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An Extension of Sylvester's Theorem on Arithmetic Progressions

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Abstract: Sylvester's theorem states that every number can be decomposed into a sum of consecutive positive integers except powers of 2. In a way, this theorem characterizes the partitions of a number as a sum of consecutive integers. The first generalization we propose of the theorem characterizes the partitions of a number as a sum of arithmetic progressions with positive terms. In addition to synthesizing and rediscovering known results, the method we propose allows us to state a second generalization and characterize the partitions of a number into parts whose differences between consecutive parts form an arithmetic progression. To achieve this, we will analyze the set of divisors in arithmetics that modify the usual definition of the multiplication operation between two integers. As we will see, symmetries arise in the set of divisors based on two parameters: t_1, being even or odd, and t_2, congruent to 0, 1, or 2 (mod 3). This approach also leads to a unique representation result of the same nature as Sylvester's theorem, i.e., a power of 3 cannot be represented as a sum of three or more terms of a positive integer sequence such that the differences between consecutive terms are consecutive integers.

Keywords: Sylvester's theorem; partition; divisor; arithmetic progression; representation

MSC: 11P81; 11A41; 11A51

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1. Introduction

Tom M. Apostol [1] stated Sylvester's theorem: "Every integer n, not a power of 2, is a sum of two or more consecutive positive integers. Moreover, the number of such representations is equal to the number of distinct odd divisors of n that exceed 1". Results about the representation of positive integers as the sum of consecutive numbers abound in the literature. These results have been rediscovered time and time again, even in recreational mathematics. This work aims to explore and expand this result in various ways. The first extension is known, but our approach will bring order and clarity to all results of this type. In addition to consecutive integers, one may wonder about the representations of a number as the sum of a sequence of integers whose difference between consecutive terms is an integer $t > 0$. We will first distinguish between representations using general finite sequences and partitions (representations with positive integers). Once we have analyzed these cases, we will deal with the case in which the difference between consecutive terms of the sequence forms an arithmetic progression (AP). With this, among other questions, we can answer the following: What must a numerical sequence satisfy so that the powers of 3 cannot be represented as partial sums of the sequence?

This work is based on the authors' previous research on partitions in APs (see [2–4]). However, this paper introduces a novel technique compared to the conventional methods studied in partition theory. This approach provides an alternative perspective on classical results and allows for the generalization of proposed problems. The main idea is as follows.

In the usual arithmetic, the set of divisors of a number allows us to divide it into equal parts. Now, we will study a new arithmetic, defining a new multiplication operation in which the set of divisors of a number n allows us to divide it into integers in AP. We will develop this idea in Section 2. Then, in Section 3, we will extend the product definition to study representations of n as a sum of sequences of integers whose differences between consecutive terms form APs. As we will see when studying the set of divisors in the new arithmetics, symmetries will occur depending on two parameters. We can see the most representative case of this symmetry in Lemma 5.

2. First Extension

Definition 1. *Given $m,\ t \in \mathbb{Z}$, for all positive integers n, we define the multiplication operation $\odot_t$ as follows:*

$$m \odot_t n = m + (m+t) + \ldots + (m + t + \overset{(n-1)}{\cdots} + t). \tag{1}$$

We can easily add the right-hand side of this equation to obtain

$$m \odot_t n = m \cdot n + \frac{n \cdot (n-1) \cdot t}{2}. \tag{2}$$

Notation 1. *Let $t \in \mathbb{Z}$. We use the notation $\mathcal{Z}_t = \{\mathbb{Z}, +, \odot_t, <\}$ to infer that we are working on the set of integers with the usual order, the usual addition, and the new multiplication operation.*

The study of $\mathcal{Z}_t$ is interesting in itself. In the article titled "Alternative varieties of integer multiplication" (F. J. de Vega, under review), we study $\mathcal{Z}_t$ as a variation of Dedekind–Peano arithmetic, where the classic axioms of multiplication are modified. The algebraic properties of this new structure are also studied.

Definition 2. *An integer $d > 0$ is called a divisor of n on $\mathcal{Z}_t$, denoted by $d|_t n$, if there exists some integer b such that $n = b \odot_t d$.*

In other words, d is the number of terms in the summation (1).

If we defined a new product mapping, we should have a new quotient.

Definition 3. *Let $a, b \in \mathbb{Z}$, $b \neq 0$. An integer c is called a quotient of a divided by b on $\mathcal{Z}_t$ if and only if $c \odot_t b = a$. We write: $a \oslash_t b = c \iff c \odot_t b = a$.*

In addition, we can use the usual quotient to study the new one:

$$a \oslash_t b = \frac{a}{b} - (b-1) \cdot \frac{t}{2}. \tag{3}$$

For instance, if we want to write 93 as the sum of 6 terms of an AP whose difference is 5, then the quotient indicates the first term of the solution: $93 \oslash_5 6 = 93/6 - 5 \cdot 5/2 = 3$. Then, $93 = 3 \odot_5 6 = 3 + 8 + 13 + 18 + 23 + 28$. Furthermore, $6|_5 93$.

