BFC, A Branch-and-Fix Coordination Algorithmic Framework for Solving Some Types of Stochastic Pure and Mixed $0-1$ Programs

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Abstract

We present a framework for solving some types of $0-1$ multi-stage scheduling/planning problems under uncertainty in the objective function coefficients and the right-hand-side. A scenario analysis scheme with full recourse is used. The solution offered for each scenario group at each stage takes into account all scenarios but without subordinating to any of them. The constraints are modelled by a splitting variables representation via scenarios. So, a $0-1$ model for each scenario is considered plus the non-anticipativity constraints that equate the $0-1$ variables from the same group of scenarios in each stage. The mathematical representation of the model is very amenable for the proposed framework to deal with the $0-1$ character of the variables. A Branch-and-Fix Coordination approach is introduced for coordinating the selection of the branching nodes and branching variables in the scenario subproblems to be jointly optimized. Some computational experience is reported for different types of problems.

Keywords: Stochastic programming, Multistage scenario tree, Mixed 0-1 programs, Splitting variables representation, Twin node families.

1 Introduction

Decision making is inherent to all aspects of industrial, business and social activities. In all of them, difficult tasks must be accomplished. One of the most reliable decision support tools available today is Optimization, a field at the confluence of Mathematics and

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Computer Science. The purpose of the field is to build and solve effectively realistic mathematical models of the situation under study, in order to allow decision makers to explore a huge variety of possible alternatives. Since reality is complex, many of these models are *large* (in terms of the number of decision variables), *stochastic* (there are parameters whose value cannot be controlled by the decision maker and are uncertain) and $0-1$ *integer* (there are binary variables to be dealt with). Each of the last two features makes the problem difficult to tackle, yet its solution is critical for many leading organizations in fields such as public policy making, supply chain planning, production and distribution planning, financial assets and liability allocation, water resources planning, investments planning, energy generation allocation planning and traffic planning and scheduling, among many other areas.

Problems with the characteristics given above are transformed into mathematical optimization models. Often there are tens of thousands of constraints and continuous and $0-1$ variables. Given today's Operations Research state-of-the-art tools, deterministic mixed $0-1$ program optimization should not present major difficulties for moderate size problems, at least. However, it has long been recognized (Beale, 1955 and Dantzig, 1955) that traditional deterministic optimization is not suitable for capturing the truly dynamic behaviour of most real-life applications. The main reason is that such applications involve data uncertainties which arise because information that will be needed in subsequent decision stages is not available to the decision maker when the decision must be made. See Kall and Wallace (1994), Higle and Sen (1996) and Birge and Louveaux (1997) for good surveys on Stochastic Linear Programming and additional references. However, Stochastic $0-1$ Programming is still in its infancy, see Johnson et al. (2000), although it has a broad application field as well, see Laporte and Louveaux (1993), Van der Vlerk (1995), Løkketangen and Woodruff (1996), Schultz et al. (1998), Stougie and Van der Vlerk (1997), Carøe and Tind (1998), Carøe and Schultz (1998, 1999), Escudero et al. (1999a, b), Ahmed et al. (2000) and Römisch and Schultz (2001) among others. For a recent survey on mixed integer linear stochastic programs, see Klein Haneveld and Van der Vlerk (1999).

The goal of this work is to present a modelling approach to deal with multi-stage linking constraints in a mixed $0-1$ optimization setting under uncertainty in some parameters as well as its algorithmic scheme for problem solving that looks very promising for the types of large-scale problems that we have experimented with. See also in Alonso-Ayuso et al. (2001) an application to a stochastic two-stage real-world problem. Moreover, the general idea of the proposed algorithmic framework can be easily extended to more general (mixed) integer problems.

The paper is organized as follows. Section 2 presents the stochastic programming settings to deal with. Two types of stochastic $0-1$ problems are considered, namely, mixed $0-1$ problems for two-stage environments, where the first stage has only $0-1$ variables, and pure $0-1$ problems for multi-stage environments. Section 3 presents the splitting variables representation of the scenario-related model, and the main concepts of the so-called *Branch-and-Fix Coordination* (*BFC*) scheme for problem solving. The mathematical representation is very amenable for the proposed *BFC* approach to deal with the $0-1$ character of the variables that are included in the non-anticipativity constraints. Its main idea consists of using this type of constraints to coordinate the selection of the branching nodes and branching variables in the scenario subproblems to be jointly optimized. Note that the non-anticipativity constraints are related to the variables from the first stage for two-stage problems and the variables from each scenario group for multi-stage problems. Section 4 presents some variables fixing and branching criteria for *BFC*. The results produced by our approach are evaluated on the basis of a computational comparison with an heuristic approach (so-called *Fix-and-Relax*) for the pure $0-1$ problems. The new approach yields very good results with reasonable computing time. Section 5 reports on the computational results for instances of pure $0-1$ problems and instances of mixed $0-1$ problems from two types of real-life applications. And, finally, section 6 draws some conclusions from the work.

2 General Approach

Consider the following (deterministic) model

$$\begin{aligned} &\min ax + cy \\ &\text{s.t. } Ax + By = b \\ &\qquad x \in \{0,1\}^n, y \geq 0 \end{aligned} \tag{1}$$

where a and c are the vectors of the objective function coefficients, b is the right-hand-side (*rhs*) m-vector, A and B are the $m \times n$ and $m \times nc$ constraint matrices, respectively, x and y are the $n-$ and $nc-$vectors of the $0-1$ and continuous variables to optimize, say, over a set of stages R, respectively, and m, n and nc are the related number of constraints, $0-1$ variables and continuous variables. The model must be extended in order to deal properly with uncertainty in the values of some parameters. Thus, an approach to model the uncertainty in the problem data is needed. We may employ the so-called *scenario analysis* technique, where the uncertainty is modelled via a set of scenarios.

Let $\mathcal{S} = \{1, \ldots, |\mathcal{S}|\}$ denote the set of scenarios to consider, and w^s the likelihood assigned to scenario s by the decision maker. (Note: Only finite sets of scenarios are considered throughout the paper). One way to deal with the uncertainty is to obtain the solution x, y that best tracks each of the scenarios, while satisfying the constraints for each scenario. This can be achieved by obtaining a solution that minimizes a norm of the weighted difference between the proposed solution value and the optimal solution value for each scenario. The resulting model does not increase the number of variables of the original representation, but now there are $m|\mathcal{S}|$ constraints. Unfortunately, this representation does not preserve the structure of the deterministic model (1) and, on the other hand, the objective function is no longer linear. Models of this form are known as scenario immunization models, or *SI* models for short, see Mulvey et al. (1995), among others.

As an alternative, we could minimize the expected value of the objective function; in this case, the model (1) becomes

$$\begin{aligned} \min & \sum_{s\in\mathcal{S}} w^s(a^s x + c^s y) + \alpha \sum_{s\in\mathcal{S}} H(z^s) \\ \text{s.t. } & Ax + By + z^s = b^s, \quad \forall s \in \mathcal{S} \\ & x \in \{0,1\}^n, y \geq 0 \end{aligned} \tag{2}$$

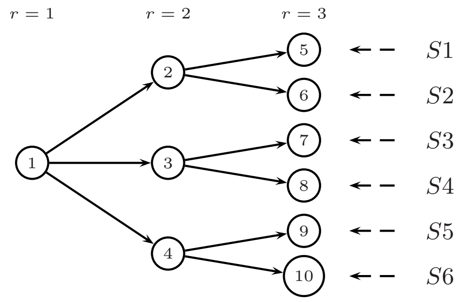
where a^s, b^s and c^s are the realizations of the vector a, b and c under scenario s, α is a nonnegative weighting factor, $H(\cdot)$ is a penalty function and z^s is a vector that expresses the constraints' infeasibility for scenario s. Note that the coefficients of the matrices A and B can also be associated with the occurrence of the scenarios but, for the sake of simplicity in the exposition, we are not considering it.

Note that in *SI* models the $z-$recourse variables seldom represent decisions to be taken at future time stages but, rather, they represent penalties to be taken into account for better decision making in the initial stage. In this case, the models will be usually reoptimized in a rolling time horizon mode. However, when spot decisions (i.e., decisions for the first stage) are the only ones to be made, the information about future uncertainty should be taken into account in a different way for a better spot decision making.

Let $\nu_r^s = (x_r^s, y_r^s)$ denote the vector of the variables related to stage r under scenario s for $r \in \mathcal{R}$, $s \in \mathcal{S}$, and let ν^s be the set of vectors $\{\nu_r^s, \forall r \in \mathcal{R}\}$ for scenario s.

