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Novel Bäcklund Transformations for Integrable Equations

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Abstract: In this paper, we construct a new matrix partial differential equation having a structure and properties which mirror those of a matrix fourth Painlevé equation recently derived by the current authors. In particular, we show that this matrix equation admits an auto-Bäcklund transformation analogous to that of this matrix fourth Painlevé equation. Such auto-Bäcklund transformations, in appearance similar to those for Painlevé equations, are quite novel, having been little studied in the case of partial differential equations. Our work here shows the importance of the underlying structure of differential equations, whether ordinary or partial, in the derivation of such results. The starting point for the results in this paper is the construction of a new completely integrable equation, namely, an inverse matrix dispersive water wave equation.

Keywords: integrable equations; Bäcklund transformations; Miura maps; inverse matrix; dispersive water wave equation

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1. Introduction

Our aim in the current paper is to present a new matrix partial differential equation having a structure and properties which mirror those of the matrix fourth Painlevé equation obtained in [1]. For this equation, we then give an auto-Bäcklund transformation of a form similar to that of an auto-Bäcklund transformation for this matrix fourth Painlevé equation. Our overarching objective is to provide further proof of the importance of the underlying structure of differential equations, not just ordinary but also partial, in the derivation of such results.

This underlying structure, for both our new matrix partial differential equation and our matrix fourth Painlevé equation, is expressed using Miura maps. Thus, our starting point here is the derivation of a new completely integrable inverse matrix dispersive water wave equation; we recall that the matrix dispersive water wave hierarchy was also first presented in [1]. We then use a Miura map to obtain a modified version of this equation, which in turn leads us to the derivation of the new matrix partial differential equation of interest to us here, and for which we subsequently give a novel auto-Bäcklund transformation.

We refer to the auto-Bäcklund transformation presented here as novel, since auto-Bäcklund transformations for partial differential equations of the form we give here, i.e., similar to those of Painlevé equations, have been little studied in the literature, especially in the (relatively troublesome) matrix case. We remark that results for an inverse matrix Korteweg–de Vries equation were obtained in [2], and results for scalar partial differential equations have been given in [3,4]. Descriptions and examples of auto-Bäcklund transformations of the kind that are usually given for integrable partial differential equations in $1+1$-dimensions, which, in contrast to those for Painlevé equations, depend on a parameter, can be found in [5,6]. For example, the potential modified Korteweg–de Vries equation $v_t + v_{xxx} + 2v_x^3 = 0$, which is related to the modified Korteweg–de Vries equation in u via $u = v_x$, has the auto-Bäcklund transformation

$$(z+w)_x = 2k\sin(z-w), \tag{1}$$
$$(z+w)_t = -2k(z_{xx}-w_{xx})\cos(z-w) - 2k(z_x^2+w_x^2)\sin(z-w); \tag{2}$$

elimination of z or w between these equations yields the potential modified Korteweg–de Vries equation in $v = w$ or $v = z$, respectively. It is the presence of the arbitrary parameter k, that allows the construction of multi-soliton solutions. For discussions of auto-Bäcklund transformations for Painlevé equations, on the other hand, we refer to [7,8]. The second Painlevé equation, for example, has the auto-Bäcklund transformation

$$u = v + \frac{\alpha-\beta}{2v_x - 2v^2 - x}, \quad \alpha = 1-\beta, \tag{3}$$

which maps from a solution of the second Painlevé equation for dependent variable v and constant coefficient β, i.e., $v_{xx} = 2v^3 + xv + \beta$, to a solution of the same equation for dependent variable u and constant coefficient α, i.e., $u_{xx} = 2u^3 + xu + \alpha$, with $\alpha = 1-\beta$. Such mappings between solutions involving changes in the values of constant parameters appearing as coefficients are typical of the Painlevé equations (except the first) and their various analogues.