The following result will allow us to study the divisors of an integer a on $\mathcal{Z}_t$.

Corollary 1. *An integer $d > 0$ is a divisor of a on $\mathcal{Z}_t \iff a \oslash_t d$ is an integer.*

What is the significance of divisors of n on $\mathcal{Z}_t$? We can see it in the following remark.

Remark 1. *If we have the divisors of n on $\mathcal{Z}_t$, then we have the arithmetic progressions whose difference is t with sum n.*

Proof. If d is a divisor of n on $\mathcal{Z}_t$, there exists an integer a such that $n = a \odot_t d$. Then $n = a + (a+t) + \ldots + (a + (d-1)t)$. Hence, n is the sum of an AP whose difference is t. On the other hand, if n is the sum of an AP whose difference is t, there must exist $a, d \in \mathbb{Z}$,

$d > 0$ such that $n = a + (a+t) + (a+2t) + \ldots + (a+(d-1)t)$. Then $n = a \odot_t d$ and d is a divisor of n on $\mathcal{Z}_t$. □

That is it! By calculating the divisors of a number on $\mathcal{Z}_t$, we obtain its representations as the sum of integers in AP.

Lemma 1. *Let $t, a \in \mathbb{Z}$. The set of divisors of a on $\mathcal{Z}_t$, denoted by $Div_t(a)$, consists of*

1. *The usual divisors of a if $t \in E = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$.*
2. *The usual divisors of $2a$ except the even usual divisors of a if $t \in O = \{\ldots, -1, 1, 3, 5, \ldots\}$.*

Proof. 1. We use Corollary 1. If $d \mid a$, then $a/d - (d-1)t/2$ is an integer because $t \in \mathrm{E}$. The reciprocal is also trivial; hence, if $t \in \mathrm{E}$, then $d \mid a \Leftrightarrow d \mid_t a$.

2. Let $\mathrm{Div}(a)$ be the set of the usual divisors of a. It is clear that $\mathrm{Div}_t(a) \subseteq \mathrm{Div}(2a)$. Indeed, $d \mid_t a \Leftrightarrow \exists b$ such that $b \odot_t d = a \Leftrightarrow 2a/d = b + (d-1)t \in \mathbb{Z}$. Therefore, we have two cases:

- $d \mid a$: if $d \in \mathrm{Div}(a)$ and $t \in \mathrm{O}$, then $a \oslash_t d \in \mathbb{Z} \Leftrightarrow a/d - (d-1)t/2 \in \mathbb{Z} \Leftrightarrow d \in \mathrm{O}$.
- $d \mid 2a$ and $d \nmid a$: we can suppose that $\exists h \in \mathbb{Z}$ such that $a/d = h/2$ with $h \equiv 1 \pmod 2$. Hence, if $t \in \mathrm{O}$, then $a \oslash_t d \in \mathbb{Z}$, which is equivalent to

$$a/d - (d-1)t/2 \in \mathbb{Z} \Leftrightarrow h/2 - (d-1)t/2 \in \mathbb{Z} \Leftrightarrow \big(1 - (d-1)t\big)/2 \in \mathbb{Z} \Leftrightarrow d \in \mathrm{E}.$$

□

As we will see, there will be two possibilities for $\mathrm{Div}_t(n)$, which we will denote as $\mathrm{Div}_{\mathrm{E}}(n)$ and $\mathrm{Div}_{\mathrm{O}}(n)$. The first will refer to the case where t is even, and the second will refer to the case where t is odd.

Example 1. *Calculate the set $Div_3(30)$.*

Solution. By Point 2 of Lemma 1, the divisors of 30 on $\mathcal{Z}_3$ are the usual divisors of 60 except the even usual divisors of 30. Hence, $Div_3(30) = \{1, 3, 4, 5, 12, 15, 20, 60\}$.

Now, by Remark 1, the statement of this example can be replaced by: "Express the number 30 in all possible ways as a sum of APs whose difference is 3". Each divisor d produces a representation, and the first term of each representation is $30 \oslash_3 d$. The solution can be seen in Table 1.

Table 1. Solution to Example 1.

d	$30 \oslash_3 d$	Representation	d	$30 \oslash_3 d$	Representation
1	30	30	12	-14	$-14 - 11 - 8 - \ldots + 13 + 16 + 19$
3	7	$7 + 10 + 13$	15	-19	$-19 - 16 - 13 - \ldots + 17 + 20 + 23$
4	3	$3 + 6 + 9 + 12$	20	-27	$-27 - 24 - 21 - \ldots + 24 + 27 + 30$
5	0	$0 + 3 + 6 + 9 + 12$	60	-88	$-88 - 85 - 82 - \ldots + 83 + 86 + 89$

So far, everything has been elementary. However, with this lemma, we can already study the representations of integers as the sum of arithmetic progressions. Let us see how to proceed and organize the already-known results.