Rockafellar and Wets (1991), see also Wets (1989), state the so-called *non-anticipativity* principle: *If two different scenarios, say, s and s' are identical up to stage r on the basis of the information available about them at that stage, then the values of the ν-variables must be identical up to stage r.* This principle guarantees that the solution obtained from the model at stage r does not depend on information that is not yet available. This type of scheme is termed *full recourse*. To illustrate this concept, consider a so-called *scenario tree* where each node represents a point in time where a decision can be made. Once a decision is made several contingencies can happen, and information related to these contingencies is available at the beginning of the next stage. This information structure is visualized as a tree, where each root-to-leaf path represents one specific scenario and corresponds to one realization of the uncertain parameters.

In order to introduce the implications of this principle in our approach, we define a set of scenario groups, say, $\mathcal{G}_r$ for each stage r, such that all scenarios having the same realizations of the uncertainty up to stage r belong to the same scenario group, say, g for $g \in \mathcal{G}_r$. See figure 1. Let $\mathcal{S}_{g,r}$ denote the set of scenarios that belong to group g at stage r, for $\mathcal{S}_{g,r} \subseteq \mathcal{S}$. Let a node, say, k, in the scenario tree for $k \in \mathcal{G}_r$, $r \in \mathcal{R}$, such that the tree



10 scenario groups. $|\mathcal{S}| = 6$ scenarios

For $r = 2$: $\mathcal{G}_2 = \{2, 3, 4\}$, $\mathcal{S}_{3,2} = \{S3, S4\}$

The path through the nodes (1,2,5) can be associated with scenario $S1$

Figure 1: Scenario tree

is defined by the set of nodes $\{k\}$ and the set of directed arcs $\mathcal{E}$, where $(k, \ell) \in \mathcal{E}$ if and only if $\mathcal{S}_{\ell,r+1} \subseteq \mathcal{S}_{k,r}$ for $k \in \mathcal{G}_r$ and $\ell \in \mathcal{G}_{r+1}$. Finally, let $\mathcal{N}$ denote the set of solutions that satisfy the so-called *non-anticipativity constraints*. That is,

$$\nu \in \mathcal{N} = \big\{\nu^s \text{ such that } \nu_r^{s'} = \nu_r^s \,|\, \forall s, s' \in \mathcal{S}_{g,r},\, g \in \mathcal{G}_r,\, r \in \mathcal{R}\big\}. \tag{3}$$

Hence, the *Deterministic Equivalent Model* (*DEM*) of the *full recourse* version of model (1) can be expressed as follows.

$$\begin{aligned} \min \quad & \sum_{s \in \mathcal{S}} w^s(a^s x^s + c^s y^s) \\ \text{s.t.} \quad & Ax^s + By^s = b^s \qquad \forall s \in \mathcal{S} \\ & x \in \mathcal{N} \\ & x^s \in \{0,1\}^n, y^s \geq 0 \quad \forall s \in \mathcal{S}. \end{aligned} \tag{4}$$

Recall that in our approach only the $0-1$ variables x^s $\forall s \in \mathcal{S}$ are allowed for multi-stage problems, and only the $0-1$ variables x_1^s are allowed in the first stage for two-stage problems. Anyway, the model (4) has a nice structure that we may exploit. Two approaches can be used to represent the non-anticipativity constraints (3). One approach is based on a *compact representation*, where (3) is used to eliminate variables in (4) as well as to reduce the model size, so that there is a single variable for each element at each scenario group from each stage, but any special structure of the constraints in (1) is destroyed.

Benders (1962) decomposition can be applied to the compact representation. Its first application to two-stage stochastic *LP* is due to Slyke and Wets (1969). Moreover, we consider some other types of mathematical formulations, specifically, the so-called *splitting* variables representations, since they are more amenable for our approach to deal with $0-1$ variables. Given the large-scale instances of the model, decomposition in smaller models is a key for success. One type, so-called *node-related* representation, requires to produce siblings of the variables with non-zero elements in the constraint blocks through time stages, so-called the *multi-stage linking constraints*. Another one, so-called *scenario-related* representation, requires siblings of all variables in the model. In both cases the non-anticipativity constraints must be added explicitly, but the second type of representation preserves the problem structure in a more amenable way for our purposes and, then, it is our chosen representation, see below.

3 Splitting variables representation of the *DEM*

Let us rewrite the *DEM* (4) in the following form, by adding explicitly the non-anticipativity constraints.

$$\begin{aligned} \min & \sum_{s\in\mathcal{S}} w^s(a^s x^s + c^s y^s) \\ \text{s.t. } & Ax^s + By^s = b^s \qquad \forall s \in \mathcal{S} \\ & x_r^s - x_r^{s+1} = 0 \qquad \forall s \in \mathcal{S}_{g,r},\ g \in \mathcal{G}_r,\ r \in \mathcal{R} \\ & x^s \in \{0,1\}^n, y \geq 0 \qquad \forall s \in \mathcal{S}. \end{aligned} \tag{5}$$

Figure 2 illustrates the block structure of the splitting variables representation (5). Notice that relaxing the non-anticipativity constraints results in $|\mathcal{S}|$ independent mixed $0-1$ models. Carøe and Schultz (1998, 1999) and Hemmecke and Schultz (2001) use a similar decomposition approach for two-stage problems. However, that approach focuses more on using Lagrangian relaxation to obtain good bounds, and less on branching and variable fixing. See also Takriti and Birge (2000). For an impression on Lagrangian approaches for multi-stage stochastic integer programming developments we refer to Römisch and Schultz (2001).

Let us consider the relaxation of the $0-1$ character of the $x-$variables as well as the non-anticipativity constraints in model (5),

$$x_r^s - x_r^{s+1} = 0 \quad \forall s \in \mathcal{S}_{g,r}, g \in \mathcal{G}_r, r \in \mathcal{R}, \quad x^s \in \{0,1\}^n \ \ \forall s \in \mathcal{S}, \tag{6}$$

such that the new problem can be expressed

$$\begin{aligned} \min & \sum_{s\in\mathcal{S}} w^s(a^s x^s + c^s y^s) \\ \text{s.t. } & Ax^s + By^s = b^s \qquad \forall s \in \mathcal{S} \\ & x^s \in [0,1]^n, y \geq 0 \quad \forall s \in \mathcal{S}. \end{aligned} \tag{7}$$

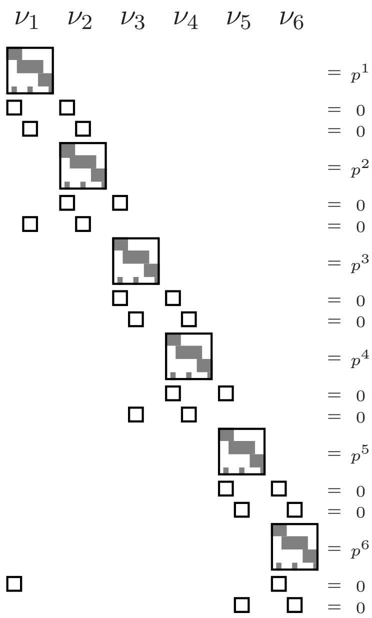
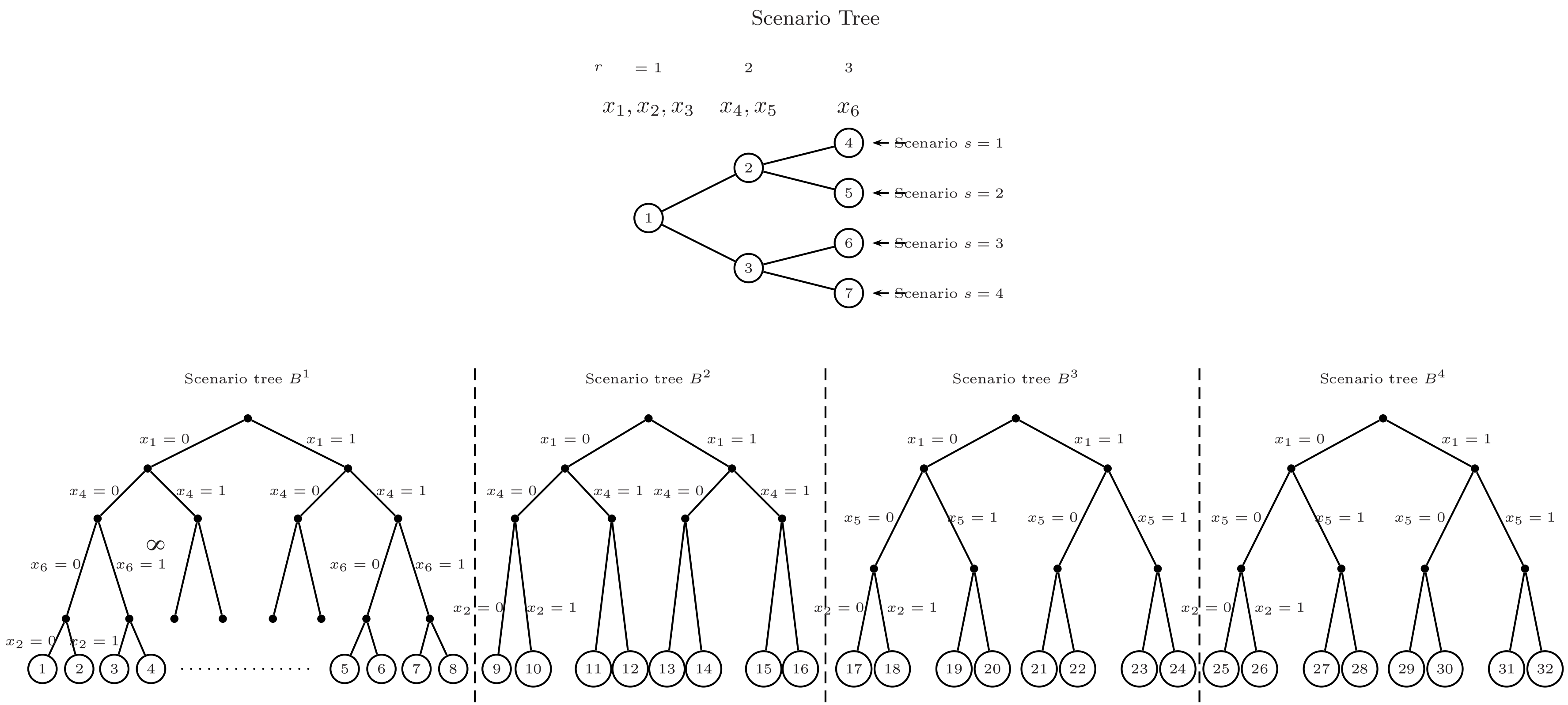