The layout of the paper is as follows. In Section 2, we derive the matrix partial differential equation of interest to us here, beginning with a new completely integrable inverse matrix dispersive water wave equation. In Section 3, we give auto-Bäcklund transformations for this equation. In Section 4, we consider the reduction of our equation and its auto-Bäcklund transformations to the ordinary differential case, i.e., to a matrix fourth Painlevé equation and corresponding results. Section 5 is devoted to a discussion and conclusions.

2. A Matrix Partial Differential Equation

In this section, we give our derivation of a new matrix equation, which, even though it is a partial rather than an ordinary differential equation, has a structure and properties which mirror those of a matrix fourth Painlevé equation presented in [1]. With this aim we begin by constructing a completely integrable non-autonomous inverse matrix dispersive water wave equation; we will then derive a modified version of this equation, which we will in turn use to obtain the equation of interest to us in this paper. We note that the matrix dispersive water wave hierarchy is as defined in [1]; for the scalar case we refer to [9–14] and in particular [15–17]. Let us consider the non-isospectral scattering problem in $(1+1)$-dimensions

$$\psi_{xx} = -u\psi, \tag{4}$$
$$\psi_t = \frac{1}{2}P[\mathbf{u}]\psi_x - \frac{1}{4}\Big[(P[\mathbf{u}])_x + (\partial_x^{-1}C_u P[\mathbf{u}])\Big]\psi, \tag{5}$$

where $u = u_0 + u_1\lambda - \lambda^2 I$, $u_0 = u_0(x,t)$ and $u_1 = u_1(x,t)$ are square matrices, $\mathbf{u} = (u_0, u_1)^T$, $\lambda = \lambda(t)$ is a scalar function of t and I is the identity matrix. We use the subscripts x and t to denote corresponding partial derivatives, ∂_x the partial differential operator with respect to x and ∂_x^{-1} its inverse. The compatibility condition of the above scattering problem reads

$$u_{1,t}\lambda + u_1\lambda_t + u_{0,t} - 2\lambda\lambda_t I = \Big[J_0[\mathbf{u}] + \lambda J_1[\mathbf{u}] - \lambda^2 J[\mathbf{u}]\Big]P[\mathbf{u}] \tag{6}$$

where the operators $J_0[\mathbf{u}]$, $J_1[\mathbf{u}]$ and $J[\mathbf{u}]$ are defined as

$$J_0[\mathbf{u}] = \frac{1}{4}\Big[\partial_x^3 + (A_{u_0}\partial_x + \partial_x A_{u_0}) + C_{u_0}\partial_x^{-1}C_{u_0}\Big], \quad (7)$$

$$J_1[\mathbf{u}] = \frac{1}{4}\Big[(A_{u_1}\partial_x + \partial_x A_{u_1}) + C_{u_0}\partial_x^{-1}C_{u_1} + C_{u_1}\partial_x^{-1}C_{u_0}\Big], \quad (8)$$

$$J[\mathbf{u}] = \partial_x - \frac{1}{4}C_{u_1}\partial_x^{-1}C_{u_1}. \quad (9)$$

In the above expressions

$$A_w = L_w + R_w, \qquad C_w = L_w - R_w, \quad (10)$$

where the left and right multiplication operators L_w and R_w act as usual:

$$L_w(z) = wz, \qquad R_w(z) = zw. \quad (11)$$

We now set in the above Lax pair

$$P[\mathbf{u}] = -\frac{1}{\lambda}\Big[J[\mathbf{u}]^{-1}u_{1,t} + 2\gamma_0 I + g_1 x I\Big], \quad (12)$$

where $\gamma_0 = \gamma_0(t)$ and $g_1 = g_1(t)$ are scalar functions of t and where we assume that λ satisfies the (relatively weak) non-isospectral condition

$$\lambda_t = -\frac{1}{2}g_1. \quad (13)$$

The compatibility condition (6) yields, collecting terms at different powers of λ,

$$J_0[\mathbf{u}]J[\mathbf{u}]^{-1}u_{1,t} + \gamma_0 u_{0,x} + J_0[\mathbf{u}](g_1 x I) = 0, \quad (14)$$

$$J_1[\mathbf{u}]J[\mathbf{u}]^{-1}u_{1,t} + u_{0,t} + \gamma_0 u_{1,x} + J_1[\mathbf{u}](g_1 x I) - \frac{1}{2}g_1 u_1 = 0. \quad (15)$$

