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Sobolev orthogonality of polynomial solutions of second-order partial differential equations

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Abstract

Given a second-order partial differential operator $\mathscr{L}$ with nonzero polynomial coefficients of degree at most 2, and a Sobolev bilinear form

$$(P, Q)_S = \sum_{i=0}^{N} \sum_{j=0}^{i} \left\langle \mathbf{u}^{(i,j)}, \partial_x^{i-j} \partial_y^{j} P \ \partial_x^{i-j} \partial_y^{j} Q \right\rangle, \quad N \geqslant 0,$$

where $\mathbf{u}^{(i,j)}$, $0 \leqslant j \leqslant i \leqslant N$, are linear functionals defined on the space of bivariate polynomials, we study the orthogonality of the polynomial solutions of the partial differential equation $\mathscr{L}[p] = \lambda_{n,m}\, p$ with respect to $(\cdot, \cdot)_S$, where $\lambda_{n,m}$ are eigenvalue parameters depending on the total and partial degree of the solutions. We show that the linear functionals in the bilinear form must satisfy Pearson equations related to the coefficients of $\mathscr{L}$. Therefore, we also study solutions of the Pearson equations that can be obtained from univariate moment functionals. In fact, the involved univariate functionals must satisfy Pearson equations in one variable. Moreover, we study polynomial solutions of $\mathscr{L}[p] = \lambda_{n,m}\, p$ obtained from univariate sequences of polynomials satisfying second-order ordinary differential equations.

Keywords Bivariate orthogonal polynomials · Sobolev orthogonal polynomials

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1 Introduction

Orthogonal polynomials with respect to a bilinear form involving both the polynomials and their derivatives are known as *Sobolev orthogonal polynomials*. The *non-standard* character of this type of orthogonality implies that the three-term relation no longer holds and, thus, it is more difficult to study these orthogonal polynomials. Therefore, *ad hoc* tools and techniques are needed; hence, there is a lack of uniformity in the theory of Sobolev orthogonal polynomials.

Sobolev orthogonal polynomials have been widely studied for the last 60 years. We refer the reader to a detailed survey by Marcellán and Xu (2015). However, the study of Sobolev orthogonal polynomials in several variables is most recent. The tools and techniques for studying these multivariate polynomials are even fewer than in the univariate case. Some references include studies on the unit ball and the unit sphere (Dai and Xu 2011; Delgado et al. 2016, 2013; Li and Xu 2014; Lizarte et al. 2021; Pérez et al. 2013; Piñar and Xu 2009; Xu 2006, 2008), the simplex (Atkas and Xu 2013; Xu 2017), and product domains (Dueñas et al. 2021, 2017; Fernández et al. 2015). We remark that most of the results involve bilinear forms with first-order derivatives (Bracciali et al. 2010; Marriaga et al. 2021).

Interestingly, some families of multivariate Sobolev orthogonal polynomials are eigenfunctions of second-order linear partial differential operators with polynomial coefficients. In Lee and Littlejohn (2006), Lee and Littlejohn study bivariate polynomials which satisfy an admissible (as defined in Krall and Sheffer (1967), see also Kim et al. (1997)) second-order partial differential equation. They find conditions for the partial differential equation to have polynomial solutions which are orthogonal with respect to a symmetric Sobolev bilinear form. However, the Lee and Littlejohn approach to Sobolev orthogonal polynomials seems to be incomplete, since there exist non-admissible partial differential equations having Sobolev orthogonal polynomial solutions. Indeed, in Koornwinder (1975), T. Koornwinder gave some interesting examples of orthogonal polynomials constructed using Jacobi polynomials in one variable, and in Kwon et al. (2001), Kwon, Lee, and Littlejohn widened the class found by (Krall and Sheffer (1967).

In this work, we study families of polynomials in two variables satisfying second-order partial differential equations, and we will connect this fact with Sobolev orthogonality. Our motivation for the study of these polynomials comes from the results obtained in Bracciali et al. (2010), Piñar and Xu (2009), Xu (2008, 2017), where families of explicit orthogonal bases are constructed from spherical harmonics and univariate Jacobi polynomials, for Sobolev inner products defined on the ball, the simplex, and the so-called parabolic biangle (on the real plane). Moreover, these Sobolev orthogonal polynomials satisfy partial differential equations with non-standard values of the parameters (i.e., values such that orthogonality with respect to linear functionals does not exist).

Our study starts with the observation that symmetry on polynomials of a partial differential operator with respect to a bilinear form involving linear functionals (and an additional hypothesis) implies the orthogonality of its polynomial eigenfunctions with respect to the bilinear form. Then, we study sufficient conditions that must be satisfied by the involved linear functionals to obtain the desired symmetry. From here, our main strategy for constructing polynomial eigenfunctions of the differential operator as well as the linear functionals involved in the bilinear form is to use the method described by Koornwinder in Koornwinder (1975) [previously introduced by Agahanov in Agahanov (1965)]. Despite its apparent simplicity, this method provides a key to the study of algebraic, differential, and analytic properties for a large class of bivariate orthogonal families of polynomials. In fact, the most usual bivariate families correspond to this scheme. For instances, eight of the nine different cases of Krall and Sheffer classical bivariate orthogonal polynomials (see Krall and Sheffer 1967) can be constructed in this way.

The main contributions of this paper are as follows. First, we systematize and unify the study of the Sobolev orthogonality of a large class of orthogonal polynomial solutions of partial differential equations. Moreover, our method handles as particular cases the orthogonal families of polynomials studied by Krall and Sheffer (1967), Koornwinder (1975), and Kwon et al. (2001), as well as polynomial families with non-standard values of the parameters such as the ones studied in Atkas and Xu (2013), Bracciali et al. (2010), Lee and Littlejohn (2006), Li and Xu (2014), Piñar and Xu (2009). Second, we describe the solutions of Pearson equations with coefficients of special shape in terms of univariate linear functionals that can be used to construct such solutions. These Pearson equations are relevant to our work since the Sobolev orthogonality of the polynomial families studied here involves bivariate linear functionals that are solutions of Pearson equations with coefficients related to the differential equations satisfied by the polynomials. Finally, to the best of our knowledge, the explicit orthogonal polynomials with non-standard parameters associated with Sobolev bilinear forms defined on the ball, simplex, and parabolic biangle presented here as examples are not found anywhere else in the literature.

The organization of this paper is as follows. In Sect. 2, we present the basic facts about univariate and bivariate orthogonal polynomials and classical linear functionals. In Sect. 3, we collect our results about second-order partial differential operators that are symmetric with respect to a Sobolev bilinear form. We show that the involved linear functionals must satisfy related distributional Pearson equations. Hence, we study solutions of certain Pearson equations in Sect. 4. These solutions are obtained from univariate linear functionals using Koornwider's method, which we also describe in this section. Then, in Sect. 5, we study eigenfunctions of second-order partial differential operators that can be obtained from polynomials satisfying second-order ordinary differential equations. Lastly, we present several illustrative examples in Sect. 6. For the sake of clarity, some technical proofs are differed to Appendix A.

2 Preliminaries

In this section, we present the basic facts needed to establish our main results.

2.1 Univariate classical moment functionals

Let us denote by Π the linear space of univariate real polynomials. We call any linear functional $\mathbf{u} : \Pi \to \mathbb{R}$ a moment functional, and we denote by $\langle \mathbf{u}, p\rangle$ the image of $p \in \Pi$ under $\mathbf{u}$.

Let $\{p_n(x)\}_{n\geqslant 0}$ be a sequence of polynomials in Π. If $\deg p_n = n$, $n \geqslant 0$, and

$$\langle \mathbf{u}, p_n\, p_m\rangle = h_n\, \delta_{n,m},$$

where $h_n \neq 0$, $n \geqslant 0$, then we say that $\{p_n(x)\}_{n\geqslant 0}$ is an orthogonal polynomial sequence (OPS) associated with $\mathbf{u}$, and such OPS is unique up to a multiplicative constant.

A moment functional $\mathbf{u}$ is said to be quasi-definite if there is an OPS associated with $\mathbf{u}$. If $\langle \mathbf{u}, p^2 \rangle > 0$ for all $p \in \Pi$ with $p(x) \not\equiv 0$, then $\mathbf{u}$ is called positive definite. If a moment functional is positive-definite, then it is quasi-definite.

Let D denote the distributional differential operator. Given a moment functional $\mathbf{u}$, its derivative $D\mathbf{u}$ is the moment functional defined as

$$\langle D\mathbf{u}, p \rangle = -\langle \mathbf{u}, p' \rangle, \quad \forall p \in \Pi,$$

and the left multiplication of $\mathbf{u}$ by a polynomial $q \in \Pi$ is the moment functional $q\mathbf{u}$ defined as

$$\langle q\mathbf{u}, p \rangle = \langle \mathbf{u}, q\,p \rangle, \quad \forall p \in \Pi.$$

Definition 2.1 A quasi-definite moment functional $\mathbf{u}$ is called classical if there are non zero polynomials $\phi(x)$ and $\psi(x)$ with $\deg \phi \leqslant 2$, $\deg \psi = 1$, such that $\mathbf{u}$ satisfies the distributional Pearson equation

$$D(\phi(x)\,\mathbf{u}) = \psi(x)\,\mathbf{u}. \tag{2.1}$$

If $\{p_n(x)\}_{n\geqslant 0}$ is a sequence of orthogonal polynomials associated with $\mathbf{u}$, then it is called a sequence of classical orthogonal polynomials.

We remark that (2.1) must be understood in the distributional sense (see Garcia-Ardila et al. (2021)). That is, for every $p \in \Pi$, the following must hold

$$\langle D(\phi\,\mathbf{u}), p \rangle = \langle \psi\,\mathbf{u}, p \rangle,$$

or, equivalently,

$$\langle \mathbf{u}, \phi\,p' + \psi\,p \rangle = 0.$$

2.2 Bivariate classical moment functionals

We denote by Π^2 the linear space of bivariate real polynomials. A bivariate moment functional is a linear mapping $\mathbf{u} : \Pi^2 \to \mathbb{R}$, and the image of a polynomial $P \in \Pi^2$ under $\mathbf{u}$ is denoted by $\langle \mathbf{u}, P \rangle$.

Let $\{P_{n,m}(x, y) : 0 \leqslant n, 0 \leqslant m \leqslant n\}$ be a polynomial sequence in Π^2 such that, for $n \geqslant 0$, $\deg P_{n,m} = n$, and $\{P_{n,m}(x, y) : 0 \leqslant m \leqslant n\}$ are $n + 1$ linearly independent polynomials. For $n \geqslant 0$, we define the polynomial column vector with $n + 1$ entries

$$\mathbb{P}_n = (P_{n,0}(x, y),\ P_{n,1}(x, y), \ldots, P_{n,n}(x, y))^\top.$$

The sequence of column vectors $\{\mathbb{P}_n\}_{n\geqslant 0}$ is called a polynomial system (PS).

A PS $\{\mathbb{P}_n\}_{n\geqslant 0}$ is called an orthogonal polynomial system (OPS) associated with a linear functional $\mathbf{u}$ if, for $n \geqslant 0$,

$$\langle \mathbf{u}, \mathbb{P}_n \mathbb{P}_m^\top \rangle = (\langle \mathbf{u}, P_{n,i} P_{m,j} \rangle)_{i,j=0}^{n,m} = \begin{cases} \mathtt{0}, & n \neq m, \\ H_n, & n = m, \end{cases}$$

where H_n is a $n + 1$ symmetric and non-singular real matrix, and $\mathtt{0}$ is the zero matrix of appropriate size. In the particular case when H_n is a diagonal matrix, we say that $\{\mathbb{P}_n\}_{n\geqslant 0}$ is a mutually orthogonal PS.

A moment functional $\mathbf{u}$ is said to be quasi-definite if there is an OPS associated with $\mathbf{u}$, and it is called positive definite if $\langle \mathbf{u}, P^2 \rangle > 0$ for all $P \in \Pi^2$ with $P(x, y) \not\equiv 0$. If a moment functional is positive definite, then it is quasi-definite.

Along this work, we will denote by

$$\partial_x P = \frac{\partial P}{\partial x} = P_x, \quad \partial_y P = \frac{\partial P}{\partial y} = P_y,$$

the respective partial derivatives of a bivariate polynomial $P(x, y) \in \Pi^2$.

We define the distributional partial derivatives of a moment functional $\mathbf{u}$ by duality as

$$\langle \partial_x \mathbf{u}, P \rangle = -\langle \mathbf{u}, \partial_x P \rangle, \quad \langle \partial_y \mathbf{u}, P \rangle = -\langle \mathbf{u}, \partial_y P \rangle, \quad \forall P \in \Pi^2,$$

and the left multiplication of $\mathbf{u}$ times a polynomial $Q \in \Pi^2$ is the new moment functional $Q\mathbf{u}$ satisfying

$$\langle Q\mathbf{u}, P \rangle = \langle \mathbf{u}, Q\,P \rangle, \quad \forall P \in \Pi^2.$$

The following product rules hold for every bivariate moment functional $\mathbf{u}$ and any polynomial $Q \in \Pi$:

$$\partial_x (Q\,\mathbf{u}) = Q_x\,\mathbf{u} + Q\,\partial_x \mathbf{u} \quad \text{and} \quad \partial_y (Q\,\mathbf{u}) = Q_y\,\mathbf{u} + Q\,\partial_y \mathbf{u}. \tag{2.2}$$

Definition 2.2 The quasi-definite moment functional $\mathbf{u}$ defined on Π^2 is said to be classical if there are nonzero polynomials $a \equiv a(x, y)$, $b \equiv b(x, y)$, $c \equiv c(x, y)$, $d \equiv d(x, y)$, and $e \equiv e(x, y)$, such that it satisfies the Pearson equations

$$\begin{aligned} \partial_x (a\,\mathbf{u}) + \partial_y (b\,\mathbf{u}) &= d\,\mathbf{u}, \\ \partial_x (b\,\mathbf{u}) + \partial_y (c\,\mathbf{u}) &= e\,\mathbf{u}, \end{aligned} \tag{2.3}$$

with $\deg a, \deg b, \deg c \leqslant 2, \deg d, \ \deg e \leqslant 1$, and

$$\det \begin{pmatrix} \langle \mathbf{u}, a \rangle & \langle \mathbf{u}, b \rangle \\ \langle \mathbf{u}, b \rangle & \langle \mathbf{u}, c \rangle \end{pmatrix} \neq 0.$$

Note that using the product rule (2.2), (2.3) can be written as

$$\begin{aligned} a\,\partial_x \mathbf{u} + b\,\partial_y \mathbf{u} &= \widetilde{d}\,\mathbf{u}, \\ b\,\partial_x \mathbf{u} + c\,\partial_y \mathbf{u} &= \widetilde{e}\,\mathbf{u}, \end{aligned}$$

where $\widetilde{d} \equiv \widetilde{d}(x, y) = d - a_x - b_y$ and $\widetilde{e} \equiv \widetilde{e}(x, y) = e - b_x - c_y$.

In the above definition, the moment functional $\mathbf{u}$ satisfies the matrix Pearson equations in the distributional sense. This means that for every polynomial $P \in \Pi^2$, we have

$$\langle \mathbf{u}, a\,\partial_x\,P + b\,\partial_y\,P + d\,P \rangle = 0, \quad \text{and} \quad \langle \mathbf{u}, b\,\partial_x\,P + c\,\partial_y\,P + e\,P \rangle = 0.$$

There are several properties that characterize OPS associated with a bivariate classical moment functional. We focus our attention on the characterization stated in the following theorem.

Theorem 2.3 (Fernández et al. 2005) *Let* $\mathbf{u}$ *be a quasi-definite moment functional and let* $\{\mathbb{P}_n\}_{n\geqslant 0}$ *be an OPS associated with* $\mathbf{u}$*. Then, the following statements are equivalent:*

1. $\mathbf{u}$ *is a classical moment functional in the sense of* Definition 2.2.
2. *There is a sequence of square matrices* $\{\Lambda_n\}_{n\geqslant 0}$ *with real numbers as entries, where* Λ_n *is of order* $n + 1$ *and* Λ_1 *is a non-singular matrix, such that*

$$\mathscr{L}[\mathbb{P}_n] \equiv a\,\partial_{xx}\mathbb{P}_n + 2\,b\,\partial_{xy}\mathbb{P}_n + c\,\partial_{yy}\mathbb{P}_n + d\,\partial_x\mathbb{P}_n + e\,\partial_y\mathbb{P}_n = \Lambda_n\,\mathbb{P}_n, \quad n \geqslant 0 \tag{2.4}$$

where $a \equiv a(x, y)$*,* $b \equiv b(x, y)$*,* $c \equiv c(x, y)$*,* $d \equiv d(x, y)$*,* $e \equiv e(x, y)$ *are fixed polynomials with* $\deg a$*,* $\deg b$*,* $\deg c \leqslant 2$*, and* $\deg d, \deg e \leqslant 1$.

The differential operator $\mathscr{L}$ in Theorem 2.3 has the following property which is of great interest in the sequel.

Theorem 2.4 (Lemma 2.6 in Marcellán et al. (2018b)) *Let* $\mathbf{u}$ *be a classical moment functional satisfying* (2.3) *and let* $\{\mathbb{P}_n\}_{n\geqslant 0}$ *be an OPS associated with* $\mathbf{u}$. *Then, the differential operator* $\mathscr{L}$ *in* Theorem 2.3 *satisfies*

$$\langle \mathbf{u}, \mathscr{L}[P]\, Q\rangle \;=\; \langle \mathbf{u}, P\,\mathscr{L}[Q]\rangle, \quad \forall P, Q \in \Pi^2.$$

Moreover, $\mathscr{L}^*[\mathbf{u}] = \mathbf{0}$, *where*

$$\mathscr{L}^*[\mathbf{u}] \equiv \partial_{xx}(a\,\mathbf{u}) + 2\,\partial_{xy}(b\,\mathbf{u}) + \partial_{yy}(c\,\mathbf{u}) - \partial_x(d\,\mathbf{u}) - \partial_y(e\,\mathbf{u}),$$

is the formal Lagrange adjoint of $\mathscr{L}$ *defined as* $\langle \mathbf{u}, \mathscr{L}[P]\rangle = \langle \mathscr{L}^*[\mathbf{u}], P\rangle$, *for every* $P \in \Pi^2$.

2.3 Differential equations belonging to the extended Lyskova class

The *Lyskova class*, studied for the first time in Lyskova (1991), and developed later in Lee et al. (2004), is the class of bivariate classical orthogonal polynomials satisfying (2.4) with $\Lambda_n = \lambda_n I_{n+1}$, $n \geqslant 0$, such that its partial derivatives are again solutions of a similar partial differential equation.

Later, the Lyskova class was extended in Álvarez de Morales et al. (2009a) for classical orthogonal polynomials in the general sense of Definition 2.2. We recall the definition of the extended Lyskova class.

Definition 2.5 (Álvarez de Morales et al. 2009a) The matrix partial differential equation (2.4) belongs to the extended Lyskova class if its polynomial coefficients have the following special shape

$$\begin{aligned}
a(x, y) &= a(x) = a_2x^2 + a_1x + a_0,\\
b(x, y) &= b_2xy + b_{10}x + b_{01}y + b_0,\\
c(x, y) &= c(y) = c_2y^2 + c_1y + c_0,\\
d(x, y) &= d(x) = d_1x + d_0,\\
e(x, y) &= e(y) = e_1y + e_0.
\end{aligned}$$

Suppose that the OPS $\{\mathbb{P}_n\}_{n\geqslant 0}$ satisfies (2.4). For each $n \geqslant 1$, the explicit expression of the entries of Λ_n can be easily obtained in terms of the polynomial coefficients of $\mathscr{L}$ by comparing both sides of (2.4) and, then, Λ_n is a diagonal matrix. The explicit expression of the diagonal entries of

$$\Lambda_n = \operatorname{diag}[\lambda_{n,0}, \lambda_{n,1}, \ldots, \lambda_{n,n}], \quad n \geqslant 1,$$

are

$$\lambda_{n,m} = (n-m)(n-m-1)a_2 + (n-m)mb_2 + m(m-1)c_2 + (n-m)d_1 + me_1.$$

In this case, each polynomial entry of $\mathbb{P}_n$ is solution of the partial differential equation (2.4), that is,

$$\mathscr{L}[P_{n,m}(x, y)] = \lambda_{n,m}\, P_{n,m}(x, y), \quad 0 \leqslant m \leqslant n.$$

For $i, j \geqslant 0$, let us take partial derivatives of (2.4) and define the second-order partial differential operator

$$\mathscr{L}^{(i,j)}[\cdot] = a\,\partial_{xx} + 2\,b\,\partial_{xy} + c\,\partial_{yy} + d^{(i,j)}\,\partial_x + e^{(i,j)}\,\partial_y, \tag{2.5}$$

where

$$d^{(i,j)} \equiv d^{(i,j)}(x) = d + i\,\partial_x a + 2\,j\,\partial_y b,$$
$$e^{(i,j)} \equiv e^{(i,j)}(y) = e + 2\,i\,\partial_x b + j\,\partial_y c.$$

Then, $\partial_x^i \partial_y^j\, \mathbb{P}_{n+i+j}$, $n \geqslant 0$, satisfies

$$\mathscr{L}^{(i,j)}[\partial_x^i \partial_y^j\, \mathbb{P}_{n+i+j}] = \Lambda_{n+i+j}^{(i,j)}\, \partial_x^i \partial_y^j\, \mathbb{P}_{n+i+j}, \tag{2.6}$$

where $\Lambda_{n+i+j}^{(i,j)}$ is a diagonal matrix whose entries are

$$\lambda_{n+i+j,m}^{(i,j)} = \lambda_{n+i+j,m} - \left(\frac{i(i-1)}{2}a_2 + 2ijb_2 + \frac{j(j-1)}{2}c_2 + id_1 + je_1\right),$$

for $0 \leqslant m \leqslant n + i + j$.

The following Theorem states the orthogonality of $\{\partial_x^i\, \partial_y^j\, \mathbb{P}_{n+i+j}\}_{n\geqslant 0}$ with respect to a modification of the moment functional $\mathbf{u}$.

Theorem 2.6 (Theorem 3.9 in Lee et al. (2004)) *Suppose that the OPS $\{\mathbb{P}_n\}_{n\geqslant 0}$ associated with the quasi-definite moment functional $\mathbf{u}$ satisfies (2.4) in the extended Lyskova class. If there exists a polynomial $f(x, y)$ satisfying*

$$a\,f_x + b\,f_y = a_x\,f,$$
$$b\,f_x + c\,f_y = 2\,b_x\,f,$$

and there exists a polynomial $g(x, y)$ satisfying

$$a\,g_x + b\,g_y = 2\,b_y\,g,$$
$$b\,g_x + c\,g_y = c_y\,g,$$

then $\{\partial_x^i\, \partial_y^j\, \mathbb{P}_{n+i+j}\}_{n\geqslant 0}$ satisfying (2.6) are orthogonal with respect to the moment functional

$$\mathbf{u}^{(i,j)} = f^i(x, y)\, g^j(x, y)\, \mathbf{u},$$

in the sense that $\langle \mathbf{u}^{(i,j)}, \partial_x^i\, \partial_y^j \mathbb{P}_{n+i+j}\, (\partial_x^i\, \partial_y^j \mathbb{P}_{m+i+j})^\top \rangle = 0$ when $n \neq m$.

