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Line graphs for directed and undirected networks: An structural and analytical comparison

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Abstract. The centrality and efficiency measures of a network G are strongly related to the respective measures on the associated line graph $L(G)$ and bipartite graph $B(G)$ as was shown in [8]. In this note we consider different ways to obtain a line graph from a given directed or undirected network and we obtain some interesting relations.

Keywords: complex networks, dual graph, line graph, line digraph.

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1. Line graphs and line digraphs

Many relevant properties of complex systems in the real world, such as social networks, Internet, the World Wide Web and other biological and technological systems may be described in terms of complex network properties [1, 3, 4, 5, 13, 10, 16, 17, 18, 20, 21, 22, 24]. In fact, the study of structural properties of complex networks is an attractive and fascinating branch of research in sociology (social networks, acquaintances or collaborations between individuals), science (metabolic and protein networks, neural networks, genetic regulatory networks, protein folding) and technology (Internet, computers in telecommunication networks, the World Wide Web,...).

The motivation behind this contribution is to consider the importance that edges have sometimes over nodes in the context of networks and graphs. An example of this comes from urbanism [11, 12], transport networks [25, 2] or urban traffic [19], where the line (dual) graph $L(G)$ associated to a given graph (network) G is considered.

For example, in the context of urban traffic, when the underlying (primal) graph is considered then intersections (or settlements) are seen as nodes while roads (or lines of relationship) are seen as edges. In contrast when the dual (line) graph is considered roads become nodes, while intersections become links between the corresponding nodes [19].

In [8] we showed some relationships between the network's efficiency and the network's Bonacich ([6], [7]) centrality of a network G and the respective measures on the dual $L(G)$ and the bipartite $B(G)$ associated networks (see below for definitions). Some other properties and relationships between the centrality of a network G and the centrality of its dual network $L(G)$ have been studied in [9]. Note that the networks considered there were undirected.

The main goal of this note is to exhibit some relations arising from the various ways in which line graphs can be obtained from a given directed network. This in turn can be applied to obtaining estimations for several parameters that measure different properties related to the network structure and performance.

In order to investigate such properties, it is necessary to understand the main structure of the underlying network [3, 20] and also to consider other complementary topological aspects.

From a schematic point of view, a complex network is a mathematical object $G = (V, E)$ composed by a set of nodes or vertices $V = \{v_1 \ldots, v_n\}$ that are pairwise joined by links or edges $\{l_1, \ldots, l_m\}$. We consider the adjacency matrix $A(G) = (a_{ij})$ determined by the conditions

$$a_{ij} = \begin{cases} 1 & \text{if} \quad \{\mathrm{v_i}, \mathrm{v_j}\} \in \mathrm{E} \\ 0 & \text{if} \quad \{\mathrm{v_i}, \mathrm{v_j}\} \notin \mathrm{E}. \end{cases}$$

The bipartite network $B(G)$ associated to G is defined by $B(G) = (X \cup E, E(B(G)))$ whose adjacency matrix is given by

$$A(B(G)) = \left(\begin{array}{c|c} 0 & I(G) \\ \hline I(G)^t & 0 \end{array} \right)$$

where $I(G)$ is the incidence matrix of G. It is shown that

$$A(B(G))^2 = \left(\begin{array}{c|c} A(G) + gr & 0 \\ \hline 0 & A(L(G)) + 2I_n \end{array} \right)$$

where $A(G) + gr$ denotes the matrix obtained by adding to $A(G)$ the diagonal matrix (b_{ij}) and b_{ii} is the degree of the vertex v_i while $L(G)$ denotes the line (or dual) network associated to G ([14], pag. 26, [15], pag. 273).

As we showed in [8], if we know the Bonacich centrality $c(L(G))$, we can recover $c(B(G))$ and reciprocally. If, in addition, G is regular then each of the three centralities can be recovered from any of the other two. Moreover, we have a relationship between the efficiencies of the dual graph $L(G)$ and the primal graph G (see [8]):

If $G = (V, E)$ and $L(G) = (E, L)$, n is the number of nodes of G, m is the number of nodes $L(G)$ and p is the number of edges of $L(G)$, we have

$$\frac{n(n-1)}{8m(m-1)}E(G) + \frac{15p-2}{8m(m-1)} \leq E(L(G))$$
$$E(L(G)) \leq \max_{i \neq j}(gr_i gr_j)\frac{n(n-1)}{m(m-1)}E(G) + \frac{2p}{m(m-1)}.$$

These results have potential interest in the context of urban streets networks ([11, 12]) and urban traffic ([19]).