The above two equations can be written as

$$\mathcal{R}[\mathbf{u}]\mathbf{u}_t + \gamma_0\mathbf{u}_x + \frac{1}{2}\,g_1\begin{pmatrix} 2u_0 + xu_{0,x} \\ u_1 + xu_{1,x} \end{pmatrix} = 0, \quad (16)$$

or alternatively as

$$\mathcal{R}[\mathbf{u}]\mathbf{u}_t + \gamma_0\mathbf{u}_x + \ g_1\mathcal{R}[\mathbf{u}]\begin{pmatrix} -\frac{1}{2}u_1 \\ I \end{pmatrix} = 0, \quad (17)$$

where the operator $\mathcal{R}[\mathbf{u}]$ is given by

$$\mathcal{R}[\mathbf{u}] = \begin{pmatrix} 0 & J_0[\mathbf{u}]J[\mathbf{u}]^{-1} \\ I & J_1[\mathbf{u}]J[\mathbf{u}]^{-1} \end{pmatrix}. \quad (18)$$

This new system of matrix equations corresponds to a non-autonomous version of the first inverse flow of the matrix dispersive wave hierarchy defined in [1]. In [1], it was shown that the operator $\mathcal{R}[\mathbf{u}]$ can be expressed as

$$\mathcal{R}[\mathbf{u}] = B_1[\mathbf{u}]B_0[\mathbf{u}]^{-1} \quad (19)$$

where

$$B_0[\mathbf{u}] = \begin{pmatrix} -J_1[\mathbf{u}] & J[\mathbf{u}] \\ J[\mathbf{u}] & 0 \end{pmatrix}, \tag{20}$$

$$B_1[\mathbf{u}] = \begin{pmatrix} J_0[\mathbf{u}] & 0 \\ 0 & J[\mathbf{u}] \end{pmatrix}. \tag{21}$$

Equation (17) can then also be written alternatively as

$$B_1[\mathbf{u}]M[\mathbf{u}] = 0 \tag{22}$$

where

$$M[\mathbf{u}] = \begin{pmatrix} J[\mathbf{u}]^{-1}u_{1,t} \\ J[\mathbf{u}]^{-1}u_{0,t} + J[\mathbf{u}]^{-1}J_1[\mathbf{u}]J[\mathbf{u}]^{-1}u_{1,t} \end{pmatrix} + \gamma_0 \begin{pmatrix} 2I \\ u_1 \end{pmatrix} + \frac{1}{2}g_1 \begin{pmatrix} 2Ix \\ xu_1 \end{pmatrix}. \tag{23}$$

We now turn to the derivation of a modified version of the new matrix partial differential Equation (17), as written in the form (22). In [1], it was shown that under the Miura map

$$\mathbf{u} = \mathbf{h}[\mathbf{\Phi}] = \begin{pmatrix} -\frac{1}{2}\phi_x - \frac{1}{4}\phi^2 \\ \phi + 2p \end{pmatrix}, \tag{24}$$

where $\mathbf{\Phi} = (\phi, p)^T$ ($\phi(x,t)$ and $p(x,t)$ being square matrices), the operator $B_1[\mathbf{u}]$ is factorized as

$$B_1[\mathbf{u}] \Big|_{\mathbf{u}=\mathbf{h}[\mathbf{\Phi}]} = \mathbf{h}'[\mathbf{\Phi}]C[\mathbf{\Phi}](\mathbf{h}'[\mathbf{\Phi}])^{\dagger}, \tag{25}$$