Figure 2: Block structure of the scenario-related splitting variables representation

Note that the *LP* model (7) consists of a set of $|S|$ independent models.

Let $\mathcal{I}$ denote the set of indices of the $x-$variables, and let $\mathcal{I}_r$ denote the set of indices of the $x-$variables associated with stage r such that $(x_r^s)_i$ will give an element of vector x_r^s for $i \in \mathcal{I}_r$. Let also B^s denote the *Branch-and-Fix* (*BF*) tree associated with scenario s, and $\mathcal{A}^s$ the set of active nodes in B^s. Any two active nodes, say, $a \in \mathcal{A}^s$ and $a' \in \mathcal{A}^{s'}$ with $s \neq s'$ are said *twin* nodes if the path from the *root* node to each of them in their own *BF* trees, say, B^s and $B^{s'}$ respectively, has branched or fixed on the same values for the branched or fixed *common variables*. The variables, say, $(x_r^s)_i$ and $(x_r^{s'})_i$ are said to be *common* variables if $s, s' \in \mathcal{S}_{g,r},\ s \neq s',$ for $g \in \mathcal{G}_r,\ i \in \mathcal{I}_r,\ r \in \mathcal{R}$. See that in order to satisfy the non-anticipativity constraints (6), the branching or fixing on the *common* variables must be on the same value $k \in \{0, 1\}$ for the *twin* nodes. A *twin* node family, say, $\mathcal{T}_f$ is a set of nodes, such that any node is a *twin* node to all the other nodes in the family. (Note. For technical reasons, all *BF* nodes should belong to one *twin* node family, at least, even if its cardinality is one). Let $\mathcal{F}$ denote the set of *twin* node families, such that it is said that the nodes a and a' are *twins* if $a, a' \in \mathcal{T}_f, f \in \mathcal{F}$. Note. $|\mathcal{T}_f| = 1$ is allowed. See figure 3, where the vector x_i is included by the elements $\{(x_r^s)_i, r \in \mathcal{R} : i \in \mathcal{I}_r, s \in \mathcal{S}\}$.

A *candidate twin* node family is a set of active nodes from different *BF* trees whose paths from their root nodes have not yet branched nor fixed on any of their *common* variables from their related attached *LP* subproblems; see section 4.3. Notice that the set of root nodes in the *BF* trees is a *candidate twin* node family.



Twin node families by scenario group

Scenario Group 1: $\{1, 9, 17, 25\}$, $\{2, 10, 18, 26\}$,..., $\{5, 15, 23, 31\}$, $\{6, 16, 24, 32\}$

$\{3, 9, 17, 25\}$, $\{4, 10, 18, 26\}$,..., $\{7, 15, 23, 31\}$, $\{8, 16, 24, 32\}$

Scenario Group 2: $\{1, 9\}$, $\{2, 10\}$, $\{3, 9\}$,$\{4, 10\}$, ..., $\{5, 15\}$, $\{6, 16\}$, $\{7, 15\}$,$\{8, 16\}$

Scenario Group 3: $\{17, 25\}$, $\{18, 26\}$,..., $\{24, 32\}$

Scenario Group 4: $\{1\}$, $\{2\}$,..., $\{8\}$

Figure 3: Branch and Fix Coordination scheme

We can observe in figure 3 that the nodes 1, 9, 17 and 25 belong to the same *twin* node family, since the path from the root node to each of them in their own *BF* has branched on the same values for the *common* variables $(x_1^s)_i$, $i \in \mathcal{I}_1$, such as $(x_1^s)_1 = 0$ and $(x_1^s)_2 = 0$ for $s = 1, 2, 3, 4$. Note that there is still one *common* variable, say $(x_1^s)_3$ to branch on, and the branching value from $\{0, 1\}$ has to be the same for all members of the family. Similarly, we can observe that the nodes 1 and 9 belong to the same *twin* node family, since the path from the *root* node to each of them in their own *BF* trees, say, $s = 1, 2$, has branched on the same values for their *common* variables $(x_r^s)_i$, $i \in \mathcal{I}_r, r = 1, 2$, say $(x_1^s)_1 = 0$, $(x_1^s)_2 = 0$ and $(x_2^s)_4 = 0$ for $s = 1, 2$. See also that the nodes 3 and 9 are *twin* nodes and, then, node 9 belongs to two *twin* node families for the same scenario group, such that some members of these families (say, the nodes 1 and 3) belong to the same scenario tree; see in section 4.3 the branching criteria for this situation.

So, the proposal is to execute $|\mathcal{S}|$ *BF* phases (one per scenario) in a coordinate way, so that the non-anticipativity constraints (6) are satisfied. For this purpose the following topics should be addressed: potential variable fixing applications from one member of a *twin* node family to any other member of the family, estimation of the lower bounds of the deterioration of the original problem's relaxation (7), selection of the branching *twin* node family and selection of the *common* branching variable.

4 Branch-and-Fix Criteria for Coordination of Scenario *BF* Phases Execution

Consider the relaxation (7), where the non-anticipativity constraints and the integrality requirement of the $0-1$ variables are not included, see (6).

4.1 Fixing variables in a *BF* tree

The problem to be solved for each scenario s is as follows,

$$(IP^s): \quad Z_{IP}^s = \min \quad a^s x^s + c^s y^s \\ \text{s.t.} \quad Ax^s + By^s = b^s \\ x^s \in \{0,1\}^n, y^s \geq 0. \tag{8}$$

Its linear relaxation will be

$$(LP^s): \quad Z_{LP}^s = \min \quad a^s x^s + c^s y^s \\ \text{s.t.} \quad Ax^s + By^s = b^s \\ x^s \in [0,1]^n, y^s \geq 0. \tag{9}$$

For simplicity and when there is not ambiguity, we will drop the index r from $(x_r^s)_i$, so the new notation for the variable will be x_i^s. Let it be a non basic variable in the optimal solution of (LP^s) and $\overline{c}_i^s$ its reduced cost. It is well known that $|\overline{c}_i^s|$ is a lower bound of the deterioration of the objective function if x_i^s is set from 0 to 1 or viceversa. Let $\overline{Z}_{IP}^s$ be an upper bound of Z_{IP}^s. The Dantzig cuts allow to fix variables as follows:

$$x_i^s \leftarrow 0 \quad \text{if } \overline{c}_i^s > \overline{Z}_{IP}^s - Z_{LP}^s \tag{10}$$

$$x_i^s \leftarrow 1 \quad \text{if } -\overline{c}_i^s > \overline{Z}_{IP}^s - Z_{LP}^s. \tag{11}$$

Our scheme allows to extend the Dantzig cuts to basic variables. For simplicity of notation let us drop the subindex s when referring to the problem related to a single scenario.

Define Z_{LPi}^1 (resp., Z_{LPi}^0) as the new optimal value of (LP) when the cut $x_i = 1$ (resp., $x_i = 0$) is appended to the LP problem. Let Δ_{LPi}^1 and Δ_{LPi}^0 denote the increase in the objective function of (LP) if the variable x_i is fixed to 1 or to 0, respectively, such that

$$\Delta_{LPi}^1 = Z_{LPi}^1 - Z_{LP} \qquad \text{and} \qquad \Delta_{LPi}^0 = Z_{LPi}^0 - Z_{LP}.$$

Note that if x_i had originally a fractional value in the optimal solution of (LP), both Δ_{LPi}^1 and Δ_{LPi}^0 can be strictly positive. Let $\underline{\Delta}_{LPi}^1$ and $\underline{\Delta}_{LPi}^0$ be lower bounds of Δ_{LPi}^1 and Δ_{LPi}^0, respectively.