Observe that we do not know *a priori* if $\mathbf{u}^{(i,j)}$ is quasi-definite. In this way, a necessary condition for the quasi-definite character of $\mathbf{u}^{(i,j)}$ is that

$$\langle \mathbf{u}^{(i,j)}, 1\rangle = \langle f^i(x, y)\, g^j(x, y)\, \mathbf{u}, 1\rangle = \langle \mathbf{u}, f^i(x, y)\, g^j(x, y)\rangle \neq 0.$$

Definition 2.7 Let $(\cdot, \cdot) : \Pi^2 \times \Pi^2 \to \mathbb{R}$ be a bilinear form. We say that $(\cdot, \cdot)$ is symmetric if $(P, Q) = (Q, P)$ for all $P, Q \in \Pi^2$. A bilinear form is called a Sobolev bilinear form if it involves partial derivatives of polynomials.

Moreover, we say that a PS $\{\mathbb{P}_n\}_{n\geqslant 0}$ is Sobolev orthogonal with respect to a Sobolev bilinear form $(\cdot, \cdot)$ if

$$\left(\mathbb{P}_n, \mathbb{P}_m^\top\right) = \left((P_{n,i}, P_{m,j})\right)_{i,j=0}^{n,m} = 0, \quad n \neq m.$$

From Theorems 2.3 and 2.6, we have that a PS satisfying a differential equation in the extended Lyskova class are Sobolev orthogonal polynomials. The following result will be of interest in the sequel.

Corollary 2.8 *Fix an integer $N \geqslant 0$. Suppose that the OPS $\{\mathbb{P}_n\}_{n\geqslant 0}$ associated with the quasi-definite moment functional* $\mathbf{u}$ *satisfies* (2.4) *in the extended Lyskova class. Then, under the hypotheses of* Theorem 2.6*, $\{\mathbb{P}_n\}_{n\geqslant 0}$ is Sobolev orthogonal with respect to the bilinear form defined as*

$$(P, Q) = \sum_{i=0}^{N}\sum_{j=0}^{i}\langle f^{i-j}\, g^{j}\mathbf{u},\, \partial_x^{i-j}\partial_y^{j} P\ \partial_x^{i-j}\partial_y^{j} Q\rangle.$$

3 Sobolev bilinear forms and partial differential equations

From Theorem 2.3, we have that a PS associated with a classical moment functional $\mathbf{u}$ satisfy a matrix second-order linear partial differential equation (2.4), and from Theorem 2.4, we have that the differential operator $\mathscr{L}$ in (2.4) satisfies

$$\langle \mathbf{u}, \mathscr{L}[P]\, Q\rangle = \langle \mathbf{u}, P\, \mathscr{L}[Q]\rangle,$$

for all polynomials $P,\ Q \in \Pi^2$. This motivates our study of second-order partial differential operators satisfying a similar symmetry condition with respect to Sobolev bilinear forms.

Definition 3.1 Fix an integer $N \geqslant 0$. Let $\mathbf{u}^{(i,j)}$, $0 \leqslant j \leqslant i \leqslant N$, be bivariate moment functionals. We can define the Sobolev bilinear form

$$(P, Q)_S = \sum_{i=0}^{N}\sum_{j=0}^{i}\langle \mathbf{u}^{(i,j)}, \partial_x^{i-j}\partial_y^{j} P\ \partial_x^{i-j}\partial_y^{j} Q\rangle, \quad \forall P, Q \in \Pi^2. \tag{3.1}$$

Moreover, let $a \equiv a(x, y)$, $b \equiv b(x, y)$, $c \equiv c(x, y)$, $d \equiv d(x, y)$, $e \equiv e(x, y)$ be fixed polynomials with $\deg a, \deg b, \deg c \leqslant 2$, and $\deg d, \deg e \leqslant 1$. We define the second-order linear partial differential operator

$$\mathscr{L}[\,\cdot\,] \equiv a\,\partial_{xx} + 2\,b\,\partial_{xy} + c\,\partial_{yy} + d\,\partial_x + e\,\partial_y. \tag{3.2}$$

3.1 Symmetric differential operators and Pearson equations

We explore the connection between the polynomial coefficients in $\mathscr{L}$ and the moment functionals in (3.1), such that $\mathscr{L}$ is symmetric in the sense that

$$(\mathscr{L}[P], Q)_S = (P, \mathscr{L}[Q])_S, \quad \forall P, Q \in \Pi^2.$$

To this end, we need the following preliminary result. Its proof is given in Appendix .

Lemma 3.2 *Let* $\mathbf{u}$ *be a bivariate moment functional, and let $\mathscr{L}$ be the operator defined in* (3.2)*. If $\mathscr{L}$ is in the extended Lyskova class, then for all $P \in \Pi^2$ and $i, j \geqslant 0$,*

$$\begin{aligned}
&\partial_x^i\partial_y^j\Big(\partial_x^i\partial_y^j\mathscr{L}[P]\,\mathbf{u}\Big) - \mathscr{L}^*\Big[\partial_x^i\partial_y^j(\partial_x^i\partial_y^j P\,\mathbf{u})\Big]\\
&\quad = -\partial_x^i\partial_y^j\Big[\partial_x^i\partial_y^j P\,\Big(\mathscr{L}^{(i,j)}\Big)^{*}[\mathbf{u}] + 2\partial_x^{i+1}\partial_y^j P\mathcal{M}_1^{(i,j)}[\mathbf{u}] + 2\partial_x^i\partial_y^{j+1} P\mathcal{M}_2^{(i,j)}[\mathbf{u}]\Big],
\end{aligned}$$

where $\mathscr{L}^{(i,j)}$ *is defined in* (2.5),

$$\mathcal{M}_1^{(i,j)}[\mathbf{u}] = (a\,\mathbf{u})_x + (b\,\mathbf{u})_y - d^{(i,j)}\,\mathbf{u},$$
$$\mathcal{M}_2^{(i,j)}[\mathbf{u}] = (b\,\mathbf{u})_x + (c\,\mathbf{u})_y - e^{(i,j)}\,\mathbf{u},$$

and

$$\left(\mathscr{L}^{(i,j)}\right)^*[\mathbf{u}] = \partial_x \mathcal{M}_1^{(i,j)}[\mathbf{u}] + \partial_y \mathcal{M}_2^{(i,j)}[\mathbf{u}],$$

is the formal Lagrange adjoint of $\mathscr{L}^{(i,j)}$.

Now, we are ready to state the main result of this section.

Theorem 3.3 *Let* $\mathscr{L}$ *be the differential operator defined in* (3.2). *If* $\mathscr{L}$ *is in the extended Lyskova class and the moment functionals* $\mathbf{u}^{(i,j)}$, $0 \leqslant j \leqslant i \leqslant N$, *satisfy*

$$\begin{cases} \partial_x(a\,\mathbf{u}^{(i,j)}) + \partial_y(b\,\mathbf{u}^{(i,j)}) = (d + (i-j)\,a_x + 2j\,b_y)\,\mathbf{u}^{(i,j)}, \\ \partial_x(b\,\mathbf{u}^{(i,j)}) + \partial_y(c\,\mathbf{u}^{(i,j)}) = (e + 2(i-j)\,b_x + j\,c_y)\,\mathbf{u}^{(i,j)}, \end{cases} \tag{3.3}$$

then $\mathscr{L}$ *is symmetric with respect to the Sobolev bilinear form* (3.1).

Proof The differential operator $\mathscr{L}$ is symmetric with respect to $(\cdot,\cdot)_S$ defined in (3.1) if and only if, for all $P, Q \in \Pi^2$, $(\mathscr{L}[P], Q)_S - (P, \mathscr{L}[Q])_S = 0$. Note that

$$\begin{aligned}&(\mathscr{L}[P], Q)_S - (P, \mathscr{L}[Q])_S \\ &\quad = \sum_{i=0}^{N}\sum_{j=0}^{i}(-1)^i \left\langle \partial_x^{i-j}\partial_y^j\left(\partial_x^{i-j}\partial_y^j \mathscr{L}[P]\,\mathbf{u}^{(i,j)}\right) - \mathscr{L}^*\left[\partial_x^{i-j}\partial_y^j(\partial_x^{i-j}\partial_y^j P\,\mathbf{u}^{(i,j)})\right], Q\right\rangle.\end{aligned}$$

Therefore, $\mathscr{L}$ is symmetric with respect to $(\cdot,\cdot)_S$ if and only if, for all $P \in \Pi^2$,

$$\sum_{i=0}^{N}\sum_{j=0}^{i}(-1)^i \left(\partial_x^{i-j}\partial_y^j\left(\partial_x^{i-j}\partial_y^j \mathscr{L}[P]\,\mathbf{u}^{(i,j)}\right) - \mathscr{L}^*\left[\partial_x^{i-j}\partial_y^j(\partial_x^{i-j}\partial_y^j P\,\mathbf{u}^{(i,j)})\right]\right) = \mathbf{0},$$

or, equivalently, using (3.2),

$$\begin{aligned}&\sum_{i=0}^{N}\sum_{j=0}^{i}(-1)^i \partial_x^{i-j}\partial_y^j\left[\partial_x^{i-j}\partial_y^j P\left(\mathscr{L}^{(i-j,j)}\right)^*[\mathbf{u}^{(i,j)}]\right. \\ &\quad \left. + 2\partial_x^{i-j+1}\partial_y^j P \mathcal{M}_1^{(i-j,j)}[\mathbf{u}^{(i,j)}] + 2\partial_x^{i-j}\partial_y^{j+1} P \mathcal{M}_2^{(i-j,j)}[\mathbf{u}^{(i,j)}]\right] = \mathbf{0}.\end{aligned} \tag{3.4}$$

If for $0 \leqslant j \leqslant i \leqslant N$, $\mathbf{u}^{(i,j)}$ satisfies (3.3), then (3.4) holds for all $P \in \Pi^2$ and therefore $\mathscr{L}$ is symmetric with respect to $(\cdot,\cdot)_S$. □

Remark 3.4 For $N = 0$, the Pearson equations (3.3) read as (2.3). In this case, (by Remark A.1) Theorem 3.3 holds even when $\mathscr{L}$ is not in the extended Lyskova class.

3.2 Sobolev orthogonality of the polynomial solutions

Let $\{\mathbb{P}_n\}_{n\geqslant 0}$ be a PS such that there is a sequence of square real matrices $\{\Lambda_n\}_{n\geqslant 0}$, where Λ_n is of order $n+1$, satisfying $\mathscr{L}[\mathbb{P}_n] = \Lambda_n\, \mathbb{P}_n$ for all $n \geqslant 0$.

The symmetry of $\mathscr{L}$ with respect to (3.1) and additional hypotheses imply the Sobolev orthogonality of $\{\mathbb{P}_n\}_{n\geqslant 0}$.

Proposition 3.5 *Assume that the partial differential operator $\mathscr{L}$ is symmetric with respect to the Sobolev bilinear form* (3.1). *Let $\{\mathbb{P}_n\}_{n\geqslant 0}$ be a PS satisfying $\mathscr{L}[\mathbb{P}_n] = \Lambda_n\, \mathbb{P}_n$ for all $n \geqslant 0$. If Λ_n and Λ_m do not share eigenvalues for $n \neq m$, then $\{\mathbb{P}_n\}_{n\geqslant 0}$ satisfies the orthogonality condition $(\mathbb{P}_n, \mathbb{P}_m^{\top})_S = 0$, $n \neq m$.*

Proof If $\mathscr{L}$ is symmetric with respect to (3.1) and $\{\mathbb{P}_n\}_{n\geqslant 0}$ satisfies $\mathscr{L}[\mathbb{P}_n] = \Lambda_n\, \mathbb{P}_n$ for all $n \geqslant 0$, then

$$\Lambda_n(\mathbb{P}_n, \mathbb{P}_m^{\top})_S = (\mathscr{L}[\mathbb{P}_n], \mathbb{P}_m^{\top})_S = (\mathbb{P}_n, \mathscr{L}[\mathbb{P}_m^{\top}])_S = (\mathbb{P}_n, \mathbb{P}_m^{\top})_S \Lambda_m^{\top}, \quad n, m \geqslant 0,$$

or, equivalently,

$$\Lambda_n(\mathbb{P}_n, \mathbb{P}_m^{\top})_S - (\mathbb{P}_n, \mathbb{P}_m^{\top})_S \Lambda_m^{\top} = 0, \quad n, m \geqslant 0. \tag{3.5}$$

Notice that (3.5) is a Sylvester matrix equation $AX - XB = C$, where $A = \Lambda_n$, $B = \Lambda_m$, $C = 0$ and the unknown is $X = (\mathbb{P}_n, \mathbb{P}_m^{\top})_S$. It is a well-known fact that a Sylvester equation has a unique solution X for all C if and only if A and B have no common eigenvalues [Horn and Johnson (1991),Th. 4.4.6]. In our case, since for $n \neq m$, Λ_n and Λ_m do not share eigenvalues, the unique solution of (3.5) is $(\mathbb{P}_n, \mathbb{P}_m^{\top})_S = 0$. □

Remark 3.6 Let $\mathscr{L}$ be the differential operator defined in (3.2). If the moment functional $\mathbf{u}$ satisfies the Pearson equations (2.3), then by Theorem 3.3 with $N = 0$, $\mathscr{L}$ satisfies the symmetry condition

$$\langle \mathbf{u}, \mathscr{L}[P]\, Q\rangle = \langle \mathbf{u}, P\, \mathscr{L}[Q]\rangle, \quad \forall P, Q \in \Pi^2.$$

Moreover, if the hypotheses of Proposition 3.5 hold, then $\{\mathbb{P}_n\}_{n\geqslant 0}$ satisfies the orthogonality condition $\langle \mathbf{u}, \mathbb{P}_n\, \mathbb{P}_m^{\top}\rangle = 0$, $n \neq m$. Observe that $\mathbf{u}$ is not a priori a quasi-definite moment functional.

If $\{\mathbb{P}_n\}_{n\geqslant 0}$ is a Sobolev OPS associated with (3.1), then the symmetry of $\mathscr{L}$ with respect to (3.1) is equivalent to $\{\mathbb{P}_n\}_{n\geqslant 0}$ satisfying $\mathscr{L}[\mathbb{P}_n] = \Lambda_n\, \mathbb{P}_n$ for all $n \geqslant 0$.

Proposition 3.7 *Let $\{\mathbb{P}_n\}_{n\geqslant 0}$ be a Sobolev OPS associated with* (3.1). *The following statements are equivalent:*

(i) *The operator $\mathscr{L}$ is symmetric with respect to the Sobolev bilinear form* (3.1).

(ii) *For $n \geqslant 0$, there is a square matrix Λ_n of order $n+1$, whose entries are real numbers, such that $\{\mathbb{P}_n\}_{n\geqslant 0}$ satisfies $\mathscr{L}[\mathbb{P}_n] = \Lambda_n\, \mathbb{P}_n$, and $\Lambda_n \widetilde{H}_{n,n} = \widetilde{H}_{n,n} \Lambda_n^{\top}$, where $\widetilde{H}_{n,m} = (\mathbb{P}_n, \mathbb{P}_m^{\top})_S$.*

Proof $(i) \Rightarrow (ii)$. Since $\mathscr{L}[\mathbb{P}_n]$ is a vector with polynomial entries of degree at most n, then we can write

$$\mathscr{L}[\mathbb{P}_n] = \sum_{j=0}^{n} M_{n,j}\mathbb{P}_j,$$

where $M_{n,j}$ is a real matrix of size $(n+1)\times(j+1)$. Using (3.1), we obtain

$$(\mathscr{L}[\mathbb{P}_n], \mathbb{P}_m^\top)_S = \sum_{j=0}^{n} M_{n,j}(\mathbb{P}_j, \mathbb{P}_m^\top)_S = M_{n,m}\widetilde{H}_{m,m}, \quad m \leqslant n.$$

On the other hand, using the symmetry of $\mathscr{L}$, we have

$$(\mathscr{L}[\mathbb{P}_n], \mathbb{P}_m^\top)_S = (\mathbb{P}_n, \mathscr{L}[\mathbb{P}_m^\top])_S = 0, \quad m \leqslant n-1.$$

This implies that $M_{n,m} = 0$ for $m \leqslant n-1$. Then (ii) holds with $\Lambda_n = M_{n,n}$. $(ii) \Rightarrow (i)$. Obviously,

$$(\mathscr{L}[\mathbb{P}_n], \mathbb{P}_m^\top)_S = \Lambda_n(\mathbb{P}_n, \mathbb{P}_m^\top)_S = \begin{cases} 0, & m \neq n, \\ \Lambda_n \widetilde{H}_{n,n}, & n = m, \end{cases}$$

$$(\mathbb{P}_n, \mathscr{L}[\mathbb{P}_m^\top])_S = (\mathbb{P}_n, \mathbb{P}_m^\top)_S \Lambda_m^\top = \begin{cases} 0, & m \neq n, \\ \widetilde{H}_{n,n}\Lambda_n^\top, & n = m, \end{cases}$$

and (i) clearly holds. □

4 Generating solutions of Pearson equations

In Sect. 3, we described the connection between the coefficients of the differential operator $\mathscr{L}$ defined in (3.2) and the Pearson equations satisfied by the moment functionals involved in the Sobolev bilinear form (3.1).

In this section, we turn our attention to generating solutions of certain Pearson equations. We consider two cases:

- Non-diagonal case:

$$\partial_x (a\,\mathbf{w}) + \partial_y (b\,\mathbf{w}) = d\,\mathbf{w},$$
$$\partial_x (b\,\mathbf{w}) + \partial_y (c\,\mathbf{w}) = e\,\mathbf{w},$$

  or, equivalently,

$$a\,\partial_x\,\mathbf{w} + b\,\partial_y\,\mathbf{w} = \widetilde{d}\,\mathbf{w}, \tag{4.1}$$
$$b\,\partial_x\,\mathbf{w} + c\,\partial_y\,\mathbf{w} = \widetilde{e}\,\mathbf{w}, \tag{4.2}$$

  where $a \equiv a(x,y)$, $b \equiv b(x,y)$, $c \equiv c(x,y)$, $d \equiv d(x,y)$, and $e \equiv e(x,y)$ are polynomials with $\deg a$, $\deg b$, $\deg c \leqslant 2$, $\deg d$, $\deg e \leqslant 1$, and $\widetilde{d} \equiv \widetilde{d}(x,y) = d - a_x - b_y$, $\widetilde{e} \equiv \widetilde{e}(x,y) = e - b_x - c_y$. Observe that in this case, each equation involves partial derivatives with respect to each variable.
- Diagonal case:

$$\partial_x(a\,\mathbf{w}) = d\,\mathbf{w},$$
$$\partial_y(c\,\mathbf{w}) = e\,\mathbf{w},$$

  or, equivalently,

$$a\,\partial_x\mathbf{w} = \widetilde{d}\,\mathbf{w}, \tag{4.3}$$
$$c\,\partial_y\mathbf{w} = \widetilde{e}\,\mathbf{w}, \tag{4.4}$$

  where $a \equiv a(x,y)$ and $c \equiv c(x,y)$ are polynomials of degree at most 2, $d \equiv d(x,y)$ and $e \equiv e(x,y)$ are polynomials of degree at most 1, and $\widetilde{d} \equiv \widetilde{d}(x,y) = d - a_x$,

$\widetilde{e} \equiv \widetilde{e}(x, y) = e - c_y$. In this case, each equation involves a partial derivative with respect to only one variable.

We focus on Pearson equations with solutions belonging to a class of bivariate functionals that can be constructed using univariate moment functionals satisfying Pearson equations in one variable.

We start by describing a method to construct bivariate moment functionals using univariate moment functionals (see Agahanov 1965; Dunkl and Xu 2014; Koornwinder 1975; Marriaga et al. 2017).

4.1 A class of bivariate moment functionals

Let $\mathbf{u}^{(s)}$ and $\mathbf{v}^{(t)}$ be univariate linear functionals acting on the variables s and t, respectively. Let $\rho(s)$ be a univariate function satisfying one of the following two conditions:

Case I: $\rho(s)$ is a polynomial of degree at most 1, that is, $\rho(s) = r_1\, s + r_0$, with $|r_1| + |r_0| > 0$,

Case II: $\rho(s)$ is the square root of a polynomial of degree at most 2, that is, $\rho(s) = \sqrt{\ell_2\, s^2 + 2\,\ell_1\, s + \ell_0}$, where $|\ell_2| + |\ell_1| + |\ell_0| > 0$, and $\mathbf{v}^{(t)}$ is symmetric (that is, $\langle \mathbf{v}^{(t)}, t^{2k+1}\rangle = 0$ for $k \geqslant 0$).

and such that $\mathbf{u}_0^{(s)} = \rho(s)\, \mathbf{u}^{(s)}$ is a well-defined moment functional in both Case I and Case II.

We define the bivariate moment functional $\mathbf{w}$ by

$$\langle \mathbf{w}, P(x, y)\rangle = \left\langle \mathbf{u}_0^{(s)}, \left\langle \mathbf{v}^{(t)},\; P(s, t\,\rho(s))\right\rangle\right\rangle, \quad \forall P \in \Pi^2, \tag{4.5}$$

where $x = s$ and $y = t\,\rho(s)$.

Suppose that $\mathbf{u}^{(s)}$ and $\mathbf{v}^{(t)}$ are quasi-definite such that, in both Case I and Case II, $\mathbf{u}_m^{(s)} = \rho(s)^{2m+1}\, \mathbf{u}^{(s)}$, $0 \leqslant m$, are quasi-definite moment functionals. For $m \geqslant 0$, let $\{p_n^{(m)}(s)\}_{n\geqslant 0}$ be an orthogonal polynomial sequence associated with the quasi-definite moment functional $\mathbf{u}_m^{(s)}$, and let $\{q_n(t)\}_{n\geqslant 0}$ be an orthogonal polynomial sequence associated with $\mathbf{v}^{(t)}$. Let define the bivariate polynomials

$$P_{n,m}(x, y) = p_{n-m}^{(m)}(x)\, \rho(x)^m q_m \left(\frac{y}{\rho(x)}\right), \quad n \geqslant 0, \quad 0 \leqslant m \leqslant n. \tag{4.6}$$

It was shown in Marriaga et al. (2017) that $\{P_{n,m}(x, y) : n \geqslant 0,\ 0 \leqslant m \leqslant n\}$ forms a mutually orthogonal basis with respect to the moment functional $\mathbf{w}$ defined in (4.5).

4.2 The non-diagonal case

We deal with (4.1) and (4.2) separately.

Throughout this section, $\mathbf{w}$ is the bivariate linear functional defined in (4.5) and $\rho'(s) \not\equiv 0$. We defer the case when $\rho(s)$ is some constant to later.

The following result deals with (4.1).

Theorem 4.1 *Let $a(s)$, $\widetilde{b}(s)$, $\widetilde{d}(s)$ be univariate polynomials with $\deg a \leqslant 2$ and $\deg \widetilde{b}$, $\deg \widetilde{d} \leqslant 1$.*

If $a(s)\,\rho'(s) = \widetilde{b}(s)\,\rho(s)$ and the univariate linear functional $\mathbf{u}_0^{(s)} = \rho(s)\,\mathbf{u}^{(s)}$ satisfies the Pearson equation

$$a(s)\, D\, \mathbf{u}_0^{(s)} = \left(\widetilde{d}(s) + \widetilde{b}(s)\right)\, \mathbf{u}_0^{(s)},$$

then $\mathbf{w}$ *satisfies* (4.1) *with* $a \equiv a(x)$, $b \equiv y\,\widetilde{b}(x)$, *and* $\widetilde{d} \equiv \widetilde{d}(x)$.