If now $A(G) = (a_{ij})$ is the adjacency matrix of the directed network G determined by the conditions

$$a_{ij} = \begin{cases} 1 & \text{if} \quad (\mathrm{v_i}, \mathrm{v_j}) \in \mathrm{E} \\ 0 & \text{if} \quad (\mathrm{v_i}, \mathrm{v_j}) \notin \mathrm{E}, \end{cases}$$

the bipartite network $B(G)$ associated to G is defined by $B(G) = (X \cup E, E(B(G)))$ whose adjacency matrix is given by

$$A(B(G)) = \left(\begin{array}{c|c} 0 & H(G) \\ \hline T(G)^t & 0 \end{array}\right)$$

where $H = H(G)$ is the incidence matrix of heads of G defined by

$$H_{ij} = \begin{cases} 1 & \text{if} \quad \mathrm{e_j} = (\mathrm{v_i}, -) \\ 0 & \text{otherwise} \end{cases}$$

and $T = T(G)$ is the incidence matrix of tails of G defined by

$$T_{ij} = \begin{cases} 1 & \text{if} \quad \mathrm{e_j} = (-, \mathrm{v_i}) \\ 0 & \text{otherwise} \end{cases}$$

it is shown that

$$A(B(G))^2 = \left(\begin{array}{c|c} A(G) & 0 \\ \hline 0 & A(\vec{L}(G)) \end{array}\right)$$

where $\vec{L}(G)$ denotes the line (or dual) network associated to G.
Recall that the Bonacich centrality of a complex network G is the non-negative normalized eigenvector $c_G \in \mathbb{R}^n$ associated to the spectral radius of the transposed adjacency matrix of G [6, 7, 20]. The following relations between the Bonacich centralities of G, $\vec{L}(G)$ and $B(G)$ are obtained:

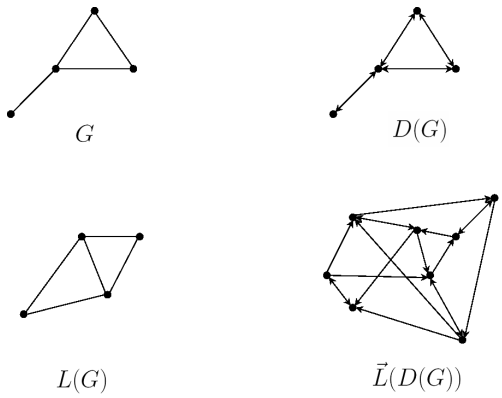

Figure 1: An example of the line-graph of a undirected network (on the left) and for a directed network (on the right).

Theorem 1 *Let $G = (V, E)$ be a connected directed graph with n vertices and m edges. Let $c_G \in \mathbb{R}^n$, $c_{\vec{L}(G)} \in \mathbb{R}^m$ and $c_{B(G)} = (c_1, c_2) \in \mathbb{R}^n \times \mathbb{R}^m$ be the Bonacich centralities of G, $\vec{L}(G)$ and $B(G)$. Then, if $\|v\|_1 = \sum_{i=1}^{n} |v_i|$ for any arbitrary $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, we have:*

(i) $c_G = \frac{c_1}{\|c_1\|_1}$ *and* $c_{\vec{L}(G)} = \frac{c_2}{\|c_2\|_1}$.

(ii) Reciprocally, $c_{B(G)} = \frac{1}{2}\left(c_G, c_{\vec{L}(G)}\right)$ *and*

$$c_G = \frac{H_G\, c_{\vec{L}(G)}}{\|H_G\, c_{\vec{L}(G)}\|_1}, \quad c_{\vec{L}(G)} = \frac{T_G^t\, c_G}{\|T_G^t\, c_G\|_1}.$$

Let $D(G)$ denote the associated symmetric digraph obtained by replacing each edge of G by an arc pair in which the two arcs are inverse to each other. Since $A(G) = A(D(G))$, the Bonacich centralities of G and $D(G)$ coincide and, in particular, the Bonacich centralities of G and $\vec{L}(D(G))$ are closely related.

There is an alternative definition for the line graph associated with G that has received relatively little attention. We will call it the oriented line graph $L^{\leadsto}(G)$ and it will be defined as follows. If $D(G) = (V(D(G)), E(D(G)))$ denotes the associated symmetric digraph, the vertices of the oriented line graph $L^{\leadsto}(G)$ are the arcs $E(D(G))$ of $D(G)$, while (e, f) is an arc in $L^{\leadsto}(G)$ if the end of e coincides with the origin of f and f is not the inverse of e. is not the inverse of e (see [23] and the references cited therein). In the same

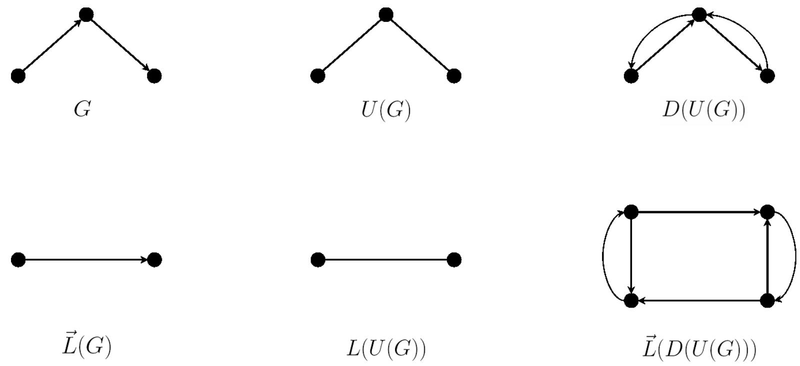

Figure 2: An example of line graphs of a directed network G, $U(G)$ and $D(U(G))$.

reference [23] the oriented line graph $L^{\rightsquigarrow}(G)$ is employed to capture graph-class structure and clustering graphs.

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