Let the following additional notation.

$\mathcal{J}_r \subseteq \mathcal{I}_r$, set of indices of the $x-$variables from stage r whose related constraints (6) are not satisfied (i.e., either they have fractional values or the non-anticipativity constraints are violated) at a given active node of a *BF* tree. Note that the variables $(x_r^s)_i$ from set $\mathcal{I}_r \setminus \mathcal{J}_r$ for $r \in \mathcal{R} : s \in \mathcal{S}_{g,r}$, where $g \in \mathcal{G}_r$, have already the same value $k \in \{0, 1\}$ in all members of the related *twin* node family.

$\mathcal{M}_{ik}^{\mathcal{T}_f,k'}$, set of indices of the x-variables whose value must be fixed to k' in any feasible solution to the problems (8) attached to the members of the *twin* node family $\mathcal{T}_f, f \in \mathcal{F}$, provided that the *common* variable $(x_r^s)_i$ is fixed to k, for $k, k' \in \{0, 1\}$, $i \in \mathcal{J}_r$, $s \in \mathcal{S}_{g,r}$, $g \in \mathcal{G}_r$, $r \in \mathcal{R}$, for $s : a \in \mathcal{A}^s$ where $a \in \mathcal{T}_f$. It can be obtained by using *probing*, see Guignard and Spielberg (1981) and *conflict graph analysis*, see Savelsbergh (1994) and Atamtürk et al. (2000), among others on the scenario-related problems (8), and using the potential x-fixings in the problem attached to any node $a \in \mathcal{T}_f$ for performing the appropriate implications in the problems attached to the other members of the family $\mathcal{T}_f$.

Procedure *BOUND* for obtaining $\underline{\Delta}_{LPi}^1$ and $\underline{\Delta}_{LPi}^0$.

1. x_i non basic.

   (a) If $x_i = 0$ then $\underline{\Delta}_{LPi}^0 = 0$, and $\underline{\Delta}_{LPi}^1 = \overline{c}_i$.

(b) If $x_i = 1$ then $\underline{\Delta}^0_{LPi} = -\bar{c}_i$, and $\underline{\Delta}^1_{LPi} = 0$.

2. x_i basic. Let c_i^D and c_i^U be lower bounds of the deterioration of the objective function when x_i is set to 0 and 1, respectively. (Note that these values can be obtained by using sensitivity analysis, such that the deterioration of the objective function up to a basis change is obtained). Note that the pair given by c_i^D and c_i^U is a different concept from the classical down- and up-pseudocost concept, since the latter is related to deterioration's estimations and the former gives deterioration's lower bounds.

   (a) If $x_i = 0$ then $\underline{\Delta}^0_{LPi} = 0$ and $\underline{\Delta}^1_{LPi} = c_i^U$.
   (b) If $x_i = 1$ then $\underline{\Delta}^0_{LPi} = c_i^D$ and $\underline{\Delta}^1_{LPi} = 0$.
   (c) If $x_i \in (0,1)$ then $\underline{\Delta}^0_{LPi} = c_i^D$ and $\underline{\Delta}^1_{LPi} = c_i^U$

A weaker but faster approach is based on the following relationship that is easy to prove,

$$\underline{\Delta}^k_{LPi} \geq \underline{\Delta}^{k'}_{LPj} \quad \text{for } j \in \mathcal{M}_{ik}^{\mathcal{T}_f,k'},\ k,k' \in \{0,1\}, \tag{12}$$

where $\mathcal{T}_f$ is the *twin* node family of the node under study. So, $\underline{\Delta}^{k'}_{LPj}$ gives a lower bound of $\underline{\Delta}^k_{LPi}$ when (12) occurs. Furthermore, if $\underline{\Delta}^k_{LPi} = 0$ then $\underline{\Delta}^{k'}_{LPj} = 0$.

In general, the following procedure can be executed.

Procedure *BOUND2* for obtaining $\underline{\Delta}^1_{LPi}$ and $\underline{\Delta}^0_{LPi}$.

Step 1: Set $\underline{\Delta}^k_{LPi} := -1\ \forall i \in \mathcal{I},\ k \in \{0,1\}$.

Step 2: If $\mathcal{I} = \emptyset$, stop. Otherwise, select $i \in \mathcal{I}$ and proceed as follows for $k \in \{0,1\}$: If $\underline{\Delta}^k_{LPi}$ has not yet been obtained (i.e., $\underline{\Delta}^k_{LPi} = -1$) and $\exists j \in \mathcal{M}_{ik}^{\mathcal{T}_f,k'}$, where $\mathcal{T}_f$ is as above, such that $\underline{\Delta}^{k'}_{LPj} > 0$ for $k' \in \{0,1\}$, then

$$\underline{\Delta}^k_{LPi} := \max_{j \in \mathcal{M}_{ik}^{\mathcal{T}_f,k'},\, k' \in \{0,1\}} \{\underline{\Delta}^{k'}_{LPj}\}. \tag{13}$$

If $\underline{\Delta}^k_{LPi} \geq 0\ \forall k \in \{0,1\}$ then $\mathcal{I} := \mathcal{I} \setminus \{i\}$ and go to Step 2.

Step 3: Apply procedure *BOUND* to obtain $\underline{\Delta}^k_{LPi}$ if $\underline{\Delta}^k_{LPi} = -1, k \in \{0,1\}$.
Update $\mathcal{I} := \mathcal{I} \setminus \{i\}$.

Step 4: If $\underline{\Delta}^k_{LPi} = 0$ then $\underline{\Delta}^{k'}_{LPj} := 0\ \forall j \in \mathcal{M}_{ik}^{\mathcal{T}_f,k'}$ with $k' \in \{0,1\}$.
If $\underline{\Delta}^k_{LPi} = 0\ \forall k \in \{0,1\}$ then $\mathcal{I} := \mathcal{I} \setminus \{j : \mathcal{M}_{ik}^{\mathcal{T}_f,k'}, k' \in \{0,1\}\}$.
In any case, go to Step 2.

By using standard arguments let the following propositions.

Proposition 4.1. *Let $\overline{Z}_{IP}$ be an upper bound of the optimal solution value for (IP). Let (LP) be the linear relaxation of (IP). If $Z_{LP} + \underline{\Delta}^k_{LPi} \leq \overline{Z}_{IP}$ and $Z_{LP} + \underline{\Delta}^{1-k}_{LPi} > \overline{Z}_{IP}$, then x_i can be fixed to k, for $k \in \{0, 1\}$.*

Then, the updating of the upper bound $\overline{Z}_{IP}$ has the following implications:

$$\begin{aligned} x_i \leftarrow 1 \quad & \text{if } \underline{\Delta}^0_{LPi} > \overline{Z}_{IP} - Z_{LP}, \\ x_i \leftarrow 0 \quad & \text{if } \underline{\Delta}^1_{LPi} > \overline{Z}_{IP} - Z_{LP}, \end{aligned}$$

Proposition 4.2. *If (IP) is feasible then $Z_{IP} \geq Z_{LP} + \max_{i \in \mathcal{I}} \big\{ \min\{\underline{\Delta}^1_{LPi}, \underline{\Delta}^0_{LPi}\} \big\}$.*

4.2 Fixing variables in a *BFC* scheme

Let us consider now our proposal for solving the stochastic $0-1$ problem (5) by using the so-called *Branch-and-Fix Coordination* scheme. Let the following additional notation.

$s, *$, root node of the *BF* tree B^s, for $s \in \mathcal{S}$.

$Z^{s,a}_{LP}$, solution value of the *LP* subproblem (9) attached to active node a in *BF* tree B^s, for $a \in \mathcal{A}^s, s \in \mathcal{S}$.

$\underline{\Delta}^{s,a,k}_{LPj}$, lower bound of the solution value $Z^{s,a}_{LP}$ increase, if the variable $(x^s_r)_j$ is fixed to k, for $k \in \{0, 1\}, j \in \mathcal{J}_r, r \in \mathcal{R}, a \in \mathcal{A}^s, s \in \mathcal{S}$. Note: It includes the weight w^s, by construction.

