 *and* a_1*.)*

Remark 4.11 *The results of this section are analogous to our results on the nesting of equations presented in [37], except that here we require that* $h_1 = 0$*. This requirement that then means that we cannot obtain lower order nested systems of the standard or generalized cases of the hierarchies (2.45)—(2.47) and (2.48)—(2.50), or of the equations which govern their basic special integrals, i.e., (3.7), (3.8), (2.47) and (3.15), (3.16), (2.50) respectively, since in these cases we have* $h_k = 0$*,* $k = 2, 3, \ldots, h_p$*, and* $h_1 \neq 0$*. However, the standard and generalized cases of these hierarchies, and of the equations which govern their basic special integrals, can be nested systems of higher order (and necessarily extended) systems.*

5 Conclusions

In a recent paper [37] we introduced a new extended second Painlevé hierarchy and studied its propeties. The approach used in [37], based on the use of nonisospectral scattering problems, is widely applicable, and we therefore expected to be able to derive extended versions of other continuous, discrete and differential-delay Painlevé hierarchies and equations. Here we have given a second example of an extended Painlevé hierarchy, namely our extended KK-SK Painlevé hierarchy, and have given results on BTs, auto-BTs and other properties of this and related ODE hierarchies. We have also discussed the nesting of equations, thus obtaining relations between systems of different orders but of the same form. In subsequent papers we will use our approach in order to derive and study further extended Painlevé hierarchies, continuous, discrete and differential-delay.

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References

[1] P. Painlevé, Bull. Soc. Math. Fr. 28 (1900) 201–261.

[2] P. Painlevé, Acta Math. (1902) 25 1–85.

[3] B. Gambier, Acta Math. 33 (1910) 1-55.

[4] E. L. Ince, "Ordinary Differential Equations," (Dover, New York, 1956).

[5] J. Chazy, Acta Math. 34 (1911) 317-385.

[6] R. Garnier, Ann. Sci. École Normale Sup. 48 (1912) 1-126.

[7] H. Exton, Rend. Mat. (6) 6 (1973) 419-462.

[8] I. P. Martynov, Differents. Uravn. 21 (1985) 764-771.

[9] I. P. Martynov, Differents. Uravn. 21 (1985) 937-946.

[10] F. J. Bureau, Ann. Mat. Pura Appl. (IV) 66 (1964) 1-116.

[11] M. J. Ablowitz and H. Segur, Phys. Rev. Lett. 38 (1977) 1103–1106.

[12] H. Airault, Stud. Appl. Math. 61 (1979) 31-53.

[13] H. Flaschka and A. C. Newell, Comm. Math. Phys. 76 (1980) 65-116.

[14] A. Pickering, Phys. Lett. A 301 (2002) 275-280.

[15] N. A. Kudryashov, Phys. Lett. A 224 (1997) 353-360.

[16] P. R. Gordoa and A. Pickering, J. Math. Phys. 40 (1999) 5749-5786.

[17] N. A. Kudryashov, Phys. Lett. A 252 (1999) 173-179.

[18] P. R. Gordoa and A. Pickering, J. Phys. A 33 (2000) 557-567.

[19] P. R. Gordoa, N. Joshi and A. Pickering, Publ. Res. Inst. Math. Sci. (Kyoto) 37 (2001) 327-347.

[20] A. Pickering, Theor. Math. Phys. 137 (2003) 1733-1742.

[21] N. A. Kudryashov, J. Math. Phys. 44 (2003) 6160-6178.

[22] N. A. Kudryashov, Theor. Math. Phys. 134 (2003) 86-93.

[23] P. R. Gordoa, N. Joshi, A. Pickering, J. Math. Phys. 47 (2006) 073504.

[24] A. Pickering, Chaos, Solitons and Fractals 45 (2012) 935-941.

[25] A. Pickering, Appl. Math. Comp. 218 (2012) 11177-11181.

[26] J. M. Conde, P. R. Gordoa and A. Pickering, Commun. Nonlinear Sci. Numer. Simulat. 19 (2014) 448-458.

[27] A. N. W. Hone, Phys. D 118 (1998) 1-16.

[28] N. Joshi and A. Pickering, preprint (1999).

[29] N. A. Kudryashov, Phys. Lett. A 273 (2000) 194-202.

[30] N. A. Kudryashov, Regul. Chaotic Dyn. 19 (2014) 48-63.

[31] P. R. Gordoa, A. Pickering and Z. N. Zhu, J. Differential Equations 261 (2016) 1128-1175.

[32] U. Muğan and F. Jrad, J. Phys. A 32 (1999) 7933–7952.

[33] U. Muğan and F. Jrad, J. Nonlinear Math. Phys. 9 (2002) 282–310.

[34] C. M. Cosgrove, Stud. Appl. Math. 104 (2000) 1-65.

[35] C. M. Cosgrove, Stud. Appl. Math. 116 (2006) 321-413.

[36] D. Levi, O. Ragnisco and M. A. Rodríguez, Theor. Math. Phys. 93 (1993) 1409-1414.

[37] P. R. Gordoa and A. Pickering, J. Differential Equations 263 (2017) 4070-4125.

[38] K. Sawada and T. Kotera, Prog. Theor. Phys. 51 (1974) 1355-1367.

[39] P. J. Caudrey, R. K. Dodd and J. D. Gibbon, Proc. R. Soc. Lond. A 351 (1976) 407-422.

[40] D. J. Kaup, Stud. Appl. Math. 62 (1980) 189-216.

[41] A. P. Fordy and J. Gibbons, Phys. Lett. A 75 (1980) 325-325.

[42] A. P. Fordy and J. Gibbons, J. Math. Phys. 21 (1980) 2508-2510.

[43] B. Fuchssteiner and W. Oevel, J. Math. Phys. 23 (1982) 358-363.

[44] J. Weiss, J. Math. Phys. 25 (1984) 13-24.

[45] D. Levi and O. Ragnisco, Inverse Problems 4 (1988) 815-828.