where $\mathbf{h}'[\mathbf{\Phi}]$ is the Fréchet derivative of the Miura map $\mathbf{u} = \mathbf{h}[\mathbf{\Phi}]$ and $(\mathbf{h}'[\mathbf{\Phi}])^{\dagger}$ is its adjoint, given by

$$\mathbf{h}'[\mathbf{\Phi}] = \begin{pmatrix} -\frac{1}{2}\partial_x - \frac{1}{4}A_\phi & 0 \\ I & 2I \end{pmatrix} \tag{26}$$

and

$$(\mathbf{h}'[\mathbf{\Phi}])^{\dagger} = \begin{pmatrix} \frac{1}{2}\partial_x - \frac{1}{4}A_\phi & I \\ 0 & 2I \end{pmatrix} \tag{27}$$

respectively; and $C[\mathbf{\Phi}]$ has the form

$$C[\mathbf{\Phi}] = \frac{1}{2}\begin{pmatrix} -2\partial_x + \frac{1}{2}C_\phi\partial_x^{-1}C_\phi & \partial_x - \frac{1}{4}C_\phi\partial_x^{-1}C_\phi \\ \partial_x - \frac{1}{4}C_\phi\partial_x^{-1}C_\phi & -\frac{1}{4}(C_\phi\partial_x^{-1}C_p + C_p\partial_x^{-1}C_\phi + 2C_p\partial_x^{-1}C_p) \end{pmatrix}. \tag{28}$$

This means that, under this Miura map, corresponding to Equation (22), we have the modified equation

$$C[\mathbf{\Phi}]\left(\mathbf{h}'[\mathbf{\Phi}]\right)^{\dagger}M[\mathbf{h}[\mathbf{\Phi}]] = 0. \tag{29}$$

In order to obtain a matrix partial differential equation which mirrors the structure of a matrix fourth Painlevé equation, we now observe that since any vector consisting of scalar x-independent multiples of the identity matrix is in the kernel of the operator $C[\mathbf{\Phi}]$, Equation (29) is equivalent to

$$C[\mathbf{\Phi}]\left[\left(\mathbf{h}'[\mathbf{\Phi}]\right)^{\dagger}\tilde{M}[\mathbf{h}[\mathbf{\Phi}]] + \begin{pmatrix} eI \\ fI \end{pmatrix}\right] = 0, \tag{30}$$

where $e = e(t)$ and $f = f(t)$ are arbitrary functions of t and

$$\tilde{M}[\mathbf{u}] = M[\mathbf{u}] + \begin{pmatrix} 0 \\ -\frac{1}{2}g_1 I \end{pmatrix}. \tag{31}$$

Making now the same step used in our work in [2], and in [1,18] on matrix Painlevé hierarchies, from Equation (30) we are led to consider the non-autonomous matrix partial differential equation

$$\left(\mathbf{h}'[\mathbf{\Phi}]\right)^{\dagger}\tilde{M}[\mathbf{h}[\mathbf{\Phi}]] + \begin{pmatrix} eI \\ fI \end{pmatrix} = 0. \tag{32}$$

The structure of this new matrix partial differential equation closely reflects that of Equation (3.35) in [1]; the latter defines a matrix fourth Painlevé hierarchy, the first member of which is a matrix fourth Painlevé equation, to which (32) corresponds, as can be seen from Section 4. By taking into account the Miura map (24), and introducing the auxiliary variables w_1 and w_2 defined by

$$w_2 = J^{-1}[\mathbf{h}[\mathbf{\Phi}]](\phi + 2p)_t, \quad w_1 = J^{-1}[\mathbf{h}[\mathbf{\Phi}]]\left\{\left(-\frac{1}{2}\phi_x - \frac{1}{4}\phi^2\right)_t + J_1[\mathbf{h}[\mathbf{\Phi}]]w_2\right\}, \tag{33}$$