$\underline{Z}^{s,a}_{IP}$, lower bound of the solution value for the scenario s-related problem (8) to be obtained from the subtree where a is the *root* node of the *BF* tree B^s and the non-anticipativity constraints (6) are considered, for $a \in \mathcal{A}^s, s \in \mathcal{S}$. It can be expressed

$$\underline{Z}^{s,a}_{IP} = Z^{s,a}_{LP} + \max_{j \in \mathcal{J}_r, r \in \mathcal{R}} \big\{ \min_{k \in \{0,1\}} \{\underline{\Delta}^{s,a,k}_{LPj}\} \big\}. \tag{14}$$

$\underline{Z}^s_{IP}$, lower bound of the solution value for the scenario s-related problem (8), where the non-anticipativity constraints (6) are considered. It can be expressed

$$\underline{Z}^s_{IP} = \min_{a \in \mathcal{A}^s} \{\underline{Z}^{s,a}_{IP}\}. \tag{15}$$

Remark 1. When solving the *LP* subproblem attached to an active node a in the *BF* tree B^s, for $a \in \mathcal{A}^s, s \in \mathcal{S}$, the fixing of the variable $(x^s_r)_j$ to $k \in \{0, 1\}$ implies that the current solution value $Z^{s,a}_{LP}$ is increased by $\underline{\Delta}^{s,a,k}_{LPj}$, at least, see Proposition 4.2.

Remark 2. In order to satisfy the non-anticipativity principle, if the *common* variable $(x^s_r)_i$ is fixed to a given value $k \in \{0, 1\}$ in an active node, say, a for $a \in \mathcal{A}^s$, then the variable $(x_r^{s'})_i$ must be fixed to the same value and the variables in set $\mathcal{M}_{ik}^{\mathcal{T}_f,k'}$ must be fixed to the value $k' \in \{0,1\}$, in all members of the *twin* node family $\mathcal{T}_f$ for $a \in \mathcal{T}_f$, $f \in \mathcal{F}$, such that $a' \in \mathcal{T}_f$ for $a' \in \mathcal{A}^{s'}$, where $s, s' \in \mathcal{S}_{g,r}$ for $g \in \mathcal{G}_r$, $r \in \mathcal{R} : i \in \mathcal{J}_r$. So, additionally, the solution value $Z_{LP}^{s',a'}$ of $(LP^{s'})$ will have an increase of $\underline{\Delta}_{LPj}^{s',a',k}$, at least, for $a' \in \mathcal{A}^{s'}$, $s' \in \mathcal{S}_{g,r}$.

Proposition 4.3. *If $(x_r^s)_i = k$ is appended to problem (7) for $k \in \{0,1\}$ and $s \in \mathcal{S}_{g,r}$ for given $r \in \mathcal{R}$, such that $g \in \mathcal{G}_r$, then, a lower bound of the solution value for the enlarged problem can be expressed*

$$\sum_{s' \in \mathcal{S}} Z_{LP}^{s'} + \sum_{s' \in \mathcal{S} \setminus \mathcal{S}_{g,r}} \max_{j \in \mathcal{J}_{r'}, r' \in \mathcal{R}} \{ \min_{l \in \{0,1\}} \underline{\Delta}_{LPj}^{s',*,l} \} + \sum_{s' \in \mathcal{S}_{g,r}} \Delta_{LPi}^{s',*,k}, \tag{16}$$

where $\underline{\Delta}_{LPj}^{s',*,l}$ gives the lower bound increase of the solution value $Z_{LP}^{s'}$ for the LP *problem attached to the root node of the* BF *tree $B^{s'}$, if the variable $(x_{r'}^{s'})_j$ is fixed to $l \in \{0,1\}$ for $j \in \mathcal{J}_{r'}$, $r' \in \mathcal{R}$.*

Proof. The problem (7) consists of $|\mathcal{S}|$ independent subproblems. The solution value increase due to fixing $(x_r^s)_i = k$, $i \in \mathcal{I}_r, r \in \mathcal{R}$ in the related problem (LP^s) is $\underline{\Delta}_{LPi}^{s,*,l}$, at least. On the other hand, the non-anticipativity constraint (6) $(x_r^s)_i - (x_r^{s'})_i = 0$ implies $(x_r^{s'})_i = k \ \forall s, s' \in \mathcal{S}_{g,r}$ for $g \in \mathcal{G}_r$, and the result follows. □

The following corollaries follow trivially from Proposition 4.3.

Corollary 4.1. (*BFC* VARIABLE FIXING). *Consider the problem (5) and its relaxation (7). Let $\overline{Z}_{IP}$ be an upper bound of the solution value for (5) and Z_{LP} be the solution value for (7). Then, the variable $(x_r^s)_i$ can be fixed to $k \in \{0,1\}$, if (5) is feasible and the following condition holds*

$$Z_{LP} + \sum_{s \in \mathcal{S}} \underline{\Delta}_{LPi}^{s,*,k} \leq \overline{Z}_{IP} \quad \text{and} \quad Z_{LP} + \sum_{s \in \mathcal{S}} \underline{\Delta}_{LPi}^{s,*,1-k} > \overline{Z}_{IP}. \tag{17}$$

Note that this result allows to fix new variables in the set $\mathcal{A}^s$ of active nodes for B^s $\forall s \in \mathcal{S}$ when the updating of the incumbent solution (and, then, $\overline{Z}_{IP}$) occurs for problem (5).

Corollary 4.2. (*BF* NODE PRUNING AND VARIABLE FIXING). *Consider a twin node family, say, $\mathcal{T}_f, f \in \mathcal{F}$. Then,*

1. *The whole set $\mathcal{T}_f$ of twin nodes can be pruned if the following condition holds*

$$\sum_{s' \in \mathcal{S}: \mathcal{A}^{s'} \cap \mathcal{T}_f = \{\emptyset\}} \underline{Z}_{IP}^{s'} + \sum_{a \in \mathcal{T}_f} \underline{Z}_{IP}^{s,a} \geq \overline{Z}_{IP}. \tag{18}$$

2. *The common variables $(x_r^s)_i$ for $i \in \mathcal{I}_r, r \in \mathcal{R}$ in the* LP *subproblems attached to the members of the twin node family $\mathcal{T}_f$ can be fixed to $k \in \{0,1\}$ if the following condition holds*

$$\sum_{s' \in \mathcal{S}: \mathcal{A}^{s'} \cap \mathcal{T}_f = \{\emptyset\}} \underline{Z}_{IP}^{s'} + \sum_{a \in \mathcal{T}_f} (Z_{LP}^{s,a} + \underline{\Delta}_{LPi}^{s,a,1-k}) \geq \overline{Z}_{IP} \qquad (19)$$

where $s : a \in \mathcal{A}^s$.

Corollary 4.3. (*BFC* INCUMBENT SOLUTION OPTIMALITY PROOF). *Consider the value, say, $\overline{Z}_{IP}$ of the incumbent solution for the original problem* (5)*. Then, the incumbent solution is optimal if the following condition holds*

$$\sum_{s \in \mathcal{S}} \underline{Z}_{IP}^s \geq \overline{Z}_{IP}. \qquad (20)$$

4.3 Branching criteria

The branching in the *BFC* scheme is performed on the same *common* variable for all members of the selected *twin* node family. Moreover, notice that in case that a given node belongs to more than one family then all the node members of these families must simultaneously branch on the same *common* variable. Let us say that a *twin* node family, say, f belongs to a *twin* node family set, say, $\mathcal{F}$, if $\exists f' \in \mathcal{F}, f \neq f'$ such that $\mathcal{T}_f \bigcap \mathcal{T}_{f'} \neq \{\emptyset\}$, i.e., there is another family member, say, f' from set $\mathcal{F}$ that has a (*twin*) node in common with family f. As an illustration the members of the *twin* node families, say, $\mathcal{T}_1 = \{1,9\}$ and $\mathcal{T}_2 = \{3,9\}$ for scenario group 2 in figure 3 should branch simultaneously on the *common* variable $\{x_2^s\}_5$, $s = 1,2,3$ and, so, the families 1 and 2 belong to the same *twin* node family set, say, $\mathcal{F}$, i.e., $1,2 \in \mathcal{F}$.