Proof Let $x = s$ and $y = t\,\rho(s)$. For any polynomial $P(x,y) \in \Pi^2$,

$$\partial_s P(s, t\,\rho(s)) = \partial_x P(x,y) + y\,\frac{\rho'(x)}{\rho(x)}\,\partial_y P(x,y),$$
$$\partial_t P(s, t\,\rho(s)) = \rho(x)\,\partial_y P(x,y).$$

Therefore,

$$\begin{aligned}\partial_x P(x,y) &= \partial_s P(s, t\,\rho(s)) - t\,\frac{\rho'(s)}{\rho(s)}\,\partial_t P(s, t\,\rho(s)),\\ \partial_y P(x,y) &= \frac{1}{\rho(s)}\,\partial_t P(s, t\,\rho(s)).\end{aligned} \tag{4.7}$$

On the other hand, for every polynomial $P \in \Pi^2$, we have

$$\left\langle -a(x)\,\partial_x \mathbf{w} - y\,\widetilde{b}(x)\,\partial_y \mathbf{w} + \widetilde{d}(x)\,\mathbf{w}, P\right\rangle = \left\langle \mathbf{w}, \partial_x (a(x)\,P) + \partial_y \left(y\,\widetilde{b}(x)\,P\right) + \widetilde{d}(x)\,P\right\rangle.$$

Thus, using (4.5) and (4.7), we obtain

$$\begin{aligned}&\left\langle -a(x)\,\partial_x \mathbf{w} - y\,\widetilde{b}(x)\,\partial_y \mathbf{w} + \widetilde{d}(x)\,\mathbf{w}, P\right\rangle\\ &= \left\langle \mathbf{u}_0^{(s)}, \left\langle \mathbf{v}^{(t)}, \partial_s (a(s)\,P) - t\,\frac{\rho'(s)}{\rho(s)}\partial_t (a(s)\,P) + \frac{1}{\rho(s)}\partial_t \left(t\,\rho(s)\,\widetilde{b}(s)\,P\right) + \widetilde{d}(s)\,P\right\rangle\right\rangle\\ &= \left\langle \mathbf{u}_0^{(s)}, \left\langle \mathbf{v}^{(t)}, \partial_s (a(s)\,P) + \left(\widetilde{b}(s) + \widetilde{d}(s)\right)P + t\left(\widetilde{b}(s) - a(s)\frac{\rho'(s)}{\rho(s)}\right)\partial_t P\right\rangle\right\rangle\\ &= \left\langle \mathbf{u}_0^{(s)}, \left\langle \mathbf{v}^{(t)}, \partial_s (a(s)\,P) + \left(\widetilde{b}(s) + \widetilde{d}(s)\right)P\right\rangle\right\rangle\\ &= \left\langle -a(s)\,D\,\mathbf{u}_0^{(s)} + \left(\widetilde{b}(s) + \widetilde{d}(s)\right)\mathbf{u}_0^{(s)}, \left\langle \mathbf{v}^{(t)}, P\right\rangle\right\rangle = 0,\end{aligned}$$

where we have used the fact that $a(s)\,\rho'(s) = \widetilde{b}(s)\,\rho(s)$. Hence, $a(x)\,\partial_x \mathbf{w} + y\,\widetilde{b}(x)\,\partial_y \mathbf{w} = \widetilde{d}(x)\,\mathbf{w}$. □

Observe that in the previous theorem, no conditions are imposed on $\mathbf{v}^{(t)}$ other than the ones mentioned in Sect. 4.1.

To deal with (4.2), we must consider the explicit expression of its polynomial coefficients and the function $\rho(s)$. In both Case I and Case II in Sect. 4.1, we set

- $c(t) = c_0 + c_1 t + c_2 t^2$,
- $\widetilde{e}(t) = e_0 + e_1 t$,
- $\widetilde{b}(s) = b_0 + b_1 s$, $|b_0| + |b_1| > 0$,
- $\widetilde{d}(s) = d_0 + d_1 s$.

We also assume that $a(s)\,\rho'(s) = \widetilde{b}(s)\,\rho(s)$ where $a(s)$ is a univariate polynomial with $\deg a \leqslant 2$. First, we state the following intermediate result, which is a consequence of the equations $a(s)\,\rho'(s) = \widetilde{b}(s)\,\rho(s)$, (4.5), and (4.7).

Proposition 4.2 *For every polynomial* $P \in \Pi^2$,

$$\begin{aligned}&\left\langle -y\,\widetilde{b}(x)\,\partial_x \mathbf{w} - c(y)\,\partial_y \mathbf{w} + \widetilde{e}(y)\,\mathbf{w}, P\right\rangle\\ &\quad= \left\langle \mathbf{u}_0^{(s)}, \left\langle t\,\mathbf{v}^{(t)}, \partial_s \left(a(s)\,\rho'(s)\,P\right) + \rho'(s)\,\widetilde{b}(s)\,P\right\rangle\right\rangle\\ &\qquad+ \left\langle \mathbf{u}_0^{(s)}, \left\langle \mathbf{v}^{(t)}, \left(c_2\rho(s) - \rho'(s)\,\widetilde{b}(s)\right)\partial_t \left(t^2 P\right) + c_1\partial_t (t\,P) + e_0 P + e_1\,\rho(s)\,t\,P\right\rangle\right\rangle\\ &\qquad+ \left\langle \mathbf{u}_0^{(s)}, \left\langle \mathbf{v}^{(t)}, c_0\,\frac{1}{\rho(s)}\partial_t P\right\rangle\right\rangle.\end{aligned}$$

For Case I, we get the following result. Its proof is given in Appendix A.

Corollary 4.3 *Let $\rho(s) = r_0 + r_1\, s$ with $r_1 \neq 0$. If the following conditions hold,*

(a) $c_0 = 0$, $c_2 = b_1$, *and* $d_1 = e_1$,
(b) $\mathbf{u}_0^{(s)}$ *satisfies the Pearson equation*

$$a(s)\, D\, \mathbf{u}_0^{(s)} = \big(\widetilde{b}(s) + \widetilde{d}(s)\big)\, \mathbf{u}_0^{(s)},$$

(c) *and* $\mathbf{v}^{(t)}$ *satisfies the Pearson equation*

$$\big[(b_1\, r_0 - r_1\, b_0)\, t^2 + c_1\, t\big]\; D\, \mathbf{v}^{(t)} = [e_0 + (e_1\, r_0 - d_0\, r_1)\, t]\; \mathbf{v}^{(t)},$$

then $\mathbf{w}$ *satisfies* (4.2) *with* $b \equiv y\, \widetilde{b}(x)$, $c \equiv c(y)$, *and* $\widetilde{e} \equiv \widetilde{e}(y)$.

Case II is more intricate. We have the following result and its proof is given in Appendix A.

Corollary 4.4 *Let* $\rho(s) = \sqrt{\ell_0 + 2\ell_1 s + \ell_2 s^2}, |\ell_1| + |\ell_2| > 0$. *If the following conditions hold:*

(a) $c_2 - b_1 = c_1 = d_1 - e_1 = e_0 = 0$,
(b) $(2\, c_2 - b_1)\ell_1 - b_0\ell_2 = d_1\ell_1 + d_0\ell_2 - 2e_1\ell_1 = 0$,
(c) $\mathbf{u}_0^{(s)}$ *satisfies the Pearson equation*

$$a(s)\, D\, \mathbf{u}_0^{(s)} = \big(\widetilde{b}(s) + \widetilde{d}(s)\big)\, \mathbf{u}_0^{(s)},$$

(d) *and* $\mathbf{v}^{(t)}$ *satisfies the following Pearson equation*

$$\big[(c_2\, \ell_0 - b_0\, \ell_1)\, t^2 + c_0\big]\; D\, \mathbf{v}^{(t)} = (e_1\, \ell_0 - d_0\, \ell_1)\, t\, \mathbf{v}^{(t)},$$

then $\mathbf{w}$ *satisfies* (4.2) *with* $b \equiv y\, \widetilde{b}(x)$, $c \equiv c(y)$, *and* $\widetilde{e} \equiv \widetilde{e}(y)$.

4.3 The diagonal case

Here, we study the moment functionals satisfying the diagonal Pearson equations (4.3) and (4.4).

We identify three cases which we organize in the following theorems. The first one deals with tensor products of univariate moment functionals.

Theorem 4.5 *Let* $\rho(s) = 1$, $a(x, y) = a(x)$, $\widetilde{d}(x, y) = \widetilde{d}(x)$, $c(x, y) = c(y)$, *and* $\widetilde{e}(x, y) = \widetilde{e}(y)$. *If* $\mathbf{u}_0^{(s)}$ *satisfies* $a(s)\, D\mathbf{u}_0^{(s)} = \widetilde{d}(s)\mathbf{u}_0^{(s)}$, *then the linear functional* $\mathbf{w}$ *defined in* (4.5) *satisfies the diagonal Pearson equation* $a(x)\, \partial_x \mathbf{w} = \widetilde{d}(x)\, \mathbf{w}$.

Similarly, if $\mathbf{v}^{(t)}$ *satisfies* $c(t)\, D\mathbf{v}^{(t)} = \widetilde{e}(t)\, \mathbf{v}^{(t)}$, *then* $\mathbf{w}$ *satisfies the diagonal Pearson equation* $c(y)\, \partial_y \mathbf{w} = \widetilde{e}(y)\, \mathbf{w}$.

Proof Using $a(x)\, \partial_x \mathbf{w} = \widetilde{d}(x)\, \mathbf{w}$, (4.7), and $\rho'(s) = 0$, for every $P \in \Pi^2$, we have,

$$\big\langle -a(x)\, \partial_x \mathbf{w} + \widetilde{d}(x)\, \mathbf{w},\, P\big\rangle = \Big\langle -a(s)\, D\mathbf{u}_0^{(s)} + \widetilde{d}(s)\, \mathbf{u}_0^{(s)}, \Big\langle \mathbf{v}^{(t)}, P\Big\rangle\Big\rangle = 0.$$

Similarly, using $c(t)\, D\mathbf{v}^{(t)} = \widetilde{e}(t)\, \mathbf{v}^{(t)}$ and (4.7), we get $c(y)\, \partial_y \mathbf{w} = \widetilde{e}(y)\, \mathbf{w}$. □

Now, we consider the case when $\rho(s) = r_0 + r_1\, s$. We need the explicit expression of the following polynomials:

- $a(s) = a_0 + a_1\, s$,
- $b(t) = b_0 + b_1\, t$,
- $c(t) = c_0 + c_1\, t$.

The proof of the following result is given in Appendix A.

Theorem 4.6 *Let $\rho(s) = r_0 + r_1\, s$ with $r_1 \neq 0$, and suppose that either one of the following two conditions hold:*

(i) $c_0 = 0$, $c_1 \neq 0$, *and* $\mathbf{v}^{(t)}$ *satisfies* $t\, b(t)\, D\mathbf{v}^{(t)} = \left[\gamma\, b(t) + \beta\, t\, b'(t)\right] \mathbf{v}^{(t)}$, *or*
(ii) $c_1 = 0$, $c_0 \neq 0$, *and* $\mathbf{v}^{(t)}$ *satisfies* $b(t)\, D\mathbf{v}^{(t)} = \beta\, b'(t)\, \mathbf{v}^{(t)}$,

where γ and β are real numbers. Then, $\mathbf{w}$ *satisfies the diagonal Pearson equation*

$$\begin{aligned} & c(y)\, \rho(x)\, b\left(\frac{y}{\rho(x)}\right) \partial_y \mathbf{w} \\ & = \left[\gamma\, c'(y)\, \rho(x)\, b\left(\frac{y}{\rho(x)}\right) + \beta\, c(y)\, \partial_y \left(\rho(x)\, b\left(\frac{y}{\rho(x)}\right)\right)\right] \mathbf{w}. \end{aligned} \tag{4.8}$$

If in addition to condition (i) (respectively, (ii)), $\mathbf{u}_0^{(s)}$ *satisfies the Pearson equation*

$$a(s)\, \rho(s)\, D\mathbf{u}_0^{(s)} = \left[\alpha\, a'(s)\, \rho(s) + (1 + \gamma + \beta)\, \rho'(s)\, a(s)\right] \mathbf{u}_0^{(s)},$$

respectively,

$$a(s)\, \rho(s)\, D\mathbf{u}_0^{(s)} = \left[\alpha\, a'(s)\, \rho(s) + (1 + \beta)\, \rho'(s)\, a(s)\right] \mathbf{u}_0^{(s)},$$

where α is a real number, then $\mathbf{w}$ *also satisfies the diagonal Pearson equation*

$$\begin{aligned} & a(x)\, \rho(x)\, b\left(\frac{y}{\rho(x)}\right) \partial_x \mathbf{w} \\ & = \left[\alpha\, a'(x)\, \rho(x)\, b\left(\frac{y}{\rho(x)}\right) + \beta\, a(x)\, \partial_x \left(\rho(x)\, b\left(\frac{y}{\rho(x)}\right)\right)\right] \mathbf{w}. \end{aligned} \tag{4.9}$$

Now, we consider the case when $\rho(s) = \sqrt{\ell_0 + 2\, \ell_1\, s + \ell_2\, s^2}$. The proof is given in Appendix A.

Theorem 4.7 *Let $\rho(s) = \sqrt{\ell_0 + 2\, \ell_1\, s + \ell_2\, s^2}$ and $b(t) = b_0 + b_1\, t + b_2\, t^2$. If* $\mathbf{u}_0^{(s)}$ *and* $\mathbf{v}^{(t)}$ *satisfy*

$$\begin{aligned} \rho(s)^2\, D\mathbf{u}_0^{(s)} &= \left(\alpha + \frac{1}{2}\right) \left(\rho(s)^2\right)'\, \mathbf{u}_0^{(s)}, \\ b(t)\, D\mathbf{v}^{(t)} &= \alpha\, b'(t)\, \mathbf{v}^{(t)}, \end{aligned}$$

where α is a real number, then $\mathbf{w}$ *satisfies the diagonal Pearson equations*

$$\rho(x)^2\, b\left(\frac{y}{\rho(x)}\right) \partial_x \mathbf{w} = \alpha\, \partial_x \left(\rho(x)^2\, b\left(\frac{y}{\rho(x)}\right)\right) \mathbf{w}, \tag{4.10}$$

$$\rho(x)^2\, b\left(\frac{y}{\rho(x)}\right) \partial_y \mathbf{w} = \alpha\, \partial_y \left(\rho(x)^2\, b\left(\frac{y}{\rho(x)}\right)\right) \mathbf{w}. \tag{4.11}$$

5 Generating polynomial solutions

In this section, we construct PS solutions $\{\mathbb{P}_n\}_{n\geqslant 0}$ of

$$\mathscr{L}[\mathbb{P}_n] = \Lambda_n \mathbb{P}_n,$$

where $\mathscr{L}$ is the differential operator defined in (3.2), and Λ_n is a square matrix of order $n+1$ whose entries are real numbers.

We are interested in solutions that can be generated using univariate polynomial sequences. That is, we seek compatible univariate polynomial sequences $\{p_n^{(m)}(x)\}_{n\geqslant 0}$, $m \geqslant 0$, and $\{q_n(y)\}_{n\geqslant 0}$, and real square matrices Λ_n, such that the entries of $\mathbb{P}_n$ satisfying $\mathscr{L}[\mathbb{P}_n] = \Lambda_n\mathbb{P}_n$ can be written as in (4.6) where $\rho(x)$ is a function satisfying the conditions in Case I or Case II described in Sect. 4.1. Before we deal with such solutions, we state the following result which will be useful in the sequel.

Proposition 5.1 *2 Let $\mathscr{L}$ be the differential operator defined in* (3.2). *Assume there is a non-zero polynomial $\Psi(x, y)$ satisfying the differential equations*

$$\mathscr{L}[\Psi] = \mu\,\Psi, \quad \mu \in \mathbb{R}, \tag{5.1}$$

and

$$\begin{aligned} a\,\Psi_x + b\,\Psi_y &= h_1\,\Psi,\\ b\,\Psi_x + c\,\Psi_y &= h_2\,\Psi, \end{aligned} \tag{5.2}$$

where h_1 and h_2 are polynomials of degree at most 1. Let $k = \deg\Psi$.

Moreover, let $\{P_{n,m}(x, y) : n \geqslant 0,\ 0 \leqslant m \leqslant n\}$ be a polynomial system satisfying the differential equation

$$\mathscr{M}[P_{n,m}] = \mathscr{L}[P_{n,m}] + 2\,h_1\,\partial_x P_{n,m} + 2\,h_2\,\partial_y P_{n,m} = \lambda_{n,m}\,P_{n,m}, \quad \lambda_{n,m} \in \mathbb{R}.$$

Then, for $k \leqslant m \leqslant n$, the polynomial $Q_{n,m} = \Psi\,P_{n-k,m-k}$ satisfies the differential equation

$$\mathscr{L}[Q_{n,m}] = (\mu + \lambda_{n-k,m-k})\,Q_{n,m}.$$

Proof Using (5.1) and (5.2), we obtain

$$\begin{aligned} \mathscr{L}[Q_{n,m}] &= \mathscr{L}[\Psi]\,P_{n-k,m-k} + \Psi\,\mathscr{L}[P_{n-k,m-k}]\\ &\quad + 2\,(a\,\Psi_x + b\,\Psi_y)\,\partial_x P_{n-k,m-k} + 2\,(b\,\Psi_x + c\,\Psi_y)\,\partial_y P_{n-k,m-k}\\ &= \mu\,\Psi\,P_{n-k,m-k} + \Psi\,\mathscr{L}[P_{n-k,m-k}]\\ &\quad + 2\,h_1\,\Psi\,\partial_x P_{n-k,m-k} + 2\,h_2\,\Psi\,\partial_y P_{n-k,m-k}\\ &= (\mu + \lambda_{n-k,m-k})\,Q_{n,m}. \end{aligned}$$

□

For the following result, we need the explicit expressions for the following functions. In both Case I and Case II in Sect. 4.1, we chose the polynomials coefficients of (3.2) with the special shapes

- $a(x, y) = a(x) = a_0 + a_1\,x + a_2\,x^2$,
- $\widetilde{b}(x) = b_0 + b_1\,x$,
- $b(x, y) = y\,\widetilde{b}(x)$,
- $c(x, y) = c(y) = c_0 + c_1\,y + c_2\,y^2$,
- $d(x, y) = d(x) = d_0 + d_1\,x$,

- $e(x, y) = e(y) = e_0 + e_1\, y$.

First, we consider the case when $\rho(x) = r_0 + r_1\, x$.

Theorem 5.2 *Let $\rho(x) = r_0 + r_1\, x$ satisfying $a(x)\,\rho'(x) = \widetilde{b}(x)\,\rho(x)$, $c_0 = 0$, $c_2 = b_1$, and $d_1 = e_1$, and let $\{P_{n,m}(x, y) : n \geqslant 0,\ 0 \leqslant m \leqslant n\}$ be the polynomials defined in (4.6). Then,*

1. *For $m \geqslant 0$, $\{p_n^{(m)}(x)\}_{n\geqslant 0}$ satisfies*
$$a(x)\,(p_n^{(m)}(x))'' + [d(x) + 2\,m\,\widetilde{b}(x)](p_n^{(m)}(x))' = \nu_{n,m}\, p_n^{(m)}(x),$$
with $\nu_{n,m} = n\ ((n-1)\,a_2 + d_1 + 2\,m\,b_1)$,
2. *$\{q_n(y)\}_{n\geqslant 0}$ satisfies*
$$[(c_2\,r_0 - b_0\,r_1)\,y^2 + c_1\,y]\,q_n''(y) + [(e_1\,r_0 - d_0\,r_1)\,y + e_0]\,q_n'(y) = \mu_n\,q_n(y),$$
with $\mu_n = -n\,[((n-1)\,b_0 + d_0)\,r_1 - ((n-1)\,b_1 + d_1)r_0]$,

if and only if
$$\mathscr{L}[\,P_{n,m}\,] = \lambda_{n,m}\,P_{n,m}, \quad n \geqslant 0, \quad 0 \leqslant m \leqslant n,$$
with $\lambda_{n,m} = \nu_{n-m,m} + m\,[(m-1)\,b_1 + d_1]$.

Proof Letting $t = y/\rho(x)$, we compute
$$\begin{aligned}\mathscr{L}[P_{n,m}] = &\left[a(x)\,(p_{n-m}^{(m)}(x))'' + [d(x) + 2\,m\,\widetilde{b}(x)](p_{n-m}^{(m)}(x))'\right]\rho(x)^m\,q_m(t)\\ &+ \left[(c_2\,r_0 - b_0\,r_1)\,t^2 + c_1\,t\right]\,p_{n-m}^{(m)}(x)\,\rho(x)^{m-1}\,q_m''(t)\\ &+ [(e_1\,r_0 - d_0\,r_1)\,t + e_0]\,p_{n-m}^{(m)}(x)\,\rho(x)^{m-1}\,q_m'(t)\\ &+ \left[a(x)\,(\rho(x)^m)'' + d(x)\,(\rho(x)^m)'\right]\,p_{n-m}^{(m)}(x)\,q_m(t),\end{aligned}$$
where we have used $a(x)\,\rho'(x) = \widetilde{b}(x)\,\rho(x)$. Moreover, taking into account that
$$a(x)\,(\rho(x)^m)'' + d(x)\,(\rho(x)^m)' = m\,[(m-1)\,b_1 + d_1]\,\rho(x)^m - \mu_m\,\rho(x)^{m-1},$$
the announced result follows. □

Now, we deal with the case when $\rho(x) = \sqrt{\ell_2\,x^2 + 2\,\ell_1\,x + \ell_0}$.

Theorem 5.3 *Let $\rho(x) = \sqrt{\ell_2\,x^2 + 2\,\ell_1\,x + \ell_0}$ satisfying $a(x)\,\rho'(x) = \widetilde{b}(x)\,\rho(x)$, and let $\{P_{n,m}(x, y) : n \geqslant 0,\ 0 \leqslant m \leqslant n\}$ be the polynomials defined in (4.6). Suppose that $c_1 = e_0 = 0$, and for $m \geqslant 0$, $\{p_n^{(m)}(x)\}_{n\geqslant 0}$ satisfies*
$$a(x)\,(p_n^{(m)}(x))'' + [d(x) + 2\,m\,\widetilde{b}(x)](p_n^{(m)}(x))' = \nu_{n,m}\,p_n^{(m)}(x),$$
with $\nu_{n,m} = n\ ((n-1)\,a_2 + d_1 + 2\,m\,b_1)$.

We have the following two cases:

(I) *$\mathscr{L}[\,P_{n,m}\,] = \lambda_{n,m}\,P_{n,m}$ for $n \geqslant 0$, and $0 \leqslant m \leqslant n$, where*
$$\lambda_{n,m} = \nu_{n-m,m} + \frac{1}{2}m\,[(m-2)\,b_1 + d_1],$$
if and only if

(a) $\ell_2 = 0$,

(b) $\ell_1 = 0$ *or* $2\,c_2 - b_1 = 2\,e_1 + b_1 - d_1 = 0$,
(c) *and* $\{q_n(y)\}_{n\geqslant 0}$ *satisfies*

$$[(c_2\,\ell_0 - b_0\,\ell_1)\,y^2 + c_0]\,q_n''(y) + [e_1\,\ell_0 + (b_0 - d_0)\,\ell_1]\,y\,q_n'(y) = \mu_n\,q_n(y),$$

with $\mu_n = -n\,[(n-2)\,(b_0\,\ell_1 - \frac{1}{2}b_1\,\ell_0) + d_0\,\ell_1 - \frac{1}{2}d_1\,\ell_0]$.