we can write our system (32) as the pair of equations

$$w_{2,x} - \frac{1}{2}(\phi w_2 + w_2\phi) - 2\gamma_0\phi - g_1 x\phi + (g_1 + 2e - f)I = 0, \tag{34}$$
$$2w_1 + 2\gamma_0(\phi + 2p) + g_1 x(\phi + 2p) + (f - g_1)I = 0, \tag{35}$$

coupled with

$$J[\mathbf{h}[\mathbf{\Phi}]]w_2 - (\phi + 2p)_t = 0, \tag{36}$$
$$J[\mathbf{h}[\mathbf{\Phi}]]w_1 + \left(\frac{1}{2}\phi_x + \frac{1}{4}\phi^2\right)_t - J_1[\mathbf{h}[\mathbf{\Phi}]]w_2 = 0. \tag{37}$$

It is this system, (34)–(37), for which we will now give auto-Bäcklund transformations, with the functions e and f playing the roles usually played by constant parameters in the case of Painlevé equations.

3. Auto-Bäcklund Transformations

The structure of the system (34)–(37), as mentioned above, mirrors that of a matrix fourth Painlevé equation given in [1]; in both cases this structure is expressed using Miura maps. The definition of an auto-Bäcklund transformation, the derivation or proof of which makes use of this structure, then proceeds similarly in the two cases. Such transformations for partial differential equations, having a form analogous to those of auto-Bäcklund transformations for the Painlevé equations, have been little studied in the literature. We now define such a novel transformation for the non-autonomous matrix partial differential system of equations given above.

Theorem 1. *The system (32) that we rewrite as the set of Equations (34)–(37), admits the auto-Bäcklund transformation*

$$\phi = \tilde{\phi} + 2(e - \tilde{e})[\tilde{w}_2 + 2\gamma_0 I + g_1 x I]^{-1}, \tag{38}$$
$$p = \tilde{p} - (e - \tilde{e})[\tilde{w}_2 + 2\gamma_0 I + g_1 x I]^{-1}, \tag{39}$$
$$w_1 = \tilde{w}_1, \tag{40}$$
$$w_2 = \tilde{w}_2, \tag{41}$$
$$e = -\tilde{e} + \tilde{f}, \tag{42}$$
$$f = \tilde{f}. \tag{43}$$

This auto-Bäcklund transformation relates solutions of (34)—(37) in $\mathbf{\Phi} = (\phi, p)^T$ *and* w_1, w_2*, with coefficient functions* (e, f) *of* t*, to solutions of the same system in* $\tilde{\mathbf{\Phi}} = (\tilde{\phi}, \tilde{p})^T$ *and* $\tilde{w}_1, \tilde{w}_2$*, with coefficient functions* $(\tilde{e}, \tilde{f})$ *of* t*, i.e.,*

$$\left(\mathbf{h}'[\tilde{\mathbf{\Phi}}]\right)^{\dagger} \tilde{M}[\mathbf{h}[\tilde{\mathbf{\Phi}}]] + (\tilde{e}I, \tilde{f}I)^T = 0, \tag{44}$$

or

$$\tilde{w}_{2,x} - \frac{1}{2}(\tilde{\phi}\tilde{w}_2 + \tilde{w}_2\tilde{\phi}) - 2\gamma_0\tilde{\phi} - g_1 x\tilde{\phi} + (g_1 + 2\tilde{e} - \tilde{f})I = 0, \tag{45}$$

$$2\tilde{w}_1 + 2\gamma_0(\tilde{\phi} + 2\tilde{p}) + g_1 x(\tilde{\phi} + 2\tilde{p}) + (\tilde{f} - g_1)I = 0, \tag{46}$$

$$J[\mathbf{h}[\tilde{\mathbf{\Phi}}]]\tilde{w}_2 - (\tilde{\phi} + 2\tilde{p})_t = 0, \tag{47}$$

$$J[\mathbf{h}[\tilde{\mathbf{\Phi}}]]\tilde{w}_1 + \left(\frac{1}{2}\tilde{\phi}_x + \frac{1}{4}\tilde{\phi}^2\right)_t - J_1[\mathbf{h}[\tilde{\mathbf{\Phi}}]]\tilde{w}_2 = 0, \tag{48}$$

provided that the first component of $\tilde{M}[\mathbf{h}[\tilde{\mathbf{\Phi}}]]$*—namely,* $\tilde{w}_2 + 2\gamma_0 I + g_1 x I$*—is nonsingular.*