Among the different criteria for selecting the next *twin* node family set, say, $\mathcal{F}$, to branch (see in Linderoth and Savelsbergh, 1999, the related criteria for single Branch-and-Bound trees), we use the *depth first search* strategy. According to this criterion, the set, say, $\widehat{\mathcal{F}}$ is the set with the *smallest lower bound of the solution value* among the two *twin* node family sets, say, $\mathcal{F}_1$ and $\mathcal{F}_2$ that have just been created, i.e.

$$\widehat{\mathcal{F}} = \text{argmin}_{\mathcal{F} \in \{\mathcal{F}_1, \mathcal{F}_2\}} \{\min_{f \in \mathcal{F}} \{\sum_{a \in \mathcal{T}_f} \underline{Z}_{IP}^{s,a}\}\}, \qquad (21)$$

where $s : a \in \mathcal{A}^s$ and $\underline{Z}_{IP}^{s,a}$ is given by (14), provided that the *LP* subproblems attached to the members of the *twin* node families $\mathcal{T}_f, f \in \mathcal{F}$ still have *common* variables that have not yet been branched, nor fixed on. Otherwise, the branching family set $\widehat{\mathcal{F}}$ is chosen according to the rule (22) from the set, say, $\mathcal{C}$ of the *candidate twin* node family sets, say, $\mathcal{F}_c$ $\forall c \in \mathcal{C}$ that can be identified from the two *twin* node family sets $\mathcal{F}_1$ and $\mathcal{F}_2$ that have just been created.

$$\widehat{\mathcal{F}} = \text{argmin}_{c\in\mathcal{C}}\{\min_{f\in\mathcal{F}_c}\{\sum_{a\in\mathcal{T}_f} \underline{Z}_{IP}^{s,a}\}\}, \tag{22}$$

Once the branching family set $\widehat{\mathcal{F}}$ has been selected, the branching $x-$variable must be the same in all members of the *twin* node family set, in order to satisfy the non-anticipativity constraints (6). We use the *most deterioration criterion* for selecting the common variable $(x_r^s)_i$ to branch on for $a \in \mathcal{T}_f$, $f \in \widehat{\mathcal{F}}$,

$$i = \text{argmax}_{j\in\mathcal{J}_r}\Big\{\min_{k\in\{0,1\}}\{\min_{f\in\widehat{\mathcal{F}}}\{\sum_{a\in\mathcal{T}_{\hat{f}}} \underline{\Delta}_{LPj}^{s,a,k}\}\}\Big\} \tag{23}$$

where $s : a \in \mathcal{A}^s$, and r is the earliest stage such that $\mathcal{J}_r \neq \emptyset$. Note that the earlier stage r is the bigger the cardinality $|\mathcal{S}_{g,r}|$ is and, then, bigger can be the global deterioration of the solution value in the original model (5). Notice that $\underline{\Delta}$ only represents a lower bound of that deterioration. Note also that $\mathcal{S}_{g,r} = \mathcal{S}$ for $r = 1$ (and, then, $|\mathcal{G}_r| = 1$) and $|\mathcal{S}_{g,r}| = 1$ for $r = |\mathcal{R}|$ (and, then, $\mathcal{G}_r = \mathcal{S}$).

Alternative strategies can be used for selecting the branching variable, by using the traditional criteria considered in MIP Branch-and-Bound schemes. Typical strategies are the most fractional value, the biggest objective function coefficient, a combination of both such that the M most fractional $x-$variables for $1 \leq M \leq n$ are considered and, then, the variable with the biggest objective function coefficient is chosen, etc.

A more sophisticated rule will be based on computing the deterioration of the solution value due to fixing each variable $(x_r^s)_i$ to zero and to one, exactly. The so-called *strong branching* approach (see Applegate et al., 1997) gives a lower bound of the up- and down-deterioration by performing a given number of dual Simplex iterations. See its performance comparison in the study reported by Linderoth and Savelsbergh (1999). The criterion (23) is in the same spirit as this approach.

5 Application cases

In this section we present two application cases of the *BFC* scheme to stochastic $0-1$ problems, the first one related to the *Air Traffic Flow Management* problem and the second one related to the *Strategic Supply Chain Management* problem.

5.1 Air Traffic Flow Management

The *Air Traffic Flow Management* (*ATFM*) problem under uncertainty in a multi-stage pure $0-1$ program environment considers a network of airports such that ground-policies for one of them have impact on the other airport schedules.

The goal is to obtain a schedule for a set of flights to minimize the total cost delay without violating the airport and air sector capacities and preserving the precedence relationships between flights, among other constraints. The main assumption on the problem described in the open literature is the deterministic character of the data. However, the airport capacity for flight arrivals and departures is a random parameter mainly due to the weather conditions; the airsector capacity is also an uncertain parameter, due to a variety of conditions.

Using a tight pure $0-1$ deterministic model for *ATFM* due to Bertsimas and Stock (1998), Alonso-Ayuso (1997) proposed a stochastic model via *scenario analysis* and a related algorithmic framework for *ATFM* to deal with the uncertainty on the airport and air sector capacities. These capacities can be assumed deterministic for the first time periods, but the weather prediction for the other periods should be considered as a random variable. A Lagrangian-based Decomposition procedure was used for problem solving but the computational effort is not worthy. Alternatively, Alonso-Ayuso et al. (2000) propose an heuristic approach to solve the problem. This approach, so-called *Fix-and-Relax*, is a Branch-and-Cut scheme that utilises a restrictive criterion for node branching selection, i.e., only the node with the best solution value in the *LP* relaxation of the associated full IP compact model representation for a given stage is chosen, provided that all variables up to the stage take $0-1$ values. The approach obtains a feasible solution that frequently can be proved optimal.

We report the computational experience obtained while optimizing the stochastic *ATFM* problem for a set of large-scale instances by using the *BFC* approach. Due to the tightness of the $0-1$ model, some perturbed cases are introduced in order to destroy the special structure of the problem. The testbed contains 24 instances grouped in 2 sets. The original instances are labelled starting with an A and the perturbed instances start with a P, followed by the number of flights in the case. Two airports and three sectors have been considered, while the time horizon has been divided into 48 time periods. Furthermore, three stages have been considered with 16 time periods each of them.

The flights are uniformly distributed over the two potential paths that are allowed. The departure time periods are randomly assigned so that the departure and arrival time periods of any flight are within the planning horizon. The precedence relationships between pairs, say, (f', f) of flights are randomly chosen, such that the time lag between the departure time of flight f and the arrival time of flight f' is constrained to be non-smaller than a given value, the so-called *turnaround time*. See in Alonso-Ayuso et al. (2000) the scheme that we have used for generating the scenario tree, included by $|\mathcal{S}| = 16$ scenarios.

5.2 Strategic Supply Chain Management

The *Strategic Supply Chain Management* (*SSCM*) problem under uncertainty in a two-stage mixed $0-1$ program environment consists of determining the topology of a given supply chain system, included by the plant sizing, single/multilevel product selection, product allocation among plants and vendor selection for raw material. The objective is the maximization of the expected benefit given by the product net profit minus the operation cost and the plant investment depreciation cost along a given time horizon. The uncertain parameters are the product demand and price, the production cost and the raw material cost along the time horizon. Alonso-Ayuso et al. (2001) present a two stage mixed $0-1$ model. The first stage is devoted to the *strategic* decisions about the plant sizing, production selection and allocation and vendor selection; the decisions are modelled with $0-1$ variables. The second stage is devoted to the *tactical* decisions about the raw material volume to be supplied from the vendors, product volume to be processed in plants, stock volume of product/raw material to be stored in warehouses, component volume to be transported from origin plants/warehouses to destination plants and product volume to be shipped from plants to market centres along the time horizon, given the topology of the supply chain system decided in the first stage. The tactical decisions are modelled by continuous variables and some strategic decisions, also modelled with $0-1$ variables, are allowed for plant capacity expansion.

We report the computational experience obtained while optimizing the stochastic *SSCM* problem for a set of large-scale instances by using the *BFC* approach. The testbed contains 12 instances, labelled starting with an S in the tables that report the computational results. The instances have the following dimensions: 4 to 6 plants/warehouses, 3 capacity levels per plant, 12 products, where 8 to 10 are subassemblies, 12 raw materials, 24 vendors, 2 markets per product and 10 time periods.

Up to 7 levels of demand for the products and 5 levels of price/cost for the products/raw materials have been considered for generating the scenario tree, resulting in $|\mathcal{S}| = 19$ significant scenarios.

5.3 Computational results

Our algorithmic approach has been implemented in a FORTRAN code, so-called *BFC*. It uses the optimization engine IBM OSL v2.1 for solving the *LP* subproblems at the active nodes in the *BF* trees. The computational experiments were conducted on a 800 MHz Pentium III Processor with 512 Mb of RAM. Note. An ad-hoc branching tree management scheme for *BFC* has been used (see Alonso, 1997), instead of the related MIP OSL routines.

Table 1 gives the number of constraints (given by the heading m) and variables together with the matrix density (%). See that the problem dimensions are very high.