(II) $\mathscr{L}[\,P_{n,m}\,] = \lambda_{n,m}\,P_{n,m}$ *for* $n \geqslant 0$, *and* $0 \leqslant m \leqslant n$, *where*

$$\lambda_{n,m} = \nu_{n-m,m} + m\,[(m-2)\,b_1 + a_2 + d_1],$$

if and only if

(d) $2\,c_2\,\ell_1 - b_1\,\ell_1 - b_0\,\ell_2 = (a_1 - b_0 - d_0)\,\ell_2 + (2\,e_1 - 3\,b_1 + 2\,a_2 - d_1)\,\ell_1 = 0$,
(e) $c_2 - b_1 = e_1 - 2\,b_1 + 2\,a_2 - d_1 = 0$,
(f) *for* $n \geqslant 0$,

$$\eta_n \equiv -n\,[(n-2)\,(b_0\,\ell_2 - b_1\,\ell_1) + a_1\,\ell_2 - 2\,a_2\,\ell_1 - d_1\,\ell_1 + d_0\,\ell_2] = 0,$$

(g) *and* $\{q_n(y)\}_{n\geqslant 0}$ *satisfies*

$$[(c_2\,\ell_0 - b_0\,\ell_1)\,y^2 + c_0]\,q_n''(y) + [(e_1 - b_1)\,\ell_0 + (a_1 - b_0 - d_0)\,\ell_1]\,y\,q_n'(y) = \widetilde{\mu}_n\,q_n(y),$$

with $\widetilde{\mu}_n = -n\,[(n-2)\,(b_0\,\ell_1 - b_1\,\ell_0) + a_0\,\ell_2 + d_0\,\ell_1 - (a_2 + d_1)\,\ell_0]$.

Proof Let $t = y/\rho(x)$. For (**I**), we compute

$$\begin{aligned}
\mathscr{L}[P_{n,m}] = &\left[a(x)\,(p_{n-m}^{(m)}(x))'' + [d(x) + 2\,m\,\widetilde{b}(x)](p_{n-m}^{(m)}(x))'\right]\rho(x)^m\,q_m(t)\\
&+ x\,\ell_1\left[(2c_2 - b_1)t^2 q_m''(y) + (2e_1 + b_1 - d_1)\,t q_m'(t)\right]p_{n-m}^{(m)}\rho(x)^{m-2}\\
&+ \left[[(c_2\ell_0 - b_0\ell_1)t^2 + c_0]q_m''(t) + [e_1\,\ell_0 + (b_0 - d_0)\ell_1]t q_m'(t)\right]p_{n-m}^{(m)}\rho(x)^{m-2}\\
&+ \left[a(x)\,(\rho(x)^m)'' + d(x)\,(\rho(x)^m)'\right]p_{n-m}^{(m)}\,q_m(t),
\end{aligned}$$

where we have used $a(x)\,\rho'(x) = \widetilde{b}(x)\,\rho(x)$. Moreover, since

$$a(x)\,(\rho(x)^m)'' + d(x)\,(\rho(x)^m)' = m\,[(m-2)\,b_1 + d_1]\,\rho(x)^m - \mu_m\,\rho(x)^{m-2},$$

the result follows.

Similarly, for (**II**), we compute

$$\begin{aligned}
\mathscr{L}[P_{n,m}] = &\left[a(x)\,(p_{n-m}^{(m)}(x))'' + [d(x) + 2\,m\,\widetilde{b}(x)](p_{n-m}^{(m)}(x))'\right]\rho(x)^m\,q_m(t)\\
&+ x^2\ell_2\left[(c_2 - b_1)\,t^2 q_m''(t) + (e_1 - 2\,b_1 + 2\,a_2 - d_1)\,t\,q_m'(t)\right]p_{n-m}^{(m)}\rho(x)^{m-2}\\
&+ x\Big[(2c_2\ell_1 - b_1\ell_1 - b_0\ell_2)t^2 q_m''(y)\\
&+ [(a_1 - b_0 - d_0)\ell_2 + (2e_1 - 3b_1 + 2a_2 - d_1)\ell_1]t q_m'(t)\Big]p_{n-m}^{(m)}\rho(x)^{m-2}\\
&+ \Big[[(c_2\ell_0 - b_0\ell_1)t^2 + c_0]q_m''(t)\\
&+ [(e_1 - b_1)\ell_0 + (a_1 - b_0 - d_0)\ell_1]t q_m'(t)\Big]p_{n-m}^{(m)}\rho(x)^{m-2}\\
&+ \left[a(x)\,(\rho(x)^m)'' + d(x)\,(\rho(x)^m)'\right]p_{n-m}^{(m)}\,q_m(t),
\end{aligned}$$

and since

$$a(x)\,(\rho(x)^m)'' + d(x)\,(\rho(x)^m)' \\ = m\,[(m-2)\,b_1 + a_2 + d_1]\,\rho(x)^m - (\eta_m\,x + \widetilde{\mu}_m)\,\rho(x)^{m-2},$$

the second result follows. □

6 Examples

The objective of this section is to explore some illustrative examples. First, we consider a partial differential equation whose polynomial solutions consist of monomials in two variables and we deduce their orthogonality with respect to a multivariate analogue of the Dirac delta functional. In the next three examples, we discuss the Sobolev orthogonality of the classical polynomials on the unit ball, simplex and parabolic biangle in $\mathbb{R}^2$, all three satisfying partial differential equations belonging to the extended Lyskova class. Then, we consider an example involving Pearson equations whose solution is a quasi-definite bivariate moment functional obtained from a non-positive definite classical moment functional, namely, the classical Bessel moment functional. In the last example, we discuss how our results can be used to find a moment functional satisfying Pearson equations with slightly more general polynomial coefficients in the sense that there is no associated differential operator in the extended Lyskova class.

In the sequel, $P_n^{(\alpha,\beta)}(x)$ denotes the univariate Jacobi polynomial of degree n. Its explicit expression is (Szegő 1975)

$$P_n^{(\alpha,\beta)}(x) \;=\; \frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(j+\alpha+1)_{n-j}\,(j+\alpha+\beta+1)_j\left(\frac{x-1}{2}\right)^j,$$

where, as usual,

$$(\nu)_0 = 1, \quad (\nu)_k = \nu\,(\nu+1)\cdots(\nu+k-1), \quad k \geqslant 1,$$

denotes Pochhammer symbol.

As shown in Szegő (1975), the expression for $P_n^{(\alpha,\beta)}(x)$ is valid for arbitrary complex values of the parameters α and β. Nevertheless, a reduction of the degree of $P_n^{(\alpha,\beta)}(x)$, $n \geqslant 1$, occurs if and only if $-(\alpha+\beta+n) \in \{1, 2, \ldots, n\}$.

For arbitrary values of the parameters, the expression for $P_n^{(\alpha,\beta)}(x)$ is a polynomial solution of the second-order linear ordinary differential equation

$$(1-x^2)\,y'' + [\beta-\alpha-(\alpha+\beta+2)\,x]\,y' \;=\; -n\,(n+\alpha+\beta+1)\,y.$$

It is important to note that if either $\alpha,\ \beta$, or $\alpha+\beta+1$ are negative integers, then the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n\geqslant 0}$ can not be associated with a quasi-definite moment functional. On the other hand, if $-\alpha,\ -\beta,\ -(\alpha+\beta+1) \notin \mathbb{N}$, then $\{P_n^{(\alpha,\beta)}(x)\}_{n\geqslant 0}$ are associated with a quasi-definite moment functional $\mathbf{u}_{\alpha,\beta}$ satisfying the Pearson equation

$$D\left[(1-x^2)\,\mathbf{u}_{\alpha,\beta}\right] \;=\; [\beta-\alpha-(\alpha+\beta+2)x]\;\mathbf{u}_{\alpha,\beta}.$$

Furthermore, for $\alpha, \beta > -1$, $\mathbf{u}_{\alpha,\beta}$ is positive definite and has the integral representation

$$\langle \mathbf{u}_{\alpha,\beta}, p\rangle \;=\; \int_{-1}^{1} p(x)\,(1-x)^\alpha(1+x)^\beta\,dx, \quad \forall p \in \Pi.$$

6.1 Multivariate Dirac delta

Consider the second-order partial differential operator

$$\mathscr{L}[\,p\,] \equiv x^2\, p_{xx} + 2\,x\,y\,p_{xy} + y^2\, p_{yy} + x\,p_x + y\,p_y.$$

For $n \geqslant 0$, define the column vector

$$\mathbb{X}_n = \left(x^{n-k}\, y^k\right)_{k=0}^{n}.$$

The PS $\{\mathbb{X}_n\}_{n\geqslant 0}$ satisfies the differential equation $\mathscr{L}[\,\mathbb{X}_n\,] = \Lambda_n\,\mathbb{X}_n$, $n \geqslant 0$, with $\Lambda_n = n^2\, I_{n+1}$.

We deduce from Theorem 3.3 (see also Remark 3.6) that if there is a bivariate moment functional $\mathbf{w}$ satisfying the Pearson equations

$$\partial_x(x^2\,\mathbf{w}) + \partial_y(x\,y\,\mathbf{w}) = x\,\mathbf{w},$$
$$\partial_x(x\,y\,\mathbf{w}) + \partial_y(y^2\,\mathbf{w}) = y\,\mathbf{w},$$

then $\mathscr{L}$ satisfies

$$\langle \mathbf{w}, \mathscr{L}[P]\,Q\rangle = \langle \mathbf{w}, P\,\mathscr{L}[Q]\rangle, \quad \forall P, Q \in \Pi^2.$$

From Proposition 3.5, we have that $\{\mathbb{X}_n\}_{n\geqslant 0}$ satisfies the orthogonality condition $\langle \mathbf{w}, \mathbb{X}_n\,\mathbb{X}_m^{\top}\rangle = \mathtt{0}$, $n \neq m$. Observe that $\{\mathbb{X}_n\}_{n\geqslant 0}$ can not be associated with a quasi-definite moment functional.

We seek univariate linear functionals $\mathbf{u}^{(s)}$ and $\mathbf{v}^{(t)}$ such that, if possible, we can construct $\mathbf{w}$ using (4.5). To this end, we rewrite the forgoing Pearson equations as

$$x^2\,\partial_x\mathbf{w} + x\,y\,\partial_y\mathbf{w} = -2\,x\,\mathbf{w},$$
$$x\,y\,\partial_x\mathbf{w} + y^2\,\partial_y\mathbf{w} = -2\,y\,\mathbf{w}.$$

First, we look at Theorem 4.1 with $a(x) = x^2$, $\widetilde{b}(x) = x$, and $d(x) = -2\,x$. A solution of $s^2\,\rho'(s) = s\,\rho(s)$ is $\rho(s) = s$. From this, we deduce that if $\mathbf{u}_0^{(s)} = \rho(s)\,\mathbf{u}^{(s)}$ satisfies

$$s^2\,D\mathbf{u}_0^{(s)} = -s\,\mathbf{u}_0^{(s)},$$

then $\mathbf{w}$ satisfies $x^2\,\partial_x\mathbf{w} + x\,y\,\partial_y\mathbf{w} = -2\,x\,\mathbf{w}$. A non-trivial solution to the univariate Pearson equation above is $\mathbf{u}_0^{(s)} = \boldsymbol{\delta}(s)$,

$$\left\langle \mathbf{u}_0^{(s)}, p(s)\right\rangle = p(0).$$

Furthermore, $\mathbf{u}^{(s)} = \boldsymbol{\delta}(s) - \boldsymbol{\delta}'(s)$ satisfies $\mathbf{u}_0^{(s)} = \rho(s)\,\mathbf{u}^{(s)}$.

From Corollary 4.3 with $\mathbf{u}_0^{(s)} = \boldsymbol{\delta}(s)$, $\rho(s) = s$, $\widetilde{b}(x) = x$, $c(y) = y^2$, and $e(y) = -2\,y$, we have that if $\mathbf{v}^{(t)}$ is an arbitrary linear functional, then $\mathbf{w}$ satisfies $x\,y\,\partial_x\,\mathbf{w} + y^2\,\partial_y\,\mathbf{w} = -2\,y\,\mathbf{w}$.

Using $\mathbf{u}_0^{(s)} = \rho(s)\,\mathbf{u}^{(s)}$ and $\mathbf{v}^{(t)} = \boldsymbol{\delta}(t)$, we obtain from (4.5) the linear functional $\mathbf{w}$ defined by

$$\langle \mathbf{w}, P(x,y)\rangle = \langle \boldsymbol{\delta}(s), \langle \boldsymbol{\delta}(t), P(s, t\,s)\rangle\rangle\,.$$

Observe that for $n \geqslant 0$ and $0 \leqslant k \leqslant n$,

$$\left\langle \mathbf{w}, x^{n-k}\,y^k\right\rangle = \left\langle \boldsymbol{\delta}(s), \left\langle \boldsymbol{\delta}(t), s^n\,t^k\right\rangle\right\rangle = \langle \boldsymbol{\delta}(s), s^n\rangle\left\langle \boldsymbol{\delta}(t), t^k\right\rangle = \begin{cases} 1, & n = 0,\\ 0, & n \neq 0.\end{cases}$$

In this case, $\mathbf{w}$ is the multivariate Dirac linear functional centered at the origin.

6.2 Orthogonal polynomials on the unit ball

Consider the second-order partial differential operator

$$\mathscr{L}_\mu[\,p\,] \equiv (1-x^2)\,p_{xx} - 2\,x\,y\,p_{xy} + (1-y^2)\,p_{yy} - (2\,\mu+2)\,x\,p_x - (2\,\mu+2)\,y\,p_y,$$

where μ is a real number. We can use Theorem 5.3 **(II)** with $a(x) = 1-x^2$, $\widetilde{b}(x) = -x$, $c(y) = 1-y^2$, $d(x) = -(2\,\mu+2)\,x$, and $e(y) = -(2\,\mu+2)\,y$ to construct polynomial solutions of the differential equation $\mathscr{L}_\mu[p] = \lambda_n\,p$.

First, a solution of $(1-x^2)\,\rho'(x) = -x\,\rho(x)$ is $\rho(x) = \sqrt{1-x^2}$. For $m \geqslant 0$, let $\{p_n^{(m)}(x)\}_{n\geqslant 0}$ be a univariate polynomial sequence satisfying

$$\begin{aligned}(1-x^2)\,(p_n^{(m)}(x))'' - (2\,\mu+2\,m+2)\,x\,(p_n^{(m)}(x))' \\ = -n\,(n+2\,\mu+2m+1)\,p_n^{(m)}(x),\end{aligned} \tag{6.1}$$

and let $\{q_n(y)\}_{n\geqslant 0}$ be a univariate polynomial sequence satisfying

$$(1-y^2)\,q_n''(y) - (2\,\mu+1)\,y\,q_n'(y) = -n\,(n+2\,\mu)\,q_n(y). \tag{6.2}$$

Then, by Theorem 5.3 **(II)**, the bivariate ball polynomials defined by

$$P_{n,m}^{(\mu)}(x,y) = p_{n-m}^{(m)}(x)\,(1-x^2)^{m/2} q_m\left(\frac{y}{\sqrt{1-x^2}}\right), \quad n \geqslant 0, \quad 0 \leqslant m \leqslant n,$$

satisfy $\mathscr{L}_\mu[P_{n,m}^{(\mu)}] = \lambda_n^{(\mu)}\,P_{n,m}^{(\mu)}$ with $\lambda_n^{(\mu)} = -n\,(n+2\,\mu+1)$.

If we denote by $\mathbb{P}_{n,\mu}$, $n \geqslant 0$, the column vector of ball polynomials

$$\mathbb{P}_{n,\mu} = (P_{n,0}^{(\mu)}(x,y),\ P_{n,1}^{(\mu)}(x,y), \ldots, P_{n,n}^{(\mu)}(x,y))^\top,$$

then we have $\mathscr{L}_\mu[\mathbb{P}_{n,\mu}] = \Lambda_n^{(\mu)}\,\mathbb{P}_{n,\mu}$ with $\Lambda_n^{(\mu)} = \lambda_n^{(\mu)}\,I_{n+1}$.

Now, we study the polynomial solutions according to the values of the parameter μ: for $\mu > -1/2$, which is the positive definite case, and, for $\mu = -1/2$, which we call the singular case. In both cases, we study the Sobolev orthogonality of the polynomial solutions.

6.2.1 Positive definite case

For $\mu > -1/2$ and $m, n \geqslant 0$, the polynomials $p_n^{(m)}(x) = P_n^{(\mu+m,\mu+m)}(x)$ and $q_n(y) = P_n^{(\mu-1/2,\mu-1/2)}(y)$ satisfy (6.1) and (6.2), respectively. Therefore, in this case, the ball polynomials are given by

$$P_{n,m}^{(\mu)}(x,y) = P_n^{(\mu+m,\mu+m)}(x)\,(1-x^2)^{m/2}\,P_n^{(\mu-1/2,\mu-1/2)}\left(\frac{y}{\sqrt{1-x^2}}\right).$$

We deduce from Theorem 3.3 (and Remark 3.6) that if the bivariate moment functional $\mathbf{w}_\mu$ satisfies the Pearson equations

$$\begin{aligned}\partial_x\left((1-x^2)\,\mathbf{w}_\mu\right) - \partial_y\left(x\,y\,\mathbf{w}_\mu\right) = -(2\,\mu+2)\,x\,\mathbf{w}_\mu, \\ -\partial_x\left(x\,y\,\mathbf{w}_\mu\right) + \partial_y\left((1-y^2)\,\mathbf{w}_\mu\right) = -(2\,\mu+2)\,y\,\mathbf{w}_\mu,\end{aligned}$$

then $\mathscr{L}_\mu$ satisfies

$$\left\langle \mathbf{w}_\mu, \mathscr{L}_\mu[P]\, Q \right\rangle = \left\langle \mathbf{w}_\mu, P\, \mathscr{L}_\mu[Q] \right\rangle, \quad \forall P, Q \in \Pi^2.$$

Since $\lambda_n^{(\mu)} \neq \lambda_m^{(\mu)}$ for $n \neq m$, $\Lambda_n^{(\mu)}$ and $\Lambda_m^{(\mu)}$ do not share eigenvalues. Then, from Proposition 3.5, we have that the PS $\{\mathbb{P}_{n,\mu}\}_{n\geqslant 0}$ satisfies the orthogonality condition $\langle \mathbf{w}_\mu, \mathbb{P}_{n,\mu}\, \mathbb{P}_{m,\mu}^\top \rangle = 0$, $n \neq m$.

We turn our attention to the solution of the foregoing Pearson equations. In this regard, we seek univariate moment functionals $\mathbf{u}^{(s)}$ and $\mathbf{v}^{(t)}$ such that, if possible, we can construct $\mathbf{w}_\mu$ using (4.5). To this end, we rewrite the Pearson equations as

$$(1-x^2)\,\partial_x \mathbf{w}_\mu - x\,y\,\partial_y \mathbf{w}_\mu = (-2\,\mu+1)\,x\,\mathbf{w}_\mu, \tag{6.3}$$

$$-x\,y\,\partial_x \mathbf{w}_\mu + (1-y^2)\,\partial_y \mathbf{w}_\mu = (-2\,\mu+1)\,y\,\mathbf{w}_\mu. \tag{6.4}$$

First, we look at Theorem 4.1 with $a(x) = 1-x^2$, $\widetilde{b}(x) = -x$, $\widetilde{d}(x) = (-2\,\mu+1)\,x$, and $\rho(s) = \sqrt{1-s^2}$. Therefore, if $\mathbf{u}_0^{(s)} = \rho(s)\,\mathbf{u}^{(s)}$ satisfies

$$(1-s^2)\,D\mathbf{u}_0^{(s)} = -2\,\mu\,s\,\mathbf{u}_0^{(s)},$$

then $\mathbf{w}_\mu$ satisfies (6.3). From here, we deduce that $\mathbf{u}_0^{(s)} \equiv \mathbf{u}_{\mu,\mu}$. Note that the univariate moment functional $\mathbf{u}^{(s)} \equiv \mathbf{u}_{\mu-1/2,\mu-1/2}$ satisfies $\mathbf{u}_0^{(s)} = \rho(s)\,\mathbf{u}^{(s)}$.

Second, from Corollary 4.4 with $\mathbf{u}_0^{(s)} \equiv \mathbf{u}_{\mu,\mu}$, $\widetilde{b}(x) = -x$, $c(y) = 1-y^2$, $\widetilde{e}(y) = (-2\,\mu+1)\,y$, $\rho(s) = \sqrt{1-s^2}$, and taking into account that $\mathbf{v}^{(t)}$ satisfies

$$(1-t^2)\,D\,\mathbf{v}^{(t)} = (-2\,\mu+1)\,t\,\mathbf{v}^{(t)},$$

we get that $\mathbf{w}_\mu$ satisfies (6.4). We can choose $\mathbf{v}^{(t)} \equiv \mathbf{u}^{(t)}$.

From (4.5) with $\mathbf{u}_0^{(s)} = \rho(s)\,\mathbf{u}^{(s)}$ and $\mathbf{v}^{(t)}$ as above, we obtain the classical moment functional $\mathbf{w}_\mu$ defined by

$$\langle \mathbf{w}_\mu, P \rangle = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} P(x,y)\,(1-x^2-y^2)^{\mu-1/2}\,dy dx, \quad \forall P \in \Pi^2.$$

Observe that $\mathbf{w}_\mu$ is defined on the unit ball $\mathbf{B}^2 = \{(x,y) \in \mathbb{R}^2 :\ 0 \leqslant 1-x^2-y^2\}$. In fact, for $\mu > -1/2$, $\mathbf{w}_\mu$ is a positive definite moment functional and, thus,

$$\langle \mathbf{w}_\mu, P_{n,m}^{(\mu)}\, P_{k,j}^{(\mu)} \rangle = H_{n,m}^{(\mu)}\,\delta_{n,k}\,\delta_{m,j},$$

where $H_{n,m}^{(\mu)} > 0$ and $\delta_{n,k}$ denotes the usual Kronecker delta.

Moreover, note that $\mathbf{u}_0^{(s)}$ and $\mathbf{v}^{(t)}$ satisfy the sufficient conditions of Theorem 4.7 with $\alpha = \mu - 1/2$ and $b(t) = 1-t^2$. Indeed, it is well known that $\mathbf{w}_\mu$ also satisfies the diagonal Pearson equations (Álvarez de Morales et al. 2009b; Lee 2000; Marcellán et al. 2018a, b)

$$(1-x^2-y^2)\,\partial_x \mathbf{w}_\mu = (-2\,\mu+1)\,x\,\mathbf{w}_\mu,$$
$$(1-x^2-y^2)\,\partial_y \mathbf{w}_\mu = (-2\,\mu+1)\,y\,\mathbf{w}_\mu.$$

The polynomials $\{P_{n,m}^{(\mu)}(x,y) :\ n \geqslant 0,\ 0 \leqslant m \leqslant n\}$ defined above are Sobolev orthogonal polynomials. Indeed, fix an integer $N \geqslant 0$. For $0 \leqslant j \leqslant i \leqslant N$, the Pearson equations (3.3) read

$$\partial_x\left((1-x^2)\,\mathbf{u}^{(i,j)}\right) - \partial_y\left(x\,y\,\mathbf{u}^{(i,j)}\right) = -[2\,(\mu+i)+2]\,x\,\mathbf{u}^{(i,j)},$$

$$-\partial_x \left(x\, y\, \mathbf{u}^{(i,j)}\right) + \partial_y \left((1-y^2)\, \mathbf{u}^{(i,j)}\right) = -[2\,(\mu+i)+2]\, y\, \mathbf{u}^{(i,j)}.$$

Therefore, $\mathbf{u}^{(i,j)} \equiv \mathbf{w}_{\mu+i}$ and, by Theorem 3.3, $\mathscr{L}_\mu$ is symmetric with respect to the Sobolev bilinear form

$$(P, Q)_\mu = \sum_{i=0}^{N} \left\langle \mathbf{w}_{\mu+i}, \sum_{j=0}^{i} \partial_x^{i-j} \partial_y^{j} P\ \partial_x^{i-j} \partial_y^{j} Q \right\rangle, \quad \forall P, Q \in \Pi^2,$$

The Sobolev orthogonality of the ball polynomials with respect to $(\cdot\,,\cdot)_\mu$ follows from Proposition 3.5 since $\lambda_n^{(\mu)} \neq \lambda_m^{(\mu)}$ for $n \neq m$.