Proof. To prove our result, first of all we note that the Miura transformation $\mathbf{h}[\mathbf{\Phi}]$, as given by (24), is invariant under transformation (38)–(43), i.e., $\mathbf{h}[\mathbf{\Phi}] = \mathbf{h}[\tilde{\mathbf{\Phi}}]$. It is trivial to check using (38) and (39) that the second component of the Miura map is invariant, i.e., that $\phi + 2p = \tilde{\phi} + 2\tilde{p}$. To prove the invariance of the first component of the Miura map, we see that (38), (42) and (45) give

$$\begin{aligned}
&\frac{1}{2}\phi_x + \frac{1}{4}\phi^2 = \frac{1}{2}\tilde{\phi}_x + \frac{1}{4}\tilde{\phi}^2 - (e - \tilde{e})(\tilde{w}_2 + 2\gamma_0 I + g_1 x I)^{-1} \times \\
&\left\{\tilde{w}_{2,x} - \frac{1}{2}(\tilde{\phi}\tilde{w}_2 + \tilde{w}_2\tilde{\phi}) - 2\gamma_0\tilde{\phi} - g_1 x\tilde{\phi} + (g_1 + \tilde{e} - e)I\right\}(\tilde{w}_2 + 2\gamma_0 I + g_1 x I)^{-1} \\
&= \frac{1}{2}\tilde{\phi}_x + \frac{1}{4}\tilde{\phi}^2 + (e - \tilde{e})(\tilde{e} + e - \tilde{f})(\tilde{w}_2 + 2\gamma_0 I + g_1 x I)^{-2} = \frac{1}{2}\tilde{\phi}_x + \frac{1}{4}\tilde{\phi}^2.
\end{aligned} \tag{49}$$

We now use this invariance of the Miura map under the auto-Bäcklund transformation to show that (38)–(43) is indeed an auto-Bäcklund transformation for (32), i.e., for the system (34)–(37). First of all, using Equations (38), (41) and (43), we see that under the above auto-Bäcklund transformation the left-hand-side of Equation (34) transforms as

$$\begin{aligned}
&w_{2,x} - \frac{1}{2}(\phi w_2 + w_2\phi) - 2\gamma_0\phi - g_1 x\phi + (g_1 + 2e - f)I \\
&= \tilde{w}_{2,x} - \frac{1}{2}(\tilde{\phi}\tilde{w}_2 + \tilde{w}_2\tilde{\phi}) - 2\gamma_0\tilde{\phi} - g_1 x\tilde{\phi} + (g_1 + 2\tilde{e} - \tilde{f})I.
\end{aligned} \tag{50}$$

Additionally, using Equations (40) and (43), and the invariance of the Miura map, in particular that $\phi + 2p = \tilde{\phi} + 2\tilde{p}$, we see that the left-hand-side of Equation (35) transforms as

$$\begin{aligned}
&2w_1 + 2\gamma_0(\phi + 2p) + g_1 x(\phi + 2p) + (f - g_1)I \\
&= 2\tilde{w}_1 + 2\gamma_0(\tilde{\phi} + 2\tilde{p}) + g_1 x(\tilde{\phi} + 2\tilde{p}) + (\tilde{f} - g_1)I.
\end{aligned} \tag{51}$$