Table 2 shows the main results of our computational experimentation obtained by using *BFC*. The headings are as follows: Z_{LP}, solution value of the *LP* relaxation; Z_{IP}, solution value of the optimal integer solution; GAP, optimality gap (%) defined as $|Z_{IP} - Z_{LP}|/Z_{LP} \times 100$ (note that Z_{LP} gives the solution value of the integer model for $GAP = 0$); T_{LP} and T_{IP}, the elapsed time (secs.) to obtain the *LP* solution and the additional time to

Table 1: Testbed model dimensions

Instance	Deterministic model				DEM Stochastic model			
	m	n	nc	density	m	n	nc	density
A150-1	2388	3993	-	0.09	17061	29631	-	0.01
A150-2	1318	2201	-	0.17	15280	26660	-	0.01
A150-3	1932	3342	-	0.12	16593	29964	-	0.01
A150-4	2158	3696	-	0.10	16660	29715	-	0.01
A150-5	2426	4224	-	0.09	16952	30690	-	0.01
A150-6	1932	3341	-	0.12	16593	29948	-	0.01
A200-1	3188	5556	-	0.07	22277	40272	-	0.01
A200-2	3268	5492	-	0.07	23875	41786	-	0.01
A200-3	3200	5548	-	0.07	22337	40087	-	0.01
A200-4	3220	5644	-	0.07	22132	40258	-	0.01
A200-5	3200	5551	-	0.07	22493	40447	-	0.01
A200-6	3220	5644	-	0.07	22132	40258	-	0.01
P150-1	2158	3695	-	0.10	16660	29699	-	0.01
P150-2	2190	3744	-	0.10	16794	30153	-	0.01
P150-3	2426	4228	-	0.09	16952	30754	-	0.01
P150-4	1318	2206	-	0.17	15280	26740	-	0.01
P150-5	2158	3692	-	0.10	16660	29675	-	0.01
P150-6	2052	3465	-	0.11	17544	30834	-	0.01
P200-1	2900	5075	-	0.08	22256	40127	-	0.01
P200-2	2150	3719	-	0.10	20870	37733	-	0.01
P200-3	2900	5075	-	0.08	22256	40127	-	0.01
P200-4	3200	5548	-	0.07	22337	40087	-	0.01
P200-5	2832	4847	-	0.08	22569	40238	-	0.01
P200-6	3130	5289	-	0.07	22843	40020	-	0.01
S-01	3406	108	2820	0.10	63436	756	54948	0.01
S-02	3059	79	2430	0.11	57041	403	46854	0.01
S-03	3458	108	2960	0.10	64388	756	57608	0.01
S-04	3101	103	2430	0.11	57731	751	47538	0.01
S-05	3145	103	2560	0.11	58495	751	50008	0.01
S-06	3933	114	3540	0.10	73323	762	68628	0.01
S-07	3718	111	3160	0.10	69328	759	61408	0.01
S-08	2959	89	2540	0.11	55213	629	49400	0.01
S-09	3081	103	2440	0.12	57351	751	47728	0.01
S-10	3678	111	3100	0.10	68568	759	60268	0.01
S-11	2819	89	2260	0.13	52553	629	44080	0.01
S-12	3405	105	2960	0.10	63435	753	57608	0.01

obtain the integer solution, respectively; T, total time; nf, number of *twin* node families that have been considered for joint branching and fixing variables in the scenario trees; nn, total number of branching nodes. Note. The A$-$ and P$-$instances are minimization problems, and the S$-$ instances are maximization problems.

Table 2: Branch-and-Fix Coordination scheme performance

Instance	Z_{LP}	Z_{IP}	GAP	T_{LP}	T_{IP}	T	nf	nn
A150-1	4128.50	4138.29	0.24	40.48	90.02	130.50	187	760
A150-2	943.92	1000.46	5.99	11.65	214.32	225.97	1021	4096
A150-3	1844.18	1913.93	3.78	44.27	21.20	65.47	26	416
A150-4	1397.25	1449.57	3.74	33.01	53.44	86.45	143	584
A150-5	1530.23	1574.55	2.90	30.87	137.75	168.62	409	1648
A150-6	1682.86	1775.61	5.51	40.70	75.08	115.78	141	1032
A200-1	6426.91	6559.22	2.06	92.82	1455.86	1548.68	961	15376
A200-2	3329.22	3336.70	0.22	79.80	10.05	89.85	7	40
A200-3	1090.78	1213.60	11.26	98.65	355.86	454.51	182	2864
A200-4	1046.61	1182.09	12.94	72.45	173.45	245.90	146	2336
A200-5	864.06	924.33	6.98	82.61	305.99	388.60	377	1664
A200-6	1303.33	1426.16	9.42	91.01	256.39	347.40	147	2352
P150-1	63768.13	84299.35	32.20	16.04	351.68	367.72	698	10352
P150-2	138193.59	146502.25	6.01	15.77	86.51	102.28	177	2496
P150-3	360703.95	379813.32	5.30	37.73	793.95	831.68	1220	17978
P150-4	74569.37	76637.22	2.77	8.08	4.28	12.36	9	72
P150-5	26784.89	40785.83	52.27	15.65	258.70	274.35	512	6752
P150-6	120795.87	122262.89	1.21	19.83	29.49	49.32	50	368
P200-1	137456.91	144081.97	4.82	32.35	232.17	264.52	272	3968
P200-2	432209.31	441757.53	2.21	24.72	165.16	189.88	209	860
P200-3	84408.66	89180.73	5.65	27.63	86.39	114.02	160	2368
P200-4	138617.09	151665.09	9.41	66.02	1085.77	1151.79	1420	10912
P200-5	79895.25	87661.45	9.72	28.56	47.02	75.58	72	840
P200-6	111106.96	118970.64	7.08	38.94	625.66	664.60	690	3552
S-01	237838.66	181997.69	23.48	1257.57	7200.00(*)	8457.57	3605	6521
S-02	163599.55	124092.48	24.15	275.61	1129.71	1405.32	1373	4631
S-03	64217.66	0.00	100.00	296.10	30.98	327.08	7	133
S-04	175033.83	140832.71	19.54	267.87	970.48	1238.35	623	4421
S-05	284638.74	227401.41	20.11	544.25	2572.44	3116.69	2273	9131
S-06	51920.95	0.00	100.00	1336.22	897.65	2233.87	39	741
S-07	114305.25	58995.35	48.39	761.04	1915.97	2677.01	1109	3827
S-08	130357.12	95229.39	26.95	282.26	5822.93	6105.19	9349	20635
S-09	180121.04	147078.30	18.34	213.71	631.53	845.24	419	3929
S-10	108551.50	50513.93	53.47	805.48	6789.40	7594.88	3277	22447
S-11	94799.09	14411.54	84.80	460.55	3425.98	3886.53	2245	6007
S-12	119917.28	68990.07	42.47	500.97	1267.96	1768.93	801	3555

(*) The CPU time limit has been reached

The first observation on the results shown in Table 2 is that the *BFC* scheme proves the solution's optimality for all instances in the testbed but one, (whose optimization was stopped due to reaching the 2 hours time limit).

We can also observe in Table 2 that the *GAP* for the (pure 0−1 scheduling) A−instances is small (giving an indication of the tightness of the Bertsimas-Stock scenario model), mainly when comparing it with the GAP for the P$-$instances (that result from perturbations in the A$-$instances) and, mainly, with the GAP for the S$-$instances. See that the elapsed time needed to obtain the optimal solution for the A$-$ and P$-$instances barely reaches 25 minutes, while most of the cases require less than 10 minutes. (Note. See in Table 3 a computational comparison of these results and the results obtained by an ad-hoc heuristic algorithm that we present elsewhere for the *Air Traffic Flow Management* problem). Notice that, although the optimality GAP between the LP and IP solution values is small for the A$-$ and P$-$instances, an important branching effort has been required (see the nn and nf columns) for obtaining the optimal solution in most of the instances.

Moreover, we can also notice in Table 2 that the GAP for the (mixed $0-1$ planning) S$-$instances is quite high. (However, see in Table 4 a smaller GAP by replacing the LP bound by a tighter one). This fact, together with the extremely high dimensions of the related stochastic models, makes more difficult to prove solution's optimality. In any case, a 2 hours time limit was only reached in one out of 12 instances before the solution's optimality was proved.

Table 3 shows the main results of our computational experimentation with the heuristic algorithm, so-called *Fix-and-Relax* (for short, *FR*) presented in Alonso-Ayuso et al. (2000) for the multi-stage stochastic pure $0-1$ approach for the *Air Traffic Flow Management* problem, and its computing time comparison with the *BFC* scheme. The headings have the same meaning as in Table 2, with the following main differences. Z_{LP} in the *BFC* scheme gives the *LP* solution value for the splitting variables representation, where the non-anticipativity constraints have been relaxed in the *LP* model, while Z'_{LP} in the *FR* scheme gives the *LP* solution value for the compact representation where, by construction, the non-anticipativity constraints are not relaxed and, as a consequence, the values of GAP' and nn' that are shown in Table 3 are much smaller than the related values that are shown in Table 2. However, the computing time T' (for the *FR* scheme) is much bigger than the computing time T (for the *BFC* scheme) in 21 out of 24 instances, in spite of the *FR* scheme does only prove the optimality in 15 out of 24 instances. See also the column *RT* where the ratio T/T' is shown. In any case, one of the main advantages of the *BFC* scheme over the *FR* scheme is that the dimensions of the scenario-related models that are used in the *Branch-and-Fix* execution phase do not change, by construction, with the number of scenarios (although obviously the number of *BF* trees increases). By the contrary, the dimensions of the compact representation used in the *FR* scheme increase with the number of scenarios and, so, the instances' optimization process may collapse more easily when using it than when using the *BFC* scheme.