6.2.2 Singular case

For $\mu > -1/2$, the polynomial solutions of $\mathscr{L}_\mu[\,p\,] = \lambda_n\, p$ described in the previous example, are orthogonal with respect to a positive definite moment functional defined on the unit ball in $\mathbb{R}^2$.

However, for $\mu_k = -k - 1/2$, $k = 0, 1, 2, \ldots$, $P(x, y)(1 - x^2 - y^2)^{\mu_k - 1/2}$ is no longer integrable for all polynomials $P \in \Pi^2$, and, therefore, $\mathbf{w}_{\mu_k}$ is not guaranteed to be a quasi-definite moment functional. Moreover, it was shown in Piñar and Xu (2009) that $\mathscr{L}_{\mu_k}[\,p\,] = \lambda_n^{(\mu_k)}\, p$ with $k \geqslant 1$ is not guaranteed to have a complete PS solution.

For $k = 0$, however, $\mathscr{L}_{-1/2}[\,p\,] = \lambda_n^{(-1/2)}\, p$ has a complete PS solution and Sobolev orthogonality can be provided for these solutions.

Indeed, the ball polynomials $\{P_{n,m}^{(-1/2)}(x, y) : n \geqslant 0,\ 0 \leqslant m \leqslant n\}$ satisfy

$$\mathscr{L}_{-1/2}[P_{n,m}^{(-1/2)}] = -n^2\, P_{n,m}^{(-1/2)}$$

if $\{p_n^{(m)}(x)\}_{n \geqslant 0}$ is a univariate polynomial sequence satisfying

$$(1-x^2)\,(p_n^{(m)}(x))'' - (2\,m+1)\,x\,(p_n^{(m)}(x))' = -n\,(n+2m)\,p_n^{(m)}(x),$$

and $\{q_n(y)\}_{n \geqslant 0}$ is a univariate polynomial sequence satisfying

$$(1-y^2)\,q_n''(y) = -n\,(n-1)\,q_n(y).$$

Since $p_n^{(m)}(x) = P_n^{(m-1/2,m-1/2)}(x)$, $n, m \geqslant 0$ and the second ordinary differential equation is satisfied by the polynomials $q_0(y) = 1$, $q_1(y) = y$, and $q_n(y) = (1-y^2)\,P_{n-2}^{(1,1)}(y)$, for $n \geqslant 2$ (see [Garcia-Ardila and Marriaga (2021),eq. (4.14)]), then for $n \geqslant 2$ and $2 \leqslant m \leqslant n$,

$$\begin{aligned}
P_{0,0}^{(-1/2)}(x, y) &= P_0^{(-1/2,-1/2)}(x),\\
P_{1,0}^{(-1/2)}(x, y) &= P_1^{(-1/2,-1/2)}(x), \quad P_{1,1}^{(-1/2)}(x, y) = y\, P_0^{(1/2,1/2)}(x),\\
P_{n,0}^{(-1/2)}(x, y) &= P_n^{(-1/2,-1/2)}(x), \quad P_{n,1}^{(-1/2)}(x, y) = y\, P_{n-1}^{(1/2,1/2)}(x),\\
P_{n,m}^{(-1/2)}(x, y) &= (1-x^2-y^2)\, P_{n-m}^{(m-1/2,m-1/2)}(x)\,(1-x^2)^{\frac{m-2}{2}}\, P_{m-2}^{(1,1)}\left(\frac{y}{\sqrt{1-x^2}}\right).
\end{aligned}$$

We note that the expression for $P_{n,m}^{(-1/2)}(x, y)$ with $2 \leqslant m \leqslant n$ can also be deduced from Proposition 5.1 with $\mathscr{L} \equiv \mathscr{L}_{-1/2}$, $\Psi(x, y) = 1 - x^2 - y^2$, and $\mathscr{M} \equiv \mathscr{L}_{3/2}$. Indeed, since in this case (5.1) and (5.2) read as

$$\mathscr{L}_{-1/2}[\Psi] = -4\,\Psi,$$

and

$$\begin{aligned}(1-x^2)\,\partial_x\Psi - x\,y\,\partial_y\Psi \quad &= -2\,x\,\Psi,\\ -x\,y\,\partial_x\Psi + (1-y^2)\,\partial_y\Psi &= -2\,y\,\Psi,\end{aligned}$$

and since the ball polynomials $\{P_{n,m}^{(3/2)}(x,y) : n \geqslant 0,\ 0 \leqslant m \leqslant n\}$ satisfy

$$\mathscr{L}_{3/2}[P_{n,m}^{(3/2)}] = \mathscr{L}_{-1/2}[P_{n,m}^{(3/2)}] - 4\,x\,\partial_x P_{n,m}^{(3/2)} - 4\,y\,\partial_y P_{n,m}^{(3/2)} = -n\,(n+4)\,P_{n,m}^{(3/2)},$$

then, for $2 \leqslant m \leqslant n$,

$$\mathscr{L}_{-1/2}[\Psi\,P_{n-2,m-2}^{(3/2)}] = [-4-(n-2)\,(n+2)]\,\Psi\,P_{n-2,m-2}^{(3/2)} = -n^2\,\Psi\,P_{n-2,m-2}^{(3/2)}.$$

Hence, for $2 \leqslant m \leqslant n$, $P_{n,m}^{(-1/2)}(x,y) = (1-x^2-y^2)\,P_{n-2,m-2}^{(3/2)}(x,y)$.

Now, we deduce the orthogonality for the foregoing ball polynomials. Fix an integer $N \geqslant 1$. Then $\mathscr{L}_{-1/2}$ is symmetric with respect to the Sobolev bilinear form (3.1) if, for $0 \leqslant j \leqslant i \leqslant N$, the moment functionals in the bilinear form satisfy the Pearson equations

$$\begin{aligned}\partial_x\left((1-x^2)\,\mathbf{u}^{(i,j)}\right) - \partial_y\left(x\,y\,\mathbf{u}^{(i,j)}\right) &= -(1+2\,i)\,x\,\mathbf{u}^{(i,j)},\\ -\partial_x\left(x\,y\,\mathbf{u}^{(i,j)}\right) + \partial_y\left((1-y^2)\,\mathbf{u}^{(i,j)}\right) &= -(1+2\,i)\,y\,\mathbf{u}^{(i,j)}.\end{aligned}$$

For $1 \leqslant i \leqslant N$ and $0 \leqslant j \leqslant i$, we have that $\mathbf{u}^{(i,j)} \equiv \mathbf{w}_{i-1/2}$ satisfies the Pearson equations. Therefore,

$$\begin{aligned}\langle \mathbf{u}^{(i,j)}, P\rangle &= \langle \mathbf{w}_{i-1/2}, P\rangle\\ &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} P(x,y)\,(1-x^2-y^2)^{i-1}\,dxdy, \quad \forall P \in \Pi^2.\end{aligned}$$

For $i = j = 0$, we seek univariate moment functionals $\mathbf{u}^{(s)}$ and $\mathbf{v}^{(t)}$ such that, if possible, we can construct $\mathbf{u}^{(0,0)}$ using (4.5). From Theorem 4.1 and Corollary 4.4 with $\rho(s) = \sqrt{1-s^2}$, we can deduce that $\mathbf{u}^{(0,0)}$ can be constructed using (4.5) with univariate linear functionals $\mathbf{u}_0^{(s)}$ and $\mathbf{v}^{(t)}$ satisfying

$$(1-s^2)\,D\,\mathbf{u}_0^{(s)} = s\,\mathbf{u}_0^{(s)},$$

and

$$(1-t^2)\,D\,\mathbf{v}^{(t)} = 2\,t\,\mathbf{v}^{(t)}.$$

Then, $\mathbf{u}_0^{(s)} \equiv \mathbf{u}_{-1/2,-1/2}$ and $\mathbf{v}^{(t)} = \boldsymbol{\delta}(1+t) + \boldsymbol{\delta}(1-t)$. It follows that

$$\langle \mathbf{u}^{(0,0)}, P\rangle = \int_0^{2\pi} P(\cos\theta, \sin\theta)\,d\theta.$$

Therefore, $\mathscr{L}_{-1/2}$ is symmetric with respect to the Sobolev bilinear form

$$\begin{aligned}(P,Q)_{-1/2} =& \int_0^{2\pi} P(\cos\theta,\sin\theta)\,Q(\cos\theta,\sin\theta)\,d\theta\\ &+ \sum_{i=1}^{N}\left\langle \mathbf{w}_{i-1/2}, \sum_{j=0}^{i}\partial_x^{i-j}\partial_y^{j}P\ \partial_x^{i-j}\partial_y^{j}Q\right\rangle, \quad \forall P, Q \in \Pi^2.\end{aligned}$$

Since $\lambda_n^{(-1/2)} \neq \lambda_m^{(-1/2)}$ for $n \neq m$, from Proposition 3.5 we deduce that the ball polynomials with $\mu = -1/2$ are Sobolev orthogonal with respect to $(\cdot, \cdot)_{-1/2}$.

Remark 6.1 As we mentioned above, for $\mu_k = -k - 1/2$, $k = 0, 1, 2, \ldots$, it was shown in Piñar and Xu (2009) that $\mathscr{L}_{\mu_k}[\,p\,] = \lambda_n^{(\mu_k)}\, p$ with $k \geqslant 1$ is not guaranteed to have a complete PS solution. Indeed, we encounter an issue if we try to construct such PS solutions. In this case, (6.1) reads

$$(1-x^2)\,(p_n^{(m)}(x))'' - (2\,\mu_k + 2\,m + 2)\,x\,(p_n^{(m)}(x))' = -n\,(n + 2\,\mu_k + 2m + 1)\,p_n^{(m)}(x).$$

Then, $p_n^{(m)}(x) = P_n^{(m-k-1/2,m-k-1/2)}(x)$. However, a reduction of the degree of $P_n^{(m-k-1/2,m-k-1/2)}(x)$, $n \geqslant 1$, occurs if and only if $-(2\,m - 2\,k - 1 + n) \in \{1, 2, \ldots, n\}$. For instance, for $k \geqslant 1$,

$$P_{2k,0}^{(\mu_k)}(x, y) = P_{2k}^{(-k-1/2,-k-1/2)}(x),$$

is not a polynomial of degree $2k$ since $-(-2k - 1 + 2k) = 1$.

6.3 Orthogonal polynomials on the simplex

Consider the second-order linear partial differential operator

$$\begin{aligned}\mathscr{L}_{\alpha,\beta,\gamma}[\,p\,] &\equiv (1-x)\,x\,p_{xx} - 2\,x\,y\,p_{xy} + (1-y)\,y\,p_{yy}\\ &\quad + [\alpha + 1 - (\alpha+\beta+\gamma+3)\,x]\,p_x + [\beta + 1 - (\alpha+\beta+\gamma+3)\,y]\,p_y.\end{aligned}$$

From Theorem 5.2 with $a(x) = (1-x)\,x$, $\widetilde{b}(x) = -x$, $c(y) = (1-y)\,y$, $d(x) = \alpha + 1 - (\alpha+\beta+\gamma+3)\,x$, and $e(y) = \beta + 1 - (\alpha+\beta+\gamma+3)\,y$, we can deduce the explicit expression of polynomial solutions of $\mathscr{L}_{\alpha,\beta,\gamma}[p] = \lambda_n\,p$. To this end, let $\rho(x) = 1 - x$. Note that $\rho(x)$ satisfies $(1-x)\,x\,\rho'(x) = -x\,\rho(x)$. Then, if the univariate polynomial sequences $\{p_n^{(m)}(x)\}_{n\geqslant 0}$, $m \geqslant 0$, and $\{q_n(y)\}_{n\geqslant 0}$ satisfy

$$\begin{aligned}&(1-x)x(p_n^{(m)}(x))'' + [\alpha + 1 - (\alpha+\beta+\gamma+2\,m+3)\,x](p_n^{(m)}(x))'\\ &= -n\,(n+\alpha+\beta+\gamma+2\,m+2)\,p_n^{(m)}(x),\end{aligned} \tag{6.5}$$

and

$$(1-y)\,y\,q_n''(y) + [\beta + 1 - (\beta+\gamma+2)\,y]\,q_n'(y) = -n\,(n+\beta+\gamma+1)\,q_n(y), \tag{6.6}$$

respectively, we have that the simplex polynomials defined by

$$P_{n,m}^{(\alpha,\beta,\gamma)}(x, y) = p_{n-m}^{(m)}(x)\,(1-x)^m\,q_m\left(\frac{y}{1-x}\right), \quad n \geqslant 0, \quad 0 \leqslant m \leqslant n,$$

satisfy $\mathscr{L}_{\alpha,\beta,\gamma}\left[P_{n,m}^{(\alpha,\beta,\gamma)}\right] = \lambda_n^{(\alpha,\beta,\gamma)}\,P_{n,m}^{(\alpha,\beta,\gamma)}$ with $\lambda_n^{(\alpha,\beta,\gamma)} = -n\,(n+\alpha+\beta+\gamma+2)$.

If for $n \geqslant 0$, we denote by $\mathbb{P}_{n,\alpha,\beta,\gamma}$ the column vector

$$\mathbb{P}_{n,\alpha,\beta,\gamma} = (P_{n,0}^{(\alpha,\beta,\gamma)}(x, y), P_{n,1}^{(\alpha,\beta,\gamma)}(x, y), \ldots, P_{n,n}^{(\alpha,\beta,\gamma)}(x, y))^{\top},$$

then the PS $\{\mathbb{P}_{n,\alpha,\beta,\gamma}\}_{n\geqslant 0}$ satisfies

$$\mathscr{L}_{\alpha,\beta,\gamma}\left[\mathbb{P}_{n,\alpha,\beta,\gamma}\right] = \Lambda_n^{(\alpha,\beta,\gamma)}\,\mathbb{P}_{n,\alpha,\beta,\gamma},$$

with $\Lambda_n^{(\alpha,\beta,\gamma)} = \lambda_n^{(\alpha,\beta,\gamma)}\,I_{n+1}$.

Now, we study the simplex polynomials and their orthogonality in terms of the parameter γ. We consider the positive definite case when $\gamma > -1$, and the singular case when $-\gamma \in \mathbb{N}$.

6.3.1 Positive definite case

For $\alpha, \beta, \gamma > -1$, we have that

$$p_n^{(m)}(x) = P_n^{(\beta+\gamma+2m+1,\alpha)}(2x-1), \quad n, m \geqslant 0,$$

and

$$q_n(y) = P_n^{(\gamma,\beta)}(2y-1), \quad n \geqslant 0,$$

satisfy (6.5) and (6.6), respectively. Therefore, in this case, the simplex polynomials are given by

$$P_{n,m}^{(\alpha,\beta,\gamma)}(x,y) = P_n^{(\beta+\gamma+2m+1,\alpha)}(2x-1)\,(1-x)^m\, P_n^{(\gamma,\beta)}\left(\frac{2y}{1-x}-1\right).$$

The orthogonality of the positive definite simplex polynomials can be deduced as follows. By Theorem 3.3 (and Remark 3.6), we have that if the bivariate moment functional $\mathbf{w}_{\alpha,\beta,\gamma}$ satisfies the Pearson equations

$$\begin{aligned}\partial_x((1-x)x\mathbf{w}_{\alpha,\beta,\gamma}) - \partial_y(x\,y\,\mathbf{w}_{\alpha,\beta,\gamma}) &= (\alpha+1-(\alpha+\beta+\gamma+3)\,x)\ \mathbf{w}_{\alpha,\beta,\gamma},\\ -\partial_x(x\,y\,\mathbf{w}_{\alpha,\beta,\gamma}) + \partial_y((1-y)\,y\,\mathbf{w}_{\alpha,\beta,\gamma}) &= (\beta+1-(\alpha+\beta+\gamma+3)\,y)\ \mathbf{w}_{\alpha,\beta,\gamma},\end{aligned}$$

then $\mathscr{L}_{\alpha,\beta,\gamma}$ satisfies

$$\langle \mathbf{w}_{\alpha,\beta,\gamma}, \mathscr{L}_{\alpha,\beta,\gamma}[\,P\,]\,Q\rangle = \langle \mathbf{w}_{\alpha,\beta,\gamma}, P\,\mathscr{L}_{\alpha,\beta,\gamma}[\,Q\,]\rangle, \quad \forall P, Q \in \Pi^2.$$

Since $\lambda_n^{(\alpha,\beta,\gamma)} \neq \lambda_m^{(\alpha,\beta,\gamma)}$ for $n \neq m$, $\Lambda_n^{(\alpha,\beta,\gamma)}$ and $\Lambda_m^{(\alpha,\beta,\gamma)}$ do not share eigenvalues. Then, from Proposition 3.5, we have that the PS $\{\mathbb{P}_{n,\alpha,\beta,\gamma}\}_{n\geqslant 0}$ satisfies the orthogonality condition $\langle \mathbf{w}_{\alpha,\beta,\gamma}, \mathbb{P}_{n,\alpha,\beta,\gamma}\,\mathbb{P}_{m,\alpha,\beta,\gamma}^{\top}\rangle = 0$, $n \neq m$.

In the same way as in the previous example, we seek univariate moment functionals $\mathbf{u}^{(s)}$ and $\mathbf{v}^{(t)}$ such that, if possible, we can construct $\mathbf{w}_{\alpha,\beta,\gamma}$ using (4.5).

First, we rewrite the forgoing Pearson equations as

$$(1-x)\,x\,\partial_x\,\mathbf{w}_{\alpha,\beta,\gamma} - x\,y\,\partial_y\,\mathbf{w}_{\alpha,\beta,\gamma} = \big[\alpha-(\alpha+\beta+\gamma)\,x\big]\ \mathbf{w}_{\alpha,\beta,\gamma}, \tag{6.7}$$

$$-x\,y\,\partial_x\,\mathbf{w}_{\alpha,\beta,\gamma} + (1-y)\,y\,\partial_y\,\mathbf{w}_{\alpha,\beta,\gamma} = \big[\beta-(\alpha+\beta+\gamma)\,y\big]\ \mathbf{w}_{\alpha,\beta,\gamma}. \tag{6.8}$$

From Theorem 4.1 with $a(x) = (1-x)\,x$, $\widetilde{b}(x) = -x$, $\widetilde{d}(x) = \alpha-(\alpha+\beta+\gamma)\,x$, and $\rho(s) = 1-s$, we get that if the univariate moment functional $\mathbf{u}_0^{(s)} = \rho(s)\mathbf{u}^{(s)}$ satisfies

$$(1-s)\,s\,D\,\mathbf{u}_0^{(s)} = \big[\alpha-(\alpha+\beta+\gamma+1)\,s\big]\ \mathbf{u}_0^{(s)}$$

then, $\mathbf{w}_{\alpha,\beta,\gamma}$ satisfies (6.7). We can choose $\mathbf{u}_0^{(s)}$ to be the Jacobi linear functional on $[0, 1]$ defined by

$$\left\langle \mathbf{u}_0^{(s)}, p(s)\right\rangle = \int_0^1 p(s)\,(1-s)^{\beta+\gamma+1}\,s^{\alpha}\,ds.$$

Therefore, we let $\mathbf{u}^{(s)}$ be the Jacobi linear functional on $[0, 1]$

$$\left\langle \mathbf{u}^{(s)}, p(s)\right\rangle = \int_0^1 p(s)\,(1-s)^{\beta+\gamma}\,s^{\alpha}\,ds,$$

since it satisfies $\mathbf{u}_0^{(s)} = \rho(s)\,\mathbf{u}^{(s)}$.

From Corollary 4.3 with $\mathbf{u}_0^{(s)}$ as above, $\rho(s) = 1 - s$, $\widetilde{b}(x) = -x$, $c(y) = (1 - y)\, y$, and $\widetilde{e}(y) = \beta - (\alpha + \beta + \gamma)\, y$, we have that if $\mathbf{v}^{(t)}$ satisfies

$$(1 - t)\, t\, D\, \mathbf{v}^{(t)} = \big[\beta - (\beta + \gamma)\, t\big]\, \mathbf{v}^{(t)},$$

then $\mathbf{w}_{\alpha,\beta,\gamma}$ satisfies (6.8). Clearly, $\mathbf{v}^{(t)}$ can be chosen to be the Jacobi moment functional on $[0, 1]$

$$\left\langle \mathbf{v}^{(t)}, q(t) \right\rangle = \int_0^1 q(t)\, (1 - t)^{\gamma} t^{\beta}\, dt.$$

From (4.5) with $\mathbf{u}_0^{(s)} = \rho(s)\, \mathbf{u}^{(s)}$ and $\mathbf{v}^{(t)}$ as above, we obtain the moment functional $\mathbf{w}_{\alpha,\beta,\gamma}$ defined as

$$\left\langle \mathbf{w}_{\alpha,\beta,\gamma}, P \right\rangle = \int_0^1 \int_0^{1-x} P(x, y)\, x^{\alpha}\, y^{\beta}\, (1 - x - y)^{\gamma}\, dx dy, \quad \forall P \in \Pi^2.$$

Observe that $\mathbf{w}_{\alpha,\beta,\gamma}$ is defined on the simplex $\mathbf{T}^2 = \{(x, y) \in \mathbb{R}^2 : x \geqslant 0,\ y \geqslant 0,\ 1 - x - y \geqslant 0\}$. In fact, for $\alpha, \beta, \gamma > -1$, $\mathbf{w}_{\alpha,\beta,\gamma}$ is a positive definite moment functional and, thus,

$$\langle \mathbf{w}_{\alpha,\beta,\gamma}, P_{n,m}^{(\alpha,\beta,\gamma)}\, P_{k,j}^{(\alpha,\beta,\gamma)} \rangle = H_{n,m}^{(\alpha,\beta,\gamma)}\, \delta_{n,k}\, \delta_{m,j},$$

where $H_{n,m}^{(\alpha,\beta,\gamma)} > 0$.