Now using Equations (40) and (41), and the invariance of the Miura map, $\mathbf{h}[\mathbf{\Phi}] = \mathbf{h}[\tilde{\mathbf{\Phi}}]$, we see that the left-hand-sides of Equations (36) and (37) transform as

$$J[\mathbf{h}[\mathbf{\Phi}]]w_2 - (\phi + 2p)_t = J[\mathbf{h}[\tilde{\mathbf{\Phi}}]]\tilde{w}_2 - (\tilde{\phi} + 2\tilde{p})_t, \tag{52}$$

$$\begin{aligned}
&J[\mathbf{h}[\mathbf{\Phi}]]w_1 + \left(\frac{1}{2}\phi_x + \frac{1}{4}\phi^2\right)_t - J_1[\mathbf{h}[\mathbf{\Phi}]]w_2 \\
&= J[\mathbf{h}[\tilde{\mathbf{\Phi}}]]\tilde{w}_1 + \left(\frac{1}{2}\tilde{\phi}_x + \frac{1}{4}\tilde{\phi}^2\right)_t - J_1[\mathbf{h}[\tilde{\mathbf{\Phi}}]]\tilde{w}_2.
\end{aligned} \tag{53}$$

Thus we see that if (45)–(48) hold, then the mapping (38)–(43) provides solutions of (34)–(37). This concludes the proof that (38)–(43) is in fact an auto-Bäcklund transformation for the system of Equations (34)–(37), or equivalently for the system (32). □

Remark 1. *It is readily seen that the auto-Bäcklund transformation (38)–(43) is an involution.*

Remark 2. *The system (34)–(37) also admits the auto-Bäcklund transformation*

$$\phi = \tilde{\phi}^T, \quad p = \tilde{p}^T, \quad w_1 = \tilde{w}_1^T, \quad w_2 = \tilde{w}_2^T, \quad e = \tilde{e}, \quad f = \tilde{f}, \tag{54}$$

as may be easily checked.

In order to consider the action of the auto-Bäcklund transformation (38)–(43), we first of all need an initial solution of the system (34)–(37). In order to find such a solution, we could consider reducing to a system of matrix ordinary differential equations, or alternatively reducing to a system of scalar equations. We will see in the next section that the system (34)–(37) admits a reduction to a matrix fourth Painlevé system given in [1] (a system of matrix ordinary differential equations), which would then provide one way of obtaining solutions of (34)–(37). However, the question of solutions of this matrix fourth Painlevé system has yet to be explored, and is beyond the scope of the present paper (as an example of the consideration of solution classes of matrix ordinary differential equations, we refer to [19]). In the case of the scalar reduction, it can be shown that our auto-Bäcklund transformation (38)–(43) can allow a new solution to be obtained from a given initial solution. This can be seen from the fact that this reduction leads us to a system of scalar equations and a corresponding scalar auto-Bäcklund transformation, which were presented in [4], and that in [4], the action of this scalar auto-Bäcklund transformation on a particular initial solution of the scalar system was shown to yield a new solution of the scalar system. For the more interesting case of initial solutions of (34)–(37) corresponding to neither an ordinary differential equation reduction nor a scalar reduction, for the purposes of the present paper it is enough to consider the solution given by

$$\tilde{\phi} = 0, \quad \tilde{p} = 0, \quad \tilde{w}_1 = \frac{1}{2}(g_1 - \tilde{f})I, \quad \tilde{w}_2 = F, \tag{55}$$

for $\tilde{e}$ and $\tilde{f}$ satisfying

$$2\tilde{e} - \tilde{f} + g_1 = 0, \tag{56}$$

where $F = F(t)$ is an arbitrary matrix function of t. The auto-Bäcklund transformation (38)–(43) then yields a new solution of (34)–(37) given by

$$\phi = 2g_1(F + 2\gamma_0 I + g_1 x I)^{-1}, \quad p = -g_1(F + 2\gamma_0 I + g_1 x I)^{-1}, \tag{57}$$

$$w_1 = \frac{1}{2}(g_1 - f)I, \quad w_2 = F, \tag{58}$$

for e and f satisfying

$$2e - f - g_1 = 0. \tag{59}$$

For $g_1 \neq 0$, we thus obtain a new solution of the matrix system (34)–(37) rational in x.