Table 4 shows some parameters, say, WS (*Wait-and-See*), $EVPI$ (*Expected Value of Perfect Information*), EEV (*Expected result of using the Expected Value solution*) and VSS (*Value of the Stochastic Solution*) (see, e.g., Birge and Louveaux, 1997 for more details) for analyzing the goodness of the stochastic approach by comparing it with the

Table 3: *FR* scheme performance for *ATFM* and its comparison with the *BFC* scheme

Instance	Z'_{LP}	$\overline{Z'}_{IP}$	*GAP*$'$	T'_{LP}	T'_{IP}	T'	nn'	RT (%)
A150-1	4134.17	4138.29(*)	0.10	419.36	4.72	424.08	5	30.77
A150-2	1000.46	1000.46	0.00	344.66	0.00	344.66	0	65.56
A150-3	1913.93	1913.93	0.00	517.72	0.00	517.72	0	12.65
A150-4	1449.57	1449.57	0.00	408.48	0.00	408.48	0	21.16
A150-5	1574.55	1574.55	0.00	452.91	0.00	452.91	0	37.23
A150-6	1775.61	1775.61	0.00	618.35	0.00	618.35	0	18.72
A200-1	6549.83	6559.22	0.14	927.26	54.15	981.41	5	157.80
A200-2	3329.22	3509.13(*)	5.40	1295.04	39.98	1335.02	9	6.73
A200-3	1213.60	1213.60	0.00	893.81	0.00	893.81	0	50.85
A200-4	1182.09	1182.09	0.00	793.34	0.00	793.34	0	31.00
A200-5	924.33	924.33	0.00	814.60	0.00	814.60	0	47.70
A200-6	1426.16	1426.16	0.00	859.69	0.00	859.69	0	40.41
P150-1	66280.89	84299.35	27.19	38.23	1161.23	1199.46	529	30.66
P150-2	139777.05	146502.25	4.81	83.10	109.08	192.18	83	53.22
P150-3	361887.12	380390.48(*)	5.11	178.73	749.68	928.41	696	89.58
P150-4	76637.22	76637.22	0.00	39.22	0.00	39.22	0	31.51
P150-5	27411.35	41135.31(*)	50.07	21.26	422.54	443.80	348	61.82
P150-6	121377.42	124865.89(*)	2.87	66.73	8.74	75.47	8	65.35
P200-1	139017.08	144081.98(*)	3.64	200.04	158.30	358.34	82	73.82
P200-2	433260.48	441757.53(*)	1.96	219.70	47.46	267.16	35	71.07
P200-3	85458.53	89180.73	4.36	72.34	57.89	130.23	67	87.55
P200-4	139013.54	155333.59(*)	11.74	56.19	321.48	377.67	267	304.97
P200-5	79895.25	88796.45(*)	11.14	48.34	97.16	145.50	85	51.95
P200-6	111250.64	118970.64	6.94	171.37	27.90	199.27	7	333.52

(*) Optimality has not been proved

more traditional *average scenario* related approach. We can observe that the parameter $EVPI$ (%) is relatively small for the A$-$ and P$-$instances. It may suggest that no further research to reduce the variability of the scenario tree is very much needed for the testbed. On the other hand, the parameter VSS (%) is not a finite number for 9 out of 36 instances. Note that $VSS = +\infty$ means that there is one scenario with an infeasible solution, at least. On the other hand, there are 13 out of 36 instances where VSS is finite and greater than 10%. Additionally, there is one S$-$instance (namely, S-03) where the *BFC* scheme does not recommend to carry out any initiative, but the *average scenario* related scheme does recommend it resulting in an expected big loss. Independently of that, there are 4 S$-$instances where both schemes recommend the same type of action. And, finally, there is one S$-$instance (namely, S-11), where both schemes recommend to carry out some activity, but the *BFC* based planning gives an expected profit, while the *average scenario* planning gives an expected loss.

Table 4: Goodness Measure of the Stochastic Solution

Instance	Z_{IP}	WS	$EVPI$	(%)	EEV	VSS	(%)
A150-1	4138.29	4136.82	1.47	(0.04)	6159.26	2020.97	(48.84)
A150-2	1000.46	943.92	56.53	(5.65)	1302.40	301.94	(30.18)
A150-3	1913.93	1844.18	69.75	(3.64)	3735.52	1821.59	(95.17)
A150-4	1449.57	1397.25	52.32	(3.61)	1529.00	79.43	(5.48)
A150-5	1574.55	1530.23	44.32	(2.81)	1782.99	208.43	(13.24)
A150-6	1775.61	1682.86	92.75	(5.22)	$+\infty$	$+\infty$	$-$
A200-1	6559.22	6432.39	126.83	(1.93)	$+\infty$	$+\infty$	$-$
A200-2	3336.70	3331.74	4.96	(0.15)	$+\infty$	$+\infty$	$-$
A200-3	1213.60	1090.77	122.83	(10.12)	$+\infty$	$+\infty$	$-$
A200-4	1182.09	1046.60	135.48	(11.46)	$+\infty$	$+\infty$	$-$
A200-5	924.33	864.05	60.28	(6.52)	$+\infty$	$+\infty$	$-$
A200-6	1426.16	1303.33	122.83	(8.61)	$+\infty$	$+\infty$	$-$
P150-1	84299.35	81479.00	2820.36	(3.35)	91655.32	7355.97	(8.73)
P150-2	146502.25	144918.79	1583.45	(1.08)	188702.05	42199.81	(28.80)
P150-3	379813.32	379006.75	806.56	(0.21)	439283.59	59470.27	(15.66)
P150-4	76637.22	74569.37	2067.84	(2.70)	81947.96	5310.74	(6.93)
P150-5	40785.83	38082.71	2703.12	(6.63)	$+\infty$	$+\infty$	$-$
P150-6	122262.89	122098.74	164.15	(0.13)	$+\infty$	$+\infty$	$-$
P200-1	144081.97	142601.36	1480.62	(1.03)	311464.58	167382.61	(116.17)
P200-2	441757.53	440706.36	1051.17	(0.24)	572504.06	130746.53	(29.60)
P200-3	89180.73	88130.86	1049.87	(1.18)	90634.59	1453.87	(1.63)
P200-4	151665.09	145740.84	5924.25	(3.91)	310844.96	159179.87	(105.95)
P200-5	87661.45	87661.45	.00	(0.00)	146046.70	58385.25	(66.60)
P200-6	118970.64	118826.96	143.68	(0.12)	118970.64	0.00	(0.00)
S-01	181997.69	202053.82	20056.12	(11.02)	178663.97	3333.72	(1.83)
S-02	124092.48	125641.89	1549.41	(1.25)	124092.48	0.00	(0.00)
S-03	0.00	21172.92	21172.92	$-$	-13908.84	13908.84	$-$
S-04	140832.71	144260.55	3427.85	(2.43)	140832.71	0.00	(0.00)
S-05	227401.41	247574.33	20172.92	(8.87)	210865.90	16535.51	(7.27)
S-06	0.00	7935.54	7935.54	$-$	0.00	0.00	$-$
S-07	58995.35	74463.54	15468.19	(26.22)	56977.70	2017.65	(3.42)
S-08	95229.39	104328.67	9099.28	(9.56)	79091.18	16138.21	(16.95)
S-09	147078.30	158895.42	11817.11	(8.03)	147078.25	0.00	(0.00)
S-10	50513.93	68759.07	18245.13	(36.12)	41393.37	9120.56	(18.06)
S-11	14411.54	34525.28	20113.75	(139.57)	-6321.80	20733.34	(143.87)
S-12	68990.07	82680.97	13690.90	(19.84)	66806.05	2184.02	(3.17)

6 Conclusions

In this paper we have presented an algorithmic framework to deal with $0-1$ programs for stochastic decision making via scenario analysis. The types of stochastic problems that are considered in the paper are multi-stage pure $0-1$ programs and two-stage mixed

$0-1$ programs (where only $0-1$ variables have nonzero elements in the first stage constraints). We present a splitting variables representation of the problem via scenario to create the appropriate siblings of the coupling variables. A so-called *BFC* (*Branch-and-Fix Coordination*) approach is introduced to coordinate the *BF* phase execution for each scenario-related model, such that the non-anticipativity constraints are also satisfied. For this purpose, the concept of *twin* node families in the set of *BF* trees is introduced. The branching node and branching variable selection rules as well as the variable fixing and the node pruning criteria that are proposed result to be very effective in the types of $0-1$ problem applications that we have experimented with. Computational results for large $0-1$ instances are reported and a comparison with previous results has been shown in some of them. Although more computational testing is required, the new approach for solving stochastic $0-1$ models seems to be very promising. We have compared the proposed approach via scenario analysis for obtaining the *full recourse stochastic solution* with the more traditional approach based on the *average scenario* $0-1$ program solving. Although we have obtained the optimal solution for the second type of problems in all instances, the stochastic solution never has worse expected performance; usually, it behaves much better than the *average scenario* related solution.

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