Moreover, observe that $\mathbf{u}_0^{(s)}$ and $\mathbf{v}^{(t)}$ satisfy the sufficient conditions of Theorem 4.6 with $c_0 = 0$, $c_1 \neq 0$, $a(s) = s$, $b(t) = 1 - t$, and $c(t) = t$. Indeed, $\mathbf{w}_{\alpha,\beta,\gamma}$ satisfies the diagonal Pearson equations

$$\begin{aligned} x\, (1 - x - y)\, \partial_x\, \mathbf{w}_{\alpha,\beta,\gamma} &= \big[\alpha\, (1 - x - y) - \gamma\, x\big]\, \mathbf{w}_{\alpha,\beta,\gamma}, \\ y\, (1 - x - y)\, \partial_y\, \mathbf{w}_{\alpha,\beta,\gamma} &= \big[\beta\, (1 - x - y) - \gamma\, y\big]\, \mathbf{w}_{\alpha,\beta,\gamma}. \end{aligned}$$

From Theorem 3.3, we deduce that $\{\mathbb{P}_{n,\alpha,\beta,\gamma}\}_{n\geqslant 0}$ is a Sobolev orthogonal PS. Indeed, fix an integer $N \geqslant 0$. In this case, for $0 \leqslant j \leqslant i \leqslant N$, the Pearson equations (3.3) read

$$\begin{aligned} \partial_x \left((1 - x)\, x\, \mathbf{u}^{(i,j)}\right) - \partial_y \left(x\, y\, \mathbf{u}^{(i,j)}\right) &= \left(\alpha_{i,j} + 1 - [\alpha_{i,j} + \beta_j + \gamma_i + 3]\, x\right) \mathbf{u}^{(i,j)}, \\ -\partial_x \left(x\, y\, \mathbf{u}^{(i,j)}\right) + \partial_y \left((1 - y)\, y\, \mathbf{u}^{(i,j)}\right) &= \left(\beta_j + 1 - [\alpha_{i,j} + \beta_j + \gamma_i + 3]\, y\right) \mathbf{u}^{(i,j)}, \end{aligned}$$

where $\alpha_{i,j} = \alpha + i - j$, $\beta_j = \beta + j$, and $\gamma_i = \gamma + i$. Therefore, $\mathbf{u}^{(i,j)} = \mathbf{w}_{\alpha_{i,j},\beta_j,\gamma_i}$, and by Theorem 3.3, $\mathscr{L}_{\alpha,\beta,\gamma}$ is symmetric with respect to the Sobolev bilinear form

$$(P, Q)_S = \sum_{i=0}^{N} \sum_{j=0}^{i} \left\langle \mathbf{w}_{\alpha_{i,j},\beta_j,\gamma_i}, \partial_x^{i-j} \partial_y^{j} P\ \partial_x^{i-j} \partial_y^{j} Q \right\rangle, \quad \forall P, Q \in \Pi^2.$$

The orthogonality of the simplex polynomials with respect to $(\cdot, \cdot)_S$ follows from Proposition 3.5 since $\lambda_n^{(\alpha,\beta,\gamma)} \neq \lambda_m^{(\alpha,\beta,\gamma)}$ for $n \neq m$.

6.3.2 Singular case

For $\alpha, \beta, \gamma > -1$, the polynomial solutions of $\mathscr{L}_{\alpha,\beta,\gamma}[\,p\,] = \lambda_n^{(\alpha,\beta,\gamma)}\, p$ described above are orthogonal with respect to a positive definite moment functional defined on the unit simplex $\mathbf{T}^2$.

However, for $\gamma = -k$, $k \in \mathbb{N}$, $P(x, y)x^{\alpha} y^{\beta}(1 - x - y)^{\gamma}$ is no longer integrable for all polynomials $P \in \Pi^2$, and, therefore, $\mathbf{w}_{\alpha,\beta,-k}$ is not guaranteed to be a quasi-definite linear functional. In spite of this, Sobolev orthogonality can be provided for the simplex polynomials satisfying $\mathscr{L}_{\alpha,\beta,-k}[\,P^{(\alpha,\beta,-k)}_{n,m}\,] = \lambda^{(\alpha,\beta,-k)}_n\, P^{(\alpha,\beta,-k)}_{n,m}$ with $\lambda^{(\alpha,\beta,-k)}_n = -n\,(n+\alpha+\beta-k+2)$. First, we construct such polynomial solutions.

For $-\alpha, -\beta, -(\alpha + \beta + 1) \notin \mathbb{N}$, we have that $\{p^{(m)}_n(x)\}_{n\geqslant 0}$, $m \geqslant 0$, with $p^{(m)}_n(x) = P^{(\beta+2m-k+1,\alpha)}_n(2\,x - 1)$, and $\{q_n(y)\}_{n\geqslant 0}$ with

$$q_n(y) = \begin{cases} P^{(-k,\beta)}_n(2y-1), & 0 \leqslant n \leqslant k-1, \\ (1-y)^k\, P^{(k,\beta)}_{n-k}(2y-1), & k \leqslant n, \end{cases}$$

satisfy (6.7) and (6.8) (see Garcia-Ardila and Marriaga (2021) and the references therein), respectively. Then, in this case, the simplex polynomials are given by

$$\begin{aligned} &P^{(\alpha,\beta,-k)}_{n,m}(x, y) \\ &= P^{(\beta+2m-k+1,\alpha)}_{n-m}(2\,x-1)\,(1-x)^m\, P^{(-k,\beta)}_m\left(\frac{2y}{1-x}-1\right), \quad 0 \leqslant m \leqslant n \leqslant k-1, \end{aligned}$$

and for $n \geqslant k$,

$$\begin{aligned} &P^{(\alpha,\beta,-k)}_{n,m}(x, y) \\ &= P^{(\beta+2m-k+1,\alpha)}_{n-m}(2\,x-1)\,(1-x)^m\, P^{(-k,\beta)}_m\left(\frac{2y}{1-x}-1\right), \quad 0 \leqslant m \leqslant k-1, \\ &P^{(\alpha,\beta,-k)}_{n,m}(x, y) \\ &= (1-x-y)^k\, P^{(\beta+2m-k+1,\alpha)}_{n-m}(2x-1)(1-x)^{m-k}\, P^{(k,\beta)}_{m-k}\left(\frac{2y}{1-x}-1\right), \quad k \leqslant m \leqslant n. \end{aligned}$$

The explicit expression of $P^{(\alpha,\beta,-k)}_{n,m}(x, y)$, $k \leqslant m \leqslant n$, can also be deduced from Proposition 5.1. Consider the polynomial $\Psi(x, y) = (1 - x - y)^k$, and note that it satisfies the differential equation

$$\mathscr{L}_{\alpha,\beta,-k}[\Psi] = -k\,(\alpha+\beta+2)\,\Psi,$$

and the Pearson equations

$$\begin{aligned} (1-x)\,x\,\Psi_x - x\,y\,\Psi_y &= -k\,x\,\Psi, \\ -x\,y\,\Psi_x + (1-y)\,y\,\Psi_y &= -k\,y\,\Psi. \end{aligned}$$

Since the simplex polynomials $P^{(\alpha,\beta,k)}_{n,m}(x, y)$ satisfy

$$\begin{aligned} \mathscr{L}_{\alpha,\beta,k}[P^{(\alpha,\beta,k)}_{n,m}] &= \mathscr{L}_{\alpha,\beta,-k}[P^{(\alpha,\beta,k)}_{n,m}] - 2\,k\,x\,\partial_x P^{(\alpha,\beta,k)}_{n,m} - 2\,k\,y\,\partial_y P^{(\alpha,\beta,k)}_{n,m} \\ &= -n\,(n+\alpha+\beta+k+2)\,P^{(\alpha,\beta,k)}_{n,m}, \end{aligned}$$

we have that for $k \leqslant m \leqslant n$,

$$\begin{aligned} \mathscr{L}_{\alpha,\beta,-k}[\Psi\, P^{(\alpha,\beta,k)}_{n-k,m-k}] &= [-k\,(\alpha+\beta+2) - (n-k)\,(n+\alpha+\beta+2)]\,\Psi\, P^{(\alpha,\beta,k)}_{n-k,m-k} \\ &= -n\,(n+\alpha+\beta-k+2)\,\Psi\, P^{(\alpha,\beta,k)}_{n-k,m-k}. \end{aligned}$$

Hence, $P^{(\alpha,\beta,-k)}_{n,m}(x, y) = (1-x-y)^k P^{(\alpha,\beta,k)}_{n-k,m-k}(x, y)$ for $k \leqslant m \leqslant n$.

Now, we deduce the Sobolev orthogonality for the singular simplex polynomials. With this in mind, we determine the moment functionals involved in (3.1) such that $\mathscr{L}_{\alpha,\beta,-k}$ is symmetric with respect to the Sobolev bilinear form.

Fix an integer $N \geqslant k$. From Theorem 3.3, if the moment functional $\mathbf{u}^{(i,j)}$, $0 \leqslant j \leqslant i \leqslant N$, satisfy the Pearson equations

$$(1-x)\,x\,\partial_x \mathbf{u}^{(i,j)} - x\,y\,\partial_y \mathbf{u}^{(i,j)} = \big(\alpha_{i,j} - [\alpha_{i,j} + \beta_j + i - k]\,x\big)\,\mathbf{u}^{(i,j)},$$
$$-x\,y\,\partial_x \mathbf{u}^{(i,j)} + (1-y)\,y\,\partial_y \mathbf{u}^{(i,j)} = \big(\beta_j - [\alpha_{i,j} + \beta_j + i - k]\,y\big)\,\mathbf{u}^{(i,j)},$$

where $\alpha_{i,j} = \alpha + i - j$, and $\beta_j = \beta + j$, then $\mathscr{L}_{\alpha,\beta,-k}$ is symmetric with respect to (3.1). For $k \leqslant j \leqslant i \leqslant N$, we have $\mathbf{u}^{(i,j)} = \mathbf{w}_{\alpha_{i,j},\beta_j,i-k}$. Therefore

$$\big\langle \mathbf{w}_{\alpha_{i,j},\beta_j,i-k}, P \big\rangle = \int_0^1 \int_0^{1-x} P(x,y)\, x^{\alpha+i-j}\, y^{\beta+j}\, (1-x-y)^{i-k}\, dxdy, \quad \forall P \in \Pi^2.$$

On the other hand, for $0 \leqslant j \leqslant i \leqslant k-1$, we can use Theorem 4.1 and Corollary 4.3 with $\rho(s) = 1 - s$ to deduce the moment functional $\mathbf{u}^{(i,j)}$. In order to construct $\mathbf{u}^{(i,j)}$ using (4.5), we must find univariate functionals $\mathbf{u}^{(s)}_{0,i,j} = \rho(s)\,\mathbf{u}^{(s)}_{i,j}$ and $\mathbf{v}^{(t)}_{i,j}$ satisfying

$$(1-s)\,s\,D\,\mathbf{u}^{(s)}_{0,i,j} = \big[\alpha_{i,j} - (\alpha_{i,j} + \beta_j + i - k + 1)\,s\big]\,\mathbf{u}^{(s)}_{0,i,j},$$

and

$$(1-t)\,t\,D\,\mathbf{v}^{(t)}_{i,j} = [\beta_j - (\beta_j + i - k)\,t]\,\mathbf{v}^{(t)}_{i,j},$$

respectively. Clearly,

$$\Big\langle \mathbf{u}^{(s)}_{0,i,j}, p(s) \Big\rangle = \int_0^1 p(s)\,(1-s)^{\beta_j+i-k+1}\, s^{\alpha_{i,j}}\, ds,$$

and

$$\Big\langle \mathbf{u}^{(s)}_{i,j}, p(s) \Big\rangle = \int_0^1 p(s)\,(1-s)^{\beta_j+i-k}\, s^{\alpha_{i,j}}\, ds.$$

Notice that $\mathbf{u}^{(s)}_{i,j}$ and $\mathbf{u}^{(s)}_{0,i,j}$ are quasi-definite moment functionals, but *a priori* not positive definite.

Moreover, $\mathbf{v}^{(t)}_{i,j}$ satisfies

$$\Big\langle \mathbf{v}^{(t)}_{i,j}, (1+\beta_j)\,(1-t)^{n+1} - (n+1+i-k)\,t\,(1-t)^n \Big\rangle = 0, \quad n \geqslant 0.$$

Hence, we deduce that

$$\Big\langle \mathbf{v}^{(t)}_{i,j}, p \Big\rangle = v_{i,j} \sum_{\nu=0}^{k-1-i} \frac{(-1)^\nu (1+i-k)_\nu}{\prod_{m=1}^{\nu} [1 + \beta_j + m\,(m+i-k)]}\, p^{(\nu)}(1),$$

where $v_{i,j}$ is a free parameter. It follows that, for $0 \leqslant j \leqslant i \leqslant k-1$,

$$\langle \mathbf{u}^{(i,j)}, P \rangle = \sum_{\nu=0}^{k-1-i} w_{i,j}(\nu) \int_0^1 \Big[\partial_y^\nu P\Big](x, 1-x)\, x^{\alpha_{i,j}}\, (1-x)^{\beta_j+i-k+\nu+1} dx,$$

where

$$w_{i,j}(\nu) = \frac{(-1)^\nu\,(1+i-k)_\nu\, v_{i,j}}{\prod_{m=1}^{\nu} [1 + \beta_j + m\,(m+i-k)]}.$$

Thus, by Theorem 3.3, the differential operator $\mathscr{L}_{\alpha,\beta,-k}$ is symmetric with respect to the Sobolev bilinear form

$$
\begin{aligned}
&(P, Q)_{\alpha,\beta,-k} \\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{i}\sum_{\nu=0}^{k-1-i} w_{i,j}(\nu)\int_0^1 \left[\partial_y^{\nu}\left(\partial_x^{i-j}\partial_y^{j}P\ \partial_x^{i-j}\partial_y^{j}Q\right)\right](x,1-x)\,x^{\alpha_{i,j}}\,(1-x)^{\beta_j+i-k+\nu+1}dx \\
&\quad + \sum_{i=k}^{N}\sum_{j=0}^{i}\left\langle \mathbf{w}_{\alpha_{i,j},\beta_j,i-k}, \partial_x^{i-j}\partial_y^{j}P\ \partial_x^{i-j}\partial_y^{j}Q\right\rangle.
\end{aligned}
$$

The orthogonality of the singular simplex polynomials $\{\mathbb{P}_{n,\alpha,\beta,-k}\}_{n\geqslant 0}$ with respect to $(\cdot,\cdot)_{\alpha,\beta,-k}$ follows from Proposition 3.5 since $\lambda_n^{(\alpha,\beta,-k)} \neq \lambda_m^{(\alpha,\beta,-k)}$ for $n \neq m$.

6.4 Moment functional on the biangle

Consider the second-order linear partial differential operator

$$
\begin{aligned}
\mathscr{L}_{\alpha,\beta}[\,p\,] \equiv (1-x)\,x\,p_{xx} + (1-x)\,y\,p_{xy} + \frac{1}{4}(1-y^2)\,p_{yy} \\
+ [\beta + 3/2 - (\alpha+\beta+5/2)\,x]\,p_x - \frac{1}{2}(\alpha+\beta+2)\,y\,p_y.
\end{aligned}
$$

Since $\rho(x) = \sqrt{x}$ is a solution of the differential equation $(1-x)x\rho'(x) = \frac{1}{2}(1-x)\rho(x)$, from Theorem 5.3 **(II)** we deduce that if the families of polynomials $\{p_n^{(m)}(x)\}_{n\geqslant 0}$, and $\{q_n(y)\}_{n\geqslant 0}$, satisfy the differential equations

$$
\begin{aligned}
(1-x)\,x\,(p_n^{(m)}(x))'' + [\beta + m + 3/2 - (\alpha+\beta+m+5/2)\,x]\,(p_n^{(m)}(x))' \\
= -n\,(n+\alpha+\beta+m+3/2)\,p_n^{(m)}(x),
\end{aligned} \tag{6.9}
$$

and

$$
(1-y^2)\,q_n''(y) - (2\,\beta+2)\,y\,q_n'(y) = -n\,(n+2\,\beta+1)\,q_n(y), \tag{6.10}
$$

respectively, then the biangle polynomials defined as

$$
P_{n,m}^{(\alpha,\beta)}(x,y) = p_{n-m}^{(m)}(x)\,\left(\sqrt{x}\right)^m\,q_m\left(\frac{y}{\sqrt{x}}\right), \quad n \geqslant 0, \quad 0 \leqslant m \leqslant n, \tag{6.11}
$$

satisfy $\mathscr{L}_{\alpha,\beta}[P_{n,m}^{(\alpha,\beta)}] = \lambda_{n,m}^{(\alpha,\beta)}\,P_{n,m}^{(\alpha,\beta)}$ with

$$
\lambda_{n,m}^{(\alpha,\beta)} = -(n-m)\,(n+\alpha+\beta+3/2) - \frac{1}{4}m\,(m+2\,\alpha+2\,\beta+3).
$$

If we denote by $\mathbb{P}_{n,\alpha,\beta}$, $n \geqslant 0$, the column vector

$$
\mathbb{P}_{n,\alpha,\beta} = (P_{n,0}^{(\alpha,\beta)}(x,y), P_{n,1}^{(\alpha,\beta)}(x,y), \ldots, P_{n,n}^{(\alpha,\beta)}(x,y))^{\top},
$$

then the PS $\{\mathbb{P}_{n,\alpha,\beta}\}_{n\geqslant 0}$ satisfies $\mathscr{L}_{\alpha,\beta,}\left[\mathbb{P}_{n,\alpha,\beta}\right] = \Lambda_n^{(\alpha,\beta)}\,\mathbb{P}_{n,\alpha,\beta}$, with

$$
\Lambda_n^{(\alpha,\beta)} = \operatorname{diag}[\lambda_{n,0}^{(\alpha,\beta)}, \lambda_{n,1}^{(\alpha,\beta)}, \ldots, \lambda_{n,n}^{(\alpha,\beta)}].
$$

In contrast to the previous cases, $\lambda_{n+1,m+2}^{(\alpha,\beta)} = \lambda_{n,m}^{(\alpha,\beta)}$ for $0 \leqslant m \leqslant n-1$. Then $\Lambda_{n+1}^{(\alpha,\beta)}$ and $\Lambda_{n}^{(\alpha,\beta)}$ share eigenvalues for $n \geqslant 1$. Therefore, the orthogonality cannot be deduced from Proposition 3.5.

As in the preceding examples, we will study the biangle polynomials in terms of the value of a parameter. Here, we will consider the parameter β in two cases: first the positive definite case, that is, when $\beta > -1$, and then when $-\beta \in \mathbb{N}$, which we call the singular case.

6.4.1 Positive definite case

For $\alpha, \beta > -1$, the univariate polynomial sequences $\{p_n^{(m)}(x)\}_{n\geqslant 0}$, and $\{q_n^{(m)}(x)\}_{n\geqslant 0}$ with

$$p_n^{(m)}(x) = P_n^{(\alpha,\beta+m+1/2)}(2x-1), \quad m, n \geqslant 0,$$

and

$$q_n(y) = P_n^{(\beta,\beta)}(y), \quad n \geqslant 0,$$

satisfy (6.9) and (6.10), respectively. Therefore, in this case, the biangle polynomials are given by

$$P_{n,m}^{(\alpha,\beta)}(x,y) = P_{n-m}^{(\alpha,\beta+m+1/2)}(2x-1)\,(\sqrt{x})^m\, P_m^{(\beta,\beta)}\left(\frac{y}{\sqrt{x}}\right).$$

Recall that the orthogonality of the biangle polynomials cannot be deduced from Proposition 3.5. However, studying the symmetry of $\mathscr{L}_{\alpha,\beta}$ with respect to a bilinear form provides information about the orthogonality.

Let $\mathbf{w}_{\alpha,\beta}$ the a moment functional satisfying the Pearson equations

$$(1-x)\,x\,\partial_x \mathbf{w}_{\alpha,\beta} + \frac{1}{2}(1-x)\,y\,\partial_y \mathbf{w}_{\alpha,\beta} = [\beta - (\alpha+\beta)\,x]\,\mathbf{w}_{\alpha,\beta}, \tag{6.12}$$

$$\frac{1}{2}(1-x)\,y\,\partial_x \mathbf{w}_{\alpha,\beta} + \frac{1}{4}(1-y^2)\,\partial_y \mathbf{w}_{\alpha,\beta} = -\frac{1}{2}(\alpha+\beta)\,y\;\mathbf{w}_{\alpha,\beta}. \tag{6.13}$$

Then, by Theorem 3.3 (and Remark 3.6), we have that the following symmetry condition holds,

$$\langle \mathbf{w}_{\alpha,\beta}, \mathscr{L}_{\alpha,\beta}[P]\,Q\rangle = \langle \mathbf{w}_{\alpha,\beta}, P\,\mathscr{L}_{\alpha,\beta}[Q]\rangle, \quad \forall P, Q \in \Pi^2.$$

The expression of the moment functional $\mathbf{w}_{\alpha,\beta}$ satisfying (6.12) and (6.13) constructed with (4.5) using univariate moment functionals $\mathbf{u}_0^{(s)}$ and $\mathbf{v}^{(t)}$ can be deduced from Theorem 4.1 and Corollary 4.3 with $a(x) = (1-x)\,x$, $\widetilde{b}(x) = \frac{1}{2}(1-x)$, $c(y) = \frac{1}{4}(1-y^2)$, $\widetilde{d}(x) = \beta - (\alpha+\beta)\,x$, $\widetilde{e}(y) = -\frac{1}{2}(\alpha+\beta)\,y$, and $\rho(x) = \sqrt{x}$ as follows.

First, if the moment functional $\mathbf{u}_0^{(s)}$ satisfies the Pearson equation

$$(1-s)\,s\,D\mathbf{u}_0^{(s)} = \left[\beta + \frac{1}{2} - \left(\alpha+\beta+\frac{1}{2}\right)s\right]\mathbf{u}_0^{(s)},$$

then, by Theorem 4.1, $\mathbf{w}_{\alpha,\beta}$ satisfies (6.12). Clearly, $\mathbf{u}_0^{(s)}$ is the Jacobi linear functional on $[0,1]$ defined as

$$\left\langle \mathbf{u}_0^{(s)}, p(s)\right\rangle = \int_0^1 p(s)\,(1-s)^\alpha\, s^{\beta+1/2}\,ds,$$

and the moment functional $\mathbf{u}^{(s)}$ defined as

$$\left\langle \mathbf{u}^{(s)}, p(s) \right\rangle = \int_0^1 p(s)\,(1-s)^\alpha\, s^\beta\, ds,$$

satisfies $\mathbf{u}_0^{(s)} = \rho(s)\,\mathbf{u}^{(s)}$. Additionally, if the moment functional $\mathbf{v}^{(t)}$ satisfies the Pearson equation

$$\left(1-t^2\right)\, D\,\mathbf{v}^{(t)} = -2\,\beta\, t\,\mathbf{v}^{(t)},$$

then $\mathbf{w}_{\alpha,\beta,\gamma}$ satisfies (6.13). We can take $\mathbf{v}^{(t)}$ to be the Jacobi moment functional $\mathbf{u}_{\beta,\beta}$. Thus, from (4.5) we get

$$\left\langle \mathbf{w}_{\alpha,\beta}, P(x,y) \right\rangle = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} P(x,y)\,(1-x)^\alpha\,(x-y^2)^\beta\, dx dy.$$

Notice that $\mathbf{w}_{\alpha,\beta}$ is defined on the parabolic biangle defined as

$$\Omega = \{(x,y) \in \mathbb{R}^2 \,:\, y^2 \leqslant x \leqslant 1\}.$$

Observe that, for $m \geqslant 0$, the Jacobi polynomials $p_n^{(m)} = P_n^{(\alpha,\beta+m+1/2)}(2x-1)$, $n \geqslant 0$, are orthogonal with respect to moment functional $\mathbf{u}_m^{(s)} = \rho(s)^{2m+1}\mathbf{u}^{(s)}$, and the Jacobi polynomials $q_n(y) = P_n^{(\beta,\beta)}(y)$, $n \geqslant 0$, are orthogonal with respect to $\mathbf{v}^{(t)}$. Hence, the biangle polynomials $\{\mathbb{P}_{n,\alpha,\beta}\}_{n\geqslant 0}$ are mutually orthogonal with respect to $\mathbf{w}_{\alpha,\beta}$ (see Sect. 4.1). In fact, for $\alpha, \beta > -1$, $\mathbf{w}_{\alpha,\beta}$ is a positive definite moment functional and, thus,

$$\langle \mathbf{w}_{\alpha,\beta}, P_{n,m}^{(\alpha,\beta)}\, P_{k,j}^{(\alpha,\beta)} \rangle = H_{n,m}^{(\alpha,\beta)}\, \delta_{n,k}\,\delta_{m,j},$$

with $H_{n,m}^{(\alpha,\beta)} > 0$.