4. Reduction to a Matrix Fourth Painlevé Equation

We now consider a reduction to a matrix fourth Painlevé equation.

Theorem 2. *The system (34)–(37) admits a reduction to a matrix fourth Painlevé system given in [1]; the corresponding reductions of the auto-Bäcklund transformations (38)–(43) and (54) give auto-Bäcklund transformations for this matrix fourth Painlevé system, as given in [1].*

Proof. In system (34)–(37), we set

$$\phi = \Phi(z), \quad p = P(z), \quad z = x + t, \quad \gamma_0 = \gamma + \frac{1}{2}gt, \quad g_1 = g, \tag{60}$$

where γ and g are constant, and assume also that e and f are constant. Under this reduction we find that

$$w_2 = \Phi + 2P, \quad w_1 = \frac{3}{4}(\Phi + 2P)^2 - \left(\frac{1}{2}\Phi_z + \frac{1}{4}\Phi^2\right), \tag{61}$$

and so the system of Equations (34)–(37) yields

$$\Phi_z + 2P_z - \Phi^2 - \Phi P - P\Phi - 2\gamma\Phi - gz\Phi + (g + 2e - f)I = 0, \tag{62}$$
$$\Phi_z - \Phi^2 - 6P^2 - 3\Phi P - 3P\Phi - 2\gamma(\Phi + 2P) - gz(\Phi + 2P) + (g - f)I = 0. \tag{63}$$

This is equivalent to the matrix system—(3.47), (3.48)—obtained in [1], given as the first member of the matrix fourth Painlevé hierarchy (3.35) defined therein, and defines for $g \neq 0$ a matrix fourth Painlevé system. The auto-Bäcklund transformation (38)–(43) then reduces to

$$\Phi = \tilde{\Phi} + 2(e - \tilde{e})\left[\tilde{\Phi} + 2\tilde{P} + 2\gamma I + gzI\right]^{-1}, \tag{64}$$
$$P = \tilde{P} - (e - \tilde{e})\left[\tilde{\Phi} + 2\tilde{P} + 2\gamma I + gzI\right]^{-1}, \tag{65}$$
$$e = -\tilde{e} + \tilde{f}, \tag{66}$$
$$f = \tilde{f}. \tag{67}$$

The auto-Bäcklund transformations of the system (62), (63) as given by (64)–(67), and by the corresponding reduced case of (54), i.e.,

$$\Phi = \tilde{\Phi}^T, \quad P = \tilde{P}^T, \quad e = \tilde{e}, \quad f = \tilde{f}, \tag{68}$$

are as given in [1]. □

Remark 3. *Whilst in the auto-Bäcklund transformation (38)–(43) for the system (34)–(37) the transformation rules $w_1 = \tilde{w}_1$ and $w_2 = \tilde{w}_2$ might at first sight seem overly restrictive, we see from (61), given the invariance of the Miura map, which holds also in the reduced case, that these transformation rules are in fact quite natural.*

5. Discussion

In this paper, we have obtained the following results:

- the derivation of a new completely integrable inverse matrix dispersive water wave equation and a modification thereof;
- the derivation of a new matrix partial differential equation having a structure and properties which mirror those of a matrix fourth Painlevé equation, in particular an auto-Bäcklund transformation;
- a reduction from our matrix partial differential equation and its auto-Bäcklund transformations to the ordinary differential case.

These results, when compared to our results in [1] for a matrix fourth Painlevé equation, allowed us to confirm that in the derivation of properties such as auto-Bäcklund transformations, and the underlying structure of the equation, here expressed using a Miura map, are of fundamental importance. This is true not just of ordinary but also partial differential equations, whether they be scalar, or as in the present paper, matrix. The resulting auto-Bäcklund transformations for partial differential equations, similar to those of Painlevé equations, are novel in nature, and they and the corresponding equations are well worth future study.

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