Now we turn our attention to the Sobolev orthogonality of the foregoing biangle polynomials. Fix some integer $N \geqslant 0$. First, note that the differential equation $\mathscr{L}_{\alpha,\beta}[p] = \lambda_{n,m}^{(\alpha,\beta)}\, p$ belongs to the extended Lyskova class. Moreover, the polynomial $g(x,y) = x - y^2$ satisfies the equations

$$\begin{aligned}(1-x)\,x\,g_x + \frac{1}{2}(1-x)\,y\,g_y &= (1-x)\,g,\\ \frac{1}{2}(1-x)\,y\,g_x + \frac{1}{4}(1-y^2)\,g_y &= -\tfrac{1}{2}y\,g,\end{aligned}$$

and the system of equations

$$\begin{aligned}(1-x)\,x\,f_x + \frac{1}{2}(1-x)\,y\,f_y &= (1-2\,x)\,f,\\ \frac{1}{2}(1-x)\,y\,f_x + \frac{1}{4}(1-y^2)\,f_y &= -y\,f,\end{aligned}$$

has only the trivial solution in Π^2. Therefore, by Corollary 2.8, the biangle polynomials are Sobolev orthogonal with respect to the bilinear form

$$(P,Q)_{\alpha,\beta} = \sum_{i=0}^{N} \left\langle \mathbf{w}_{\alpha,\beta+i}, \partial_y^i P\, \partial_y^i Q \right\rangle, \quad \forall P, Q \in \Pi^2,$$

where, obviously, $\mathbf{w}_{\alpha,\beta+i} = (x-y^2)^i\,\mathbf{w}_{\alpha,\beta}$, $0 \leqslant i \leqslant N$. Furthermore, these moment functionals satisfy the Pearson equations

$$\partial_x((1-x)\,x\,\mathbf{w}_{\alpha,\beta_j}) + \frac{1}{2}\partial_y((1-x)\,y\,\mathbf{w}_{\alpha,\beta_j}) = \left(\beta_j + 3/2 - (\alpha+\beta_j+5/2)\,x\right)\mathbf{w}_{\alpha,\beta_j},$$

$$\frac{1}{2}\partial_x((1-x)\,y\,\mathbf{w}_{\alpha,\beta_j})+\frac{1}{4}\partial_y((1-y)^2\,\mathbf{w}_{\alpha,\beta_j})=-\frac{1}{2}(\alpha+\beta_j+2)\,y\,\mathbf{w}_{\alpha,\beta_j},$$

where $\beta_j=\beta+j$. Therefore, by Theorem 3.3, we have that $\mathscr{L}_{\alpha,\beta}$ is symmetric with respect to $(\cdot,\cdot)_{\alpha,\beta}$.

6.4.2 Singular case

For $\alpha,\beta>-1$, the polynomial solutions of $\mathscr{L}_{\alpha,\beta}[\,p\,]=\lambda_{n,m}^{(\alpha,\beta)}\,p$ described in the previous example, are orthogonal with respect to a linear functional defined on the biangle Ω.

However, for $\beta=-1$, $P(x,y)(1-x)^\alpha(x-y^2)^\beta$ is no longer integrable for all polynomials $P\in\Pi^2$, and, therefore, $\mathbf{w}_{\alpha,-1}$ is not guaranteed to be a quasi-definite linear functional. With this in mind, Sobolev orthogonality can be provided for the polynomial solutions of $\mathscr{L}_{\alpha,-1}[\,p\,]=\lambda_{n,m}^{(\alpha,-1)}\,p$. First, we construct such polynomial solutions.

It follows from Theorem 5.3 **(II)** that the biangle polynomials defined in (6.11) satisfy the differential equation with

$$\lambda_{n,m}^{(\alpha,-1)}=-(n-m)\,(n+\alpha+1/2)-\frac{1}{4}m\,(m+2\,\alpha+1),$$

if $\{p_n^{(m)}(x)\}_{n\geqslant0}$ is a univariate polynomial sequence satisfying

$$\begin{aligned}(1-x)\,x\,(p_n^{(m)}(x))''+[m+1/2-(\alpha+m+3/2)\,x]\,(p_n^{(m)}(x))'\\=-n\,(n+\alpha+m+1/2)\,p_n^{(m)}(x),\end{aligned}$$

and $\{q_n(y)\}_{n\geqslant0}$ is a univariate polynomial sequence satisfying

$$(1-y^2)\,q_n''(y)=-n\,(n-1)\,q_n(y).$$

Clearly, for $n,m\geqslant0$, $p_n^{(m)}(x)=P_n^{(\alpha,m-1/2)}(2x-1)$. Moreover, the second ordinary differential equation is satisfied by (see Proposition 3 in Garcia-Ardila and Marriaga (2021)) $q_0(y)=1$, $q_1(y)=y$, and $q_n(y)=(1-y^2)\,P_{n-2}^{(1,1)}(y)$, $n\geqslant2$.

Thus, the singular biangle polynomials are given by

$$\begin{aligned}
P_{0,0}^{(\alpha,-1)}(x,y)&=P_0^{(\alpha,-1/2)}(2x-1),\\
P_{1,0}^{(\alpha,-1)}(x,y)&=P_1^{(\alpha,-1/2)}(2x-1),\quad P_{1,1}^{(\alpha,-1)}(x,y)=y,\\
P_{n,0}^{(\alpha,-1)}(x,y)&=P_n^{(\alpha,-1/2)}(2x-1),\quad P_{n,1}^{(\alpha,-1)}(x,y)=y\,P_{n-1}^{(\alpha,1/2)}(2x-1),\\
P_{n,m}^{(\alpha,-1)}(x,y)&=(x-y^2)\,P_{n-m}^{(\alpha,m-1/2)}(2x-1)\,(\sqrt{x})^{m-2}\,P_{m-2}^{(1,1)}\left(\frac{y}{\sqrt{x}}\right),\quad 2\leqslant m\leqslant n.
\end{aligned}$$

The explicit expression of $P_{n,m}^{(\alpha,-1)}(x,y)$, $2\leqslant m\leqslant n$, can also be deduced from Proposition 5.1. Consider the polynomial $\Psi(x,y)=x-y^2$, and note that it satisfies the differential equation

$$\mathscr{L}_{\alpha,-1}[\Psi]=-(\alpha+3/2)\,\Psi,$$

and the Pearson equations

$$\begin{aligned}
(1-x)\,x\,\Psi_x+\frac{1}{2}(1-x)\,y\,\Psi_y&=(1-x)\,\Psi,\\
\frac{1}{2}(1-x)\,y\,\Psi_x+\frac{1}{4}(1-y^2)\,\Psi_y&=-\frac{1}{2}\,y\,\Psi.
\end{aligned}$$

Since the biangle polynomials $P_{n,m}^{(\alpha,1)}(x,y)$ satisfy

$$\begin{aligned}\mathscr{L}_{\alpha,1}[P_{n,m}^{(\alpha,1)}] &= \mathscr{L}_{\alpha,-1}[P_{n,m}^{(\alpha,1)}] + 2\,(1-x)\,\partial_x P_{n,m}^{(\alpha,1)} - y\,\partial_y P_{n,m}^{(\alpha,1)}\\ &= [-(n-m)\,(n+\alpha+5/2) - \frac{1}{4}m\,(m+2\,\alpha+5)]\,P_{n,m}^{(\alpha,1)},\end{aligned}$$

we have that for $2 \leqslant m \leqslant n$,

$$\begin{aligned}&\mathscr{L}_{\alpha,\beta,-1}[\Psi\, P_{n-2,m-2}^{(\alpha,1)}]\\ &\quad= [-(\alpha+3/2) - (n-m)\,(n+\alpha+1/2) - \frac{1}{4}(m-2)\,(m+2\,\alpha+3)]\,\Psi\, P_{n-2,m-2}^{(\alpha,1)}\\ &\quad= [-(n-m)\,(n+\alpha+1/2) - \frac{1}{4}m\,(m+2\,\alpha+1)]\,\Psi\, P_{n-2,m-2}^{(\alpha,1)}.\end{aligned}$$

Hence, $P_{n,m}^{(\alpha,-1)}(x,y) = (x-y^2)\,P_{n-2,m-2}^{(\alpha,1)}(x,y)$ for $2 \leqslant m \leqslant n$.

As in the positive definite case, the orthogonality of the biangle polynomials in the singular case cannot be deduced from Proposition 3.5. However, studying the symmetry of $\mathscr{L}_{\alpha,-1}$ with respect to a Sobolev bilinear form provides important information about the orthogonality.

Fix an integer $N \geqslant 1$. From Theorem 3.3, we have that if the bivariate linear functionals $\mathbf{u}^{(j,j)}$, $0 \leqslant j \leqslant N$, satisfy the Pearson equations

$$(1-x)\,x\,\partial_x\mathbf{u}^{(j,j)} + \frac{1}{2}(1-x)\,y\,\partial_y\mathbf{u}^{(j,j)} = [j-1-(\alpha+j-1)x]\,\mathbf{u}^{(j,j)}, \tag{6.14}$$

$$\frac{1}{2}(1-x)\,y\,\partial_x\mathbf{u}^{(j,j)} + \frac{1}{4}(1-y^2)\,\partial_y\mathbf{u}^{(j,j)} = -\frac{1}{2}(\alpha+j-1)\,y\,\mathbf{u}^{(j,j)}, \tag{6.15}$$

then $\mathscr{L}_{\alpha,-1}$ is symmetric with respect to the bilinear form

$$(P,Q)_{\alpha,-1} = \sum_{j=0}^{N}\left\langle \mathbf{u}^{(j,j)}, \partial_y^j P\,\partial_y^j Q\right\rangle, \quad \forall P, Q \in \Pi^2.$$

Observe that for $1 \leqslant j \leqslant N$, $\mathbf{u}^{(j,j)} = \mathbf{w}_{\alpha,j-1}$. It remains to find the moment functional $\mathbf{u}^{(0,0)}$.

The expression of the moment functional $\mathbf{u}^{(0,0)}$ can be deduced from Theorem 4.1 and Corollary 4.3 with $a(x) = (1-x)\,x$, $\widetilde{b}(x) = \frac{1}{2}(1-x)$, $c(y) = \frac{1}{4}(1-y^2)$, $\widetilde{d}(x) = -1-(\alpha-1)\,x$, $\widetilde{e}(y) = -\frac{1}{2}(\alpha-1)\,y$, and $\rho(x) = \sqrt{x}$. We will find univariate moment functionals $\mathbf{u}_0^{(s)}$ and $\mathbf{v}^{(t)}$ such that $\mathbf{u}^{(0,0)}$ can be constructed using (4.5).

On one hand, from Theorem 4.1, we have that if $\mathbf{u}_0^{(s)}$ satisfies

$$(1-s)\,s\,D\,\mathbf{u}_0^{(s)} = [-1/2-(\alpha-1/2)\,s]\,\mathbf{u}_0^{(s)},$$

then $\mathbf{u}^{(0,0)}$ satisfies (6.14) with $j=0$. Hence, $\mathbf{u}_0^{(s)}$ is the Jacobi moment functional on $[0,1]$ defined as

$$\left\langle \mathbf{u}_0^{(s)}, p(s)\right\rangle = \int_0^1 p(s)\,(1-s)^\alpha\,s^{-1/2}\,ds.$$

On the other hand, from Corollary 4.4 with $\mathbf{u}_0^{(s)}$ as above, we have that if $\mathbf{v}^{(t)}$ satisfies

$$(1-t^2)\,D\,\mathbf{v}^{(t)} = 2\,t\,\mathbf{v}^{(t)},$$

then $\mathbf{u}^{(0,0)}$ satisfies (6.15) with $j = 0$. Hence, $\mathbf{v}^{(t)} = \boldsymbol{\delta}(1-t) + \boldsymbol{\delta}(1+t)$. Using (4.5), we get

$$\langle \mathbf{u}^{(0,0)}, P \rangle = \int_{-1}^{1} P(y^2, y)\,(1-y^2)^{\alpha}\,dy.$$

It follows that

$$(P, Q)_{\alpha,-1} = \int_{-1}^{1} P(y^2, y)\,Q(y^2, y)\,(1-y^2)^{\alpha}\,dy \\ + \sum_{j=1}^{N} \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \partial_y^j P(x, y)\,\partial_y^j Q(x, y)\,(1-x)^{\alpha}(x-y^2)^{j-1}\,dydx.$$

Using the well-known properties of the Jacobi polynomials, we get that the biangle polynomials satisfy $\partial_y P_{n,0}^{(\alpha,-1)}(x, y) = 0$ for $n \geqslant 0$, $\partial_y P_{n,1}^{(\alpha,-1)}(x, y) = P_{n-1,0}^{(\alpha,0)}(x, y)$ for $n \geqslant 1$, and

$$\partial_y^j P_{n,m}^{(\alpha,-1)}(x, y) = -\frac{(m-1)_j}{2^{j-2}}\,P_{n-j,m-j}^{(\alpha,j-1)}(x, y), \quad n \geqslant 2, \quad 2 \leqslant m \leqslant n, \quad 1 \leqslant j \leqslant m.$$

Using this fact and a direct computation, we have the following result on the orthogonality of the singular biangle polynomials.

Proposition 6.2 *The biangle polynomials* $\{P_{n,m}^{(\alpha,-1)}(x, y) : n \geqslant 0,\ 0 \leqslant m \leqslant n\}$ *are Sobolev orthogonal with respect to* $(\cdot,\cdot)_{\alpha,-1}$. *Moreover,*

$$(P_{n,m}^{(\alpha,-1)}, P_{k,j}^{(\alpha,-1)})_{\alpha,-1} = \widetilde{H}_{n,m}^{(\alpha,-1)}\,\delta_{n,k}\,\delta_{m,j},$$

where $\widetilde{H}_{n,m}^{(\alpha,-1)} > 0$.

Remark 6.3 Here we justify our study of the singular case with $\beta = -1$ instead of taking $\beta = -k$ for any integer $k \geqslant 1$. Observe that for $\beta = -k$ with $k \geqslant 2$, (6.9) and (6.10) read

$$(1-x)\,x\,(p_n^{(m)}(x))'' + [m - k + 3/2 - (\alpha + m - k + 5/2)\,x]\,(p_n^{(m)}(x))' \\ = -n\,(n + \alpha + m - k + 3/2)\,p_n^{(m)}(x),$$

and

$$(1-y^2)\,q_n''(y) - (2 - 2\,k)\,y\,q_n'(y) = -n\,(n - 2\,k + 1)\,q_n(y),$$

respectively. Then,

$$p_n^{(m)}(x) = P_n^{(\alpha,m-k+1/2)}(2x-1), \quad m, n \geqslant 0,$$

and (Section 4.2.5 of Garcia-Ardila and Marriaga (2021))

$$q_n(y) = \begin{cases} P_n^{(-k,-k)}(y), & 0 \leqslant n \leqslant k-1, \\ (1+y)^k\,P_{n-k}^{(-k,k)}(y), & k \leqslant n \leqslant 2\,k-1, \\ (1-y^2)^k\,P_{n-2}^{(k,k)}(y), & 2\,k \leqslant n. \end{cases}$$

However, for $n \geqslant 2k$ and $k \leqslant m \leqslant 2k-1$,

$$P_{n,m}^{(\alpha,-k)}(x, y) = (\sqrt{x} + y)^k\,P_n^{(\alpha,m-k+1/2)}(2x-1)\,(\sqrt{x})^{m-k}\,P_{m-k}^{(-k,k)}\left(\frac{y}{\sqrt{x}}\right),$$

is not a polynomial. Hence, we restrict our study of the singular case for $\beta = -1$ since, in this case, $q_0(y) = 1$, $q_1(y) = y$, $q_n(y) = (1 - y^2)\, P_{n-2}^{(1,1)}(y)$, $n \geqslant 2$, and, consequently, we have that $P_{n,m}^{(\alpha,-1)}$ are polynomials for $n \geqslant 0$ and $0 \leqslant m \leqslant n$.

6.5 Bessel–Laguerre moment functional

In this example, we apply our results concerning solutions of Pearson equations to a non-positive definite case studied in Kwon et al. (2001) (see also Marriaga et al. (2017)).

Consider the second-order linear partial differential operator

$$\mathscr{L}[p] \equiv x^2 p_{xx} + 2x\, y\, p_{xy} + (y-1)\, y\, p_{yy} + g\,(x-1)\, p_x + g\,(y-\gamma)\, p_y.$$

From Theorem 5.2 with $a(x) = x^2$, $\widetilde{b}(x) = x$, $c(y) = (y-1)\, y$, $d(x) = g\,(x-1)$, $e(y) = g\,(y-\gamma)$, and $\rho(x) = x/g$ satisfying $x^2\, \rho'(x) = x\, \rho(x)$, we deduce that the polynomials defined as

$$P_{n,m}(x,y) = p_{n-m}^{(m)}(x) \left(\frac{x}{g}\right)^m q_m\left(\frac{g\,y}{x}\right), \quad n \geqslant 0, \quad 0 \leqslant m \leqslant n,$$

satisfy $\mathscr{L}[P_{n,m}] = \lambda_n\, P_{n,m}$ with $\lambda_n = n + g - 1$, if $\{p_n^{(m)}(x)\}_{n\geqslant 0}$ and $\{q_n(y)\}_{n\geqslant 0}$ satisfy

$$x^2\,(p_n^{(m)}(x))'' + [(g+2m)\,x - g]\,(p_n^{(m)}(x))' = n\,(n-1+g+2m)\, p_n^{(m)}(x),$$

and

$$x\, q_n''(y) + (g\,\gamma - y)\, q_n'(y) = -n\, q_n(y),$$

respectively. Then $p_n^{(m)}(x) = B_n^{(g+2m,-g)}(x)$ are classical Bessel polynomials (Krall and Frink 1949) and $q_n(y) = L^{(g\,\gamma-1)}(y)$ are Laguerre polynomials. Therefore, for $g + n \neq 0$ and $g\,\gamma + n \neq 0$, $n \geqslant 0$,

$$P_{n,m}(x,y) = B_{n-m}^{(g+2m,-g)}(x) \left(\frac{x}{g}\right)^m L_m^{(g\,\gamma-1)}\left(\frac{g\,y}{x}\right), \quad n \geqslant 0, \quad 0 \leqslant m \leqslant n.$$

Let $\mathbf{w}$ be a bivariate moment functional. By Theorem 3.3 (and Remark 3.6), the differential operator $\mathscr{L}$ satisfies

$$\langle \mathbf{w}, \mathscr{L}[P]\, Q\rangle = \langle \mathbf{w}, P\, \mathscr{L}[Q]\rangle, \quad \forall P, Q \in \Pi^2,$$

if $\mathbf{w}$ satisfies by the Pearson equations

$$x^2\, \partial_x\, \mathbf{w} + x\, y\, \partial_y\, \mathbf{w} = [(g-3)\, x - g]\, \mathbf{w},$$
$$x\, y\, \partial_x\, \mathbf{w} + (y^2 - y)\, \partial_y\, \mathbf{w} = [(g-3)\, y - (g\,\gamma - 1)]\, \mathbf{w}.$$

We seek univariate moment functionals $\mathbf{u}^{(s)}$ and $\mathbf{v}^{(t)}$ such that $\mathbf{w}$ can be constructed, if possible, using (4.5) with $\mathbf{u}^{(s)}$ and $\mathbf{v}^{(t)}$.

First, we deal with $\mathbf{u}_0^{(s)}$. Hence, we look at Theorem 4.1 with $a(x) = x^2$, $\widetilde{b}(x) = x$, $\widetilde{d}(x) = (g-3)\, x - g$, and $\rho(s) = s/g$. If $\mathbf{u}_0^{(s)}$ satisfies the Pearson equation

$$s^2\, D\, \mathbf{u}_0^{(s)} = [(g-2)\, s - g]\, \mathbf{u}_0^{(s)},$$

then, from Theorem 4.1, we deduce that $\mathbf{w}$ satisfies

$$x^2\, \partial_x\, \mathbf{w} + x\, y\, \partial_y\, \mathbf{w} = [(g-3)\, x - g]\, \mathbf{w}.$$

Then, $\mathbf{u}_0^{(s)}$ is the Bessel linear functional on the unit circle defined by

$$\left\langle \mathbf{u}_0^{(s)}, p(s) \right\rangle = \frac{1}{2\pi i} \int_c p(s)\, s^{g-2}\, e^{g/s} ds,$$

where c is the unit circle oriented in the counter-clockwise direction. The Bessel linear functional

$$\left\langle \mathbf{u}^{(s)}, p(s) \right\rangle = \frac{g}{2\pi i} \int_c p(s)\, s^{g-3}\, e^{g/s} ds,$$

satisfies $\mathbf{u}_0^{(s)} = \rho(s)\, \mathbf{u}^{(s)}$.

From Corollary 4.3 with $\mathbf{u}_0^{(s)}$ as above, $\rho(s) = s/g$, $\widetilde{b}(x) = x$, $c(y) = y^2 - y$, and $\widetilde{e}(y) = (g-3)\, y - (g\,\gamma - 1)$, we have that if $\mathbf{v}^{(t)}$ satisfies

$$t\, D\, \mathbf{v}^{(t)} = \left[(g\,\gamma - 1) - t\right] \mathbf{v}^{(t)},$$

then $\mathbf{w}$ satisfies $x\, y\, \partial_x\, \mathbf{w} + (y^2 - y)\, \partial_y\, \mathbf{w} = [(g-3)\, y - (g\,\gamma - 1)]\, \mathbf{w}$. Observe that $\mathbf{v}^{(t)}$ is the Laguerre linear functional

$$\left\langle \mathbf{v}^{(t)}, q(t) \right\rangle = \int_0^{\infty} q(t)\, t^{g\,\gamma - 1}\, e^{-t}\, dt.$$

The orthogonality of the Bessel–Laguerre polynomials with respect to $\mathbf{w}$ follows from Proposition 3.5 since $\lambda_n \neq \lambda_m$ for $n \neq m$.

6.6 Laguerre–Jacobi moment functional

In Fernández et al. (2012), some new examples of bivariate quasi-definite moment functionals constructed with (4.5) were introduced, and the Pearson equations satisfied by these moment functionals were studied in Marcellán et al. (2018a). The orthogonal polynomials associated with these moment functionals satisfy second-order differential equations not in the extended Lyskova class. Here, we use our results to construct solutions of the associated Pearson equations by solving auxiliary Pearson equations with the same moment functionals as solutions.

Consider the bivariate moment functional $\mathbf{w}$ satisfying the following Pearson equations:

$$\begin{aligned} x\, \partial_x\, \mathbf{w} + x\, \partial_y\, \mathbf{w} &= (\alpha - \beta - x)\, \mathbf{w}, \\ x\, \partial_x\, \mathbf{w} + (x^2 - y^2 + x)\, \partial_y\, \mathbf{w} &= [-\beta\, (x + y) + (\alpha - \beta - x)]\, \mathbf{w}. \end{aligned}$$

The Laguerre–Jacobi polynomials (Fernández et al. 2012; Marcellán et al. 2018a) defined as

$$P_{n,m}(x, y) = L_{n-m}^{(\alpha+2m-1)}(x)\, x^m\, P_m^{(0,\beta)}\left(\frac{y}{x}\right), \quad n \geqslant 0, \quad 0 \leqslant m \leqslant n,$$

are mutually orthogonal with respect to $\mathbf{w}$, and satisfy the differential equation

$$\mathscr{L}[P_{n,m}] = \lambda_{n,m} P_{n,m} + \lambda_{n,m-1} P_{n,m-1} + \lambda_{n,m-2} P_{n,m-2},$$

where

$$\begin{aligned} \mathscr{L}[p] \equiv\; & x\, p_x x + 2x\, p_{xy} + (x^2 - y^2 + x)\, p_{yy} \\ & + (1 + \alpha - \beta - x) p_x + [\alpha - \beta + 1 - (1 + \beta) x - (2 + \beta) y] p_y, \end{aligned}$$

and

$$\begin{aligned}\lambda_{n,m} &= -n - m\,(m+\beta),\\ \lambda_{n,m-1} &= -(m-1)\,(\beta+1),\\ \lambda_{n,m-2} &= m\,(m-1).\end{aligned}$$

We seek univariate moment functionals $\mathbf{u}^{(s)}$ and $\mathbf{v}^{(t)}$ such that $\mathbf{w}$ can be constructed, if possible, using (4.5) with $\mathbf{u}^{(s)}$ and $\mathbf{v}^{(t)}$.
To use our results, we consider the following system of equations

$$\begin{aligned} x\,\partial_x\,\mathbf{w} + y\,\partial_y\,\mathbf{w} &= (\alpha - x)\,\mathbf{w},\\ (x-y)\,\partial_y\,\mathbf{w} &= -\beta\,\mathbf{w}.\end{aligned}$$

Observe that $x\,\partial_x\,\mathbf{w} + x\,\partial_y\,\mathbf{w} = (\alpha-\beta-x)\,\mathbf{w}$ is obtained by adding the two equations, and we obtain $x\,\partial_x\,\mathbf{w} + (x^2 - y^2 + x)\,\partial_y\,\mathbf{w} = [-\beta\,(x+y) + (\alpha-\beta-x)]\,\mathbf{w}$ by adding the first equation to $(x+y+1)$ times the second. Therefore, if $\mathbf{w}$ satisfies the second system of equations, then it also satisfies the first system.

First, we deal with $\mathbf{u}_0^{(s)}$. Hence, we use Theorem 4.1 with $a(x) = x$, $\widetilde{b}(x) = 1$, and $\widetilde{d}(x) = \alpha - x$. A solution of $s\,\rho'(s) = \rho(s)$ is $\rho(s) = s$. Since the Laguerre moment functional on $[0, +\infty)$ defined by

$$\left\langle \mathbf{u}_0^{(s)}, p(s)\right\rangle = \int_0^{+\infty} p(s)\,s^{\alpha+1}\,e^{-s}\,ds,$$

satisfies the Pearson equation

$$s\,D\,\mathbf{u}_0^{(s)} = (\alpha+1-s)\;\mathbf{u}_0^{(s)},$$

then, from Theorem 4.1, we deduce that $\mathbf{w}$ satisfies $x\,\partial_x\,\mathbf{w} + y\,\partial_y\,\mathbf{w} = (\alpha - x)\,\mathbf{w}$. On the other hand, from Theorem 4.6 with $\mathbf{u}_0^{(s)}$ as above, $\rho(s) = s$, $\widetilde{b}(t) = 1 - t$, $c(y) = 1$, and $\widetilde{e}(y) = -\beta$, we have that if $\mathbf{v}^{(t)}$ satisfies

$$(1-t)\,D\,\mathbf{v}^{(t)} = -\beta\,\mathbf{v}^{(t)},$$

then $\mathbf{w}$ satisfies $(x-y)\,\partial_y\,\mathbf{w} = -\beta\,\mathbf{w}$. Observe that $\mathbf{v}^{(t)}$ is the Jacobi linear functional on $[-1, 1]$

$$\left\langle \mathbf{v}^{(t)}, q(t)\right\rangle = \int_{-1}^{1} q(t)\,(1-t)^{\beta}\,dt.$$

From (4.5) with $\mathbf{u}_0^{(s)} = \rho(s)\,\mathbf{u}^{(s)}$ and $\mathbf{v}^{(t)}$ as above, we obtain the linear functional $\mathbf{w}$ defined by

$$\langle \mathbf{w}, P(x,y)\rangle = \int\!\!\int_{\Omega} P(x,y)\,x^{\alpha-\beta}\,(x-y)^{\beta}\,e^{-x}\,dydx,$$

where $\Omega = \{(x,y) \in \mathbb{R}^2 : -x \leqslant y \leqslant y,\ x \geqslant 0\}$.

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Appendix A

A.1 Proof of Lemma 3.2

Let $\mathbf{u}$ be a bivariate moment functional and let $P \in \Pi^2$.

For $i = j = 0$, we have

$$\mathscr{L}[P]\,\mathbf{u} = [a\,P_{xx} + 2b\,P_{xy} + c\,P_{yy} + d\,P_x + e\,P_y]\,\mathbf{u},$$

and

$$\mathscr{L}^*[P\,\mathbf{u}] = (a\,P\,\mathbf{u})_{xx} + 2(b\,P\,\mathbf{u})_{xy} + (c\,P\,\mathbf{u})_{yy} - (d\,P\,\mathbf{u})_x - (e\,P\,\mathbf{u})_y.$$

Using the product rule and then simplifying, we get

$$\mathscr{L}[P]\mathbf{u} - \mathscr{L}^*[P\mathbf{u}] = -\,P\,\left(\mathscr{L}^{(0,0)}\right)^*[\mathbf{u}] - 2P_x\,\mathcal{M}_1^{(0,0)}[\mathbf{u}] - 2P_y\,\mathcal{M}_2^{(0,0)}[\mathbf{u}]. \tag{A.1}$$

Note that $\mathscr{L}^{(0,0)} \equiv \mathscr{L}$.

Now, let $i + j \geqslant 1$. On one hand, using Leibniz rule and the fact that $\mathscr{L}$ is in the extended Lyskova class, we can expand $\partial_x^i \partial_y^j \mathscr{L}[P]$. We get

$$\begin{aligned}
&\partial_x^i \partial_y^j \left(\partial_x^i \partial_y^j \mathscr{L}[P]\,\mathbf{u}\right) \\
&= \partial_x^i \partial_y^j \Bigg[\Bigg(\mathscr{L}[\partial_x^i \partial_y^j P] + i\,a_x\,\partial_x^{i+1}\partial_y^j P + 2\,i\,b_x\,\partial_x^i \partial_y^{j+1} P + 2\,j\,b_y\,\partial_x^{i+1}\partial_y^j P \\
&\quad + j\,c_y \partial_x^i \partial_y^{j+1} P + \frac{i(i-1)}{2} a_{xx} \partial_x^i \partial_y^j P + 2ij b_{xy} \partial_x^i \partial_y^j P + \frac{j(j-1)}{2} c_{yy} \partial_x^i \partial_y^j P \\
&\quad + i\,d_x\,\partial_x^i \partial_y^j P + j\,e_y\,\partial_x^i \partial_y^j P\Bigg)\,\mathbf{u}\Bigg].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathscr{L}^*\left[\partial_x^i \partial_y^j (\partial_x^i \partial_y^j P\,\mathbf{u})\right] \\
&= \left[a\,\partial_x^i \partial_y^j (\partial_x^i \partial_y^j P\,\mathbf{u})\right]_{xx} + 2\left[b\,\partial_x^i \partial_y^j (\partial_x^i \partial_y^j P\,\mathbf{u})\right]_{xy} + \left[c\,\partial_x^i \partial_y^j (\partial_x^i \partial_y^j P\,\mathbf{u})\right]_{yy} \\
&\quad - \left[d\,\partial_x^i \partial_y^j (\partial_x^i \partial_y^j P\,\mathbf{u})\right]_x - \left[e\,\partial_x^i \partial_y^j (\partial_x^i \partial_y^j P\,\mathbf{u})\right]_y \\
&= \partial_x^i \partial_y^j \Bigg[\mathscr{L}^*[\partial_x^i \partial_y^j P\,\mathbf{u}] + i\,d_x \partial_x^i \partial_y^j P\,\mathbf{u} + j\,e_y \partial_x^i \partial_y^j P\,\mathbf{u} + \frac{i(i-1)}{2} a_{xx} \partial_x^i \partial_y^j P\,\mathbf{u} \\
&\quad + 2ij\,b_{xy} \partial_x^i \partial_y^j P\,\mathbf{u} + \frac{j(j-1)}{2} c_{yy} \partial_x^i \partial_y^j P\,\mathbf{u} - i\partial_x\left(a_x \partial_x^i \partial_y^j P\,\mathbf{u}\right) \\
&\quad - 2i\partial_y\left(b_x \partial_x^i \partial_y^j P\,\mathbf{u}\right) - 2j\partial_x\left(b_y \partial_x^i \partial_y^j P\,\mathbf{u}\right) - j\partial_y\left(c_y \partial_x^i \partial_y^j P\,\mathbf{u}\right)\Bigg].
\end{aligned}$$

Then, using (A.1), we get

$$\begin{aligned}
&\partial_x^i\partial_y^j\Big(\partial_x^i\partial_y^j\mathscr{L}[P]\,\mathbf{u}\Big)-\mathscr{L}^*\Big[\partial_x^i\partial_y^j(\partial_x^i\partial_y^j P\,\mathbf{u})\Big]\\
&=-\partial_x^i\partial_y^j\Bigg[\partial_x^i\partial_y^j P\,\Big([(a\,\mathbf{u})_x+(b\,\mathbf{u})_y-(d+i\,a_x+2j\,b_y)\,\mathbf{u}]_x\\
&\quad+[(b\,\mathbf{u})_x+(c\,\mathbf{u})_y-(e+2i\,b_x+j\,c_y)\,\mathbf{u}]_y\Big)\\
&\quad+2\,\partial_x^{i+1}\partial_y^j P\,[(a\,\mathbf{u})_x+(b\,\mathbf{u})_y-(d+i\,a_x+2j\,b_y)\,\mathbf{u}]\\
&\quad+2\,\partial_x^i\partial_y^{j+1} P\,[(b\,\mathbf{u})_x+(c\,\mathbf{u})_y-(e+2i\,b_x+j\,c_y)\,\mathbf{u}]\Bigg].
\end{aligned}$$

Remark A.1 Observe that (A.1) is satisfied even when $\mathscr{L}$ is not in the extended Lyskova class.

A.2 Proof of Corollary 4.3

From Proposition 4.2 with $\rho(s)=r_0+r_1\,s$, we have for all $P\in\Pi^2$,

$$\begin{aligned}
&\big\langle -y\,\widetilde{b}(x)\,\partial_x\,\mathbf{w}-c(y)\,\partial_y\,\mathbf{w}+\widetilde{e}(y)\,\mathbf{w},P\big\rangle+(r_1-r_1)\Big\langle\widetilde{d}(s)\,\mathbf{u}_0^{(s)},\Big\langle t\,\mathbf{v}^{(t)},P\Big\rangle\Big\rangle\\
&=r_1\,\Big\langle -a(s)\,D\,\mathbf{u}_0^{(s)}+\big(\widetilde{b}(s)+\widetilde{d}(s)\big)\,\mathbf{u}_0^{(s)},\Big\langle t\,\mathbf{v}^{(t)},P\Big\rangle\Big\rangle\\
&\quad+r_1\Big\langle (e_1-d_1)\,s\,\mathbf{u}_0^{(s)},\Big\langle t\,\mathbf{v}^{(t)},P\Big\rangle\Big\rangle\\
&\quad+\Big\langle \mathbf{u}_0^{(s)},\Big\langle \mathbf{v}^{(t)},(c_2-b_1)\,r_1\,s\,\partial_t\,\big(t^2\,P\big)\Big\rangle\Big\rangle\\
&\quad+\Big\langle \mathbf{u}_0^{(s)},\Big\langle \mathbf{v}^{(t)},(c_2\,r_0-b_0\,r_1)\partial_t\,\big(t^2 P\big)+c_1\partial_t(t\,P)+[e_0+(e_1\,r_0-d_0\,r_1)\,t]\,P\Big\rangle\Big\rangle\\
&\quad+\Big\langle \mathbf{u}_0^{(s)},\Big\langle \mathbf{v}^{(t)},c_0\,\frac{1}{\rho(s)}\partial_t\,P\Big\rangle\Big\rangle.
\end{aligned}$$

Using conditions (a) and (b), we obtain

$$\begin{aligned}
&\big\langle -y\,\widetilde{b}(x)\,\partial_x\,\mathbf{w}-c(y)\,\partial_y\,\mathbf{w}+\widetilde{e}(y)\,\mathbf{w},P\big\rangle\\
&=\Big\langle \mathbf{u}_0^{(s)},\Big\langle \mathbf{v}^{(t)},(c_2\,r_0-b_0\,r_1)\partial_t\,\big(t^2 P\big)+c_1\partial_t(t\,P)+[e_0+(e_1\,r_0-d_0\,r_1)\,t]\,P\Big\rangle\Big\rangle,\\
&=\Big\langle \mathbf{u}_0^{(s)},\Big\langle -\big[(b_1\,r_0-r_1\,b_0)\,t^2+c_1\,t\big]\,D\,\mathbf{v}^{(t)}+[e_0+(e_1\,r_0-d_0\,r_1)\,t]\,\mathbf{v}^{(t)},P\Big\rangle\Big\rangle,
\end{aligned}$$

and from condition (c) we get $\big\langle -y\,\widetilde{b}(x)\,\partial_x\,\mathbf{w}-c(y)\,\partial_y\,\mathbf{w}+\widetilde{e}(y)\,\mathbf{w},P\big\rangle=0$, and the promised result follows.

A.3 Proof of Corollary 4.4

From Proposition 4.2 with $\rho(s)=\sqrt{\ell_0+2\,\ell_1\,s+\ell_2\,s^2}$, we get

$$\begin{aligned}
&\big\langle -y\,\widetilde{b}(x)\,\partial_x\,\mathbf{w}-c(y)\,\partial_y\,\mathbf{w}+\widetilde{e}(y)\,\mathbf{w},P\big\rangle\\
&=\Big\langle -a(s)\,D\,\mathbf{u}_0^{(s)}+\big(\widetilde{b}(s)+\widetilde{d}(s)\big)\,\mathbf{u}_0^{(s)},\Big\langle t\,\mathbf{v}^{(t)},\rho'(s)\,P\Big\rangle\Big\rangle\\
&\quad+\Big\langle \mathbf{u}_0^{(s)},\Big\langle -(c_2-b_1)\,\ell_2\,t^2\,D\,\mathbf{v}^{(t)}+\ell_2\,(e_1-d_1)\,t\,\mathbf{v}^{(t)},\frac{s^2}{\rho(s)}\,P\Big\rangle\Big\rangle
\end{aligned}$$

$$- \left\langle \mathbf{u}_0^{(s)}, \left\langle [(2\,c_2 - b_1)\,\ell_1 - b_0\,\ell_2]\,t^2\,D\,\mathbf{v}^{(t)} + (2e_1\ell_1 - d_1\ell_1 - d_0\ell_2)\,t\,\mathbf{v}^{(t)}, \frac{s}{\rho(s)}P \right\rangle \right\rangle$$
$$+ \left\langle \mathbf{u}_0^{(s)}, \left\langle -\left[(c_2\,\ell_0 - b_0\,\ell_1)\,t^2 + c_0\right]\,D\,\mathbf{v}^{(t)} + (e_1\,\ell_0 - d_0\,\ell_1)\,t\,\mathbf{v}^{(t)}, \frac{1}{\rho(s)}P \right\rangle \right\rangle.$$

The result follows from the conditions (a)–(d).

A.4 Proof of Theorem 4.6

We begin with (4.8). Using (4.7), we have

$$\left\langle \mathbf{w}, \partial_y\left(c(y)\rho(x)b\left(\frac{y}{\rho(x)}\right)P\right) + \gamma c'(y)\rho(x)b\left(\frac{y}{\rho(x)}\right)P + \beta c(y)\partial_y\left(\rho(x)b\left(\frac{y}{\rho(x)}\right)\right)P \right\rangle$$
$$= c_1\left\langle \rho(s)\,\mathbf{u}_0^{(s)}, \left\langle \mathbf{v}^{(t)}, \partial_t(t\,b(t)\,P) + \left[\gamma\,b(t) + \beta\,t\,b'(t)\right]P \right\rangle \right\rangle$$
$$+ c_0\left\langle \mathbf{u}_0^{(s)}, \left\langle \mathbf{v}^{(t)}, \partial_t(b(t)\,P) + \beta\,b'(t)\,P \right\rangle \right\rangle.$$

If either one of the two cases in the statement of the theorem holds, then we obtain that **w** satisfies the Pearson equation (4.8).

Using (4.9) and (4.7), we compute as follows.

$$\left\langle \mathbf{w}, \partial_x\left(a(x)\,\rho(x)\,b\left(\frac{y}{\rho(x)}\right)P\right) + \alpha\,a'(x)\,\rho(x)\,b\left(\frac{y}{\rho(x)}\right)P + \beta\,a(x)\,b_0\,\rho'(x)\,P \right\rangle$$
$$= \left\langle \mathbf{u}_0^{(s)}, \left\langle b(t)\,\mathbf{v}^{(t)}, \partial_s\left(a(s)\,\rho(s)\,P\right) + \alpha\,a'(s)\,\rho(s)\,P \right\rangle \right\rangle$$
$$+ \left\langle \mathbf{u}_0^{(s)}, \left\langle \mathbf{v}^{(t)}, \beta\,b_0\,\rho'(s)\,a(s)\,P - \rho'(s)\,a(s)\,t\,b'(t)\,P \right\rangle \right\rangle$$
$$+ \left\langle \mathbf{u}_0^{(s)}, \left\langle D\left[t\,b(t)\,\mathbf{v}^{(t)}\right], \rho'(s)\,a(s)\,P \right\rangle \right\rangle. \tag{A.2}$$

Now, we consider each of the conditions in the statement of the theorem. Suppose that (i) holds and that $\mathbf{u}_0^{(s)}$ satisfies

$$a(s)\,\rho(s)\,D\mathbf{u}_0^{(s)} = \left[\alpha\,a'(s)\,\rho(s) + (1+\gamma+\beta)\,\rho'(s)\,a(s)\right]\mathbf{u}_0^{(s)},$$

and $\mathbf{v}^{(t)}$ satisfies $t\,b(t)\,D\mathbf{v}^{(t)} = \left[\gamma\,b(t) + \beta\,t\,b'(t)\right]\mathbf{v}^{(t)}$ or, equivalently,

$$D\left[t\,b(t)\,\mathbf{v}^{(t)}\right] = \left[(\gamma+1)\,b(t) + (\beta+1)\,t\,b'(t)\right]\mathbf{v}^{(t)}.$$

Substituting this in the last equality of (A.2), we get

$$\left\langle \mathbf{u}_0^{(s)}, \left\langle b(t)\,\mathbf{v}^{(t)}, \partial_s\left(a(s)\,\rho(s)\,P\right) + \alpha\,a'(s)\,\rho(s)\,P \right\rangle \right\rangle$$
$$+ \left\langle \mathbf{u}_0^{(s)}, \left\langle \mathbf{v}^{(t)}, \beta\,b_0\,\rho'(s)\,a(s)\,P + \rho'(s)\,a(s)\left[(\gamma+1)\,b(t) + \beta\,t\,b'(t)\right]P \right\rangle \right\rangle$$
$$= \left\langle \mathbf{u}_0^{(s)}, \left\langle b(t)\,\mathbf{v}^{(t)}, \partial_s\left(a(s)\,\rho(s)\,P\right) + \left[\alpha\,a'(s)\,\rho(s) + (1+\beta+\gamma)\,a(s)\,\rho'(s)\right]P \right\rangle \right\rangle$$
$$= 0.$$

Therefore, **w** satisfies the first Pearson equation (4.8).

On the other hand, if (ii) holds, and $\mathbf{u}_0^{(s)}$ satisfies

$$a(s)\,\rho(s)\,D\mathbf{u}_0^{(s)} = \left[\alpha\,a'(s)\,\rho(s) + (1+\beta)\,\rho'(s)\,a(s)\right]\mathbf{u}_0^{(s)}$$

and $\mathbf{v}^{(t)}$ satisfies $b(t)\,D\mathbf{v}^{(t)} = \beta\, b'(t)\,\mathbf{v}^{(t)}$, or, equivalently,

$$D\left[b(t)\,\mathbf{v}^{(t)}\right] = (\beta+1)\,b'(t)\,\mathbf{v}^{(t)}.$$

Substituting this in the last equality of (A.2), we get

$$\begin{aligned}
&\left\langle \mathbf{u}_0^{(s)}, \left\langle b(t)\,\mathbf{v}^{(t)}, \partial_s\,(a(s)\,\rho(s)\,P) + \alpha\,a'(s)\,\rho(s)\,P\right\rangle\right\rangle \\
&\quad + \left\langle \mathbf{u}_0^{(s)}, \left\langle \mathbf{v}^{(t)}, \beta\,b_0\,\rho'(s)\,a(s)\,P + \rho'(s)\,a(s)\left[\beta\,t\,b'(t) + b(t)\right]P\right\rangle\right\rangle \\
&= \left\langle \mathbf{u}_0^{(s)}, \left\langle b(t)\,\mathbf{v}^{(t)}, \partial_s\,(a(s)\,\rho(s)\,P) + \left[\alpha\,a'(s)\,\rho(s) + (1+\beta)\,a(s)\,\rho'(s)\right]\right\rangle\right\rangle \\
&= 0.
\end{aligned}$$

Therefore, in this case we also conclude that $\mathbf{w}$ satisfies (4.8).

A.5 Proof of Theorem 4.7

We start with (4.11). For every polynomial $P \in \Pi^2$,

$$\begin{aligned}
&\left\langle -\rho(x)^2\,b\left(\frac{y}{\rho(x)}\right)\mathbf{w}_y + \alpha\,\partial_y\left(\rho(x)^2\,b\left(\frac{y}{\rho(x)}\right)\right)\mathbf{w}, P\right\rangle \\
&\quad = \left\langle \rho(s)\,\mathbf{u}_0^{(s)}, \left\langle -b(t)\,D\mathbf{v}^{(t)} + \alpha\,b'(t)\,\mathbf{v}^{(t)}, P\right\rangle\right\rangle = 0.
\end{aligned}$$

For (4.10), we get

$$\begin{aligned}
&\left\langle -\rho(x)^2\,b\left(\frac{y}{\rho(x)}\right)\mathbf{w}_x + \alpha\,\partial_x\left(\rho(x)^2\,b\left(\frac{y}{\rho(x)}\right)\right)\mathbf{w}, P\right\rangle \\
&\quad = \left\langle \mathbf{u}_0^{(s)}, \left\langle \mathbf{v}^{(t)}, b(t)\left[\partial_s\left(\rho(s)^2\,P\right) + \frac{1}{2}\left(\rho(s)^2\right)'\,P\right]\right\rangle\right\rangle \\
&\quad\quad + \left\langle \mathbf{u}_0^{(s)}, \left\langle t\,b(t)\,D\mathbf{v}^{(t)}, \frac{1}{2}\left(\rho(s)^2\right)'\,P\right\rangle\right\rangle \\
&\quad\quad + \left\langle \mathbf{u}_0^{(s)}, \left\langle \mathbf{v}^{(t)}, \alpha\,\left(\rho(s)^2\right)'\left(b_0 + \frac{1}{2}b_1\,t\right)P\right\rangle\right\rangle.
\end{aligned}$$

Using $b(t)\,D\mathbf{v}^{(t)} = \alpha\,b'(t)\,\mathbf{v}^{(t)}$, we obtain

$$\begin{aligned}
&\left\langle -\rho(x)^2\,b\left(\frac{y}{\rho(x)}\right)\mathbf{w}_x + \alpha\,\partial_x\left(\rho(x)^2\,b\left(\frac{y}{\rho(x)}\right)\right)\mathbf{w}, P\right\rangle \\
&\quad = \left\langle \mathbf{u}_0^{(s)}, \left\langle b(t)\,\mathbf{v}^{(t)}, \partial_s\left(\rho(s)^2\,P\right) + \left(\alpha + \frac{1}{2}\right)\left(\rho(s)^2\right)'\right\rangle\right\rangle = 0.
\end{aligned}$$

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