2.1. An Integer as a Sum of an AP

The following corollaries appear in [5]. Let us apply our approach to prove them.

Corollary 2. *Let $n = 2^e p_1^{e_1} \cdots p_r^{e_r}$ be any positive integer, where $p_1, \ldots, p_r$ are distinct odd primes. The number of ways n can be expressed as the sum of an arithmetic series of integers with a specified odd common difference, t, is twice the number of distinct positive odd divisors of n.*

Proof. Let $\tau_O(n) = (e_1+1)\cdot\ldots\cdot(e_r+1)$ denote the number of odd usual divisors of n. Let $\tau_E(n) = e\cdot(e_1+1)\cdot\ldots\cdot(e_r+1)$ denote the number of even usual divisors of n. We have to calculate the number of elements of $\mathrm{Div}_t(n)$, $t \in O$, denoted by $|\mathrm{Div}_t(n)|$. By Lemma 1,

$$|\mathrm{Div}_t(n)| = \tau(2n) - \tau_E(n) = (e+2)\prod_{i=1}^{r}(e_i+1) - e\prod_{i=1}^{r}(e_i+1) = 2\tau_O(n).$$

Furthermore, if $t \in O$, $|\mathrm{Div}_t(n)| = (\tau_E(2n) - \tau_E(n)) + \tau_O(2n) = (\tau_O(n)) + \tau_O(n)$. Hence, exactly half of the elements of $\mathrm{Div}_t(n)$ are even, and the other half are odd. □

Corollary 3. *In the conditions of the previous corollary, half of the representations have an even number of terms, and the other half have an odd number of terms.*

Corollaries 2 and 3 can be verified with Example 1.

Corollary 4. *Let n be any positive integer. The number of ways n can be expressed as the sum of an arithmetic sequence of integers with a specified even common difference t is the number of distinct positive divisors of n, denoted by $\tau(n)$.*

Proof. This corollary is a consequence of Point 1 of Lemma 1. □

Example 2. *Express the number 30 in all possible ways as a sum of an AP whose difference is $t = 4$.*

Solution. By Corollary 4, each usual divisor d of 30 produces a representation. The first term of this representation is $30 \oslash_4 d$ and the solution can be seen in Table 2.

Table 2. Solution to Example 2.

d	$30 \oslash_4 d$	Representation	d	$30 \oslash_4 d$	Representation
1	30	30	6	-5	$-5-1+3+7+11+15$
2	13	$13+17$	10	-15	$-15-11-\ldots+13+17+21$
3	6	$6+10+14$	15	-26	$-26-22-\ldots+22+26+30$
5	-2	$-2+2+6+10+14$	30	-57	$-57-53-\ldots+51+55+59$

2.2. An Integer as a Sum of Positive Integers in an AP

This problem was almost solved in the previous section. We need to select the representations with a positive first term. If we want to study the representation of an integer $n > 0$ as a sum of positive integers in an AP whose difference is t, then the first term must be greater than 0. If $d|_t n$, then the first term of the representation will be $n \oslash_t d$. Hence,

$$n \oslash_t d > 0 \Leftrightarrow n/d - (d-1)t/2 > 0 \Leftrightarrow t < \frac{2n}{d(d-1)}. \tag{4}$$

Remark 2. *Given an integer $n > 0$ and $t \in O$, the divisors $d \in Div_t(n)$ that produce representations of n as a sum of positive integers in an AP with a common odd difference t satisfy $d < \sqrt{2n/t}$.*

Proof. By (4), if $t \in O$, $t > 0$, and $\frac{2n/t}{d(d-1)} \leq 1$, then there will be no representations. Hence, if $d > \sqrt{2n/t}$ we will not have an arithmetic progression of positive integers with sum n.

If $d = \sqrt{2n/t}$, then $\sqrt{2n/t}$ is even and $\sqrt{2n/t} \mid n$. Thus, by Lemma 1, this case should be excluded. □

The difference between this result and those studied, for example, in [5], is that here, in addition to having an upper bound on the number of terms in the representation, we also know exactly which lengths we have to consider (Lemma 1).

Example 3. *Express the number* 100 *in all possible ways as a sum of an AP of positive integers whose common difference is* $t = 7$.

Solution. The divisors of 100 *on* $\mathcal{Z}_7$ *are* $Div_O(100) = \{1, 5, 8, 25, 40, 200\}$. *As* $\sqrt{2 \cdot 100/7} \approx 5.35$, *there is a unique nontrivial representation produced by the divisor* $d = 5$: $100 \oslash_7 5 = 6$, *hence,* $100 = 6 \odot_7 5 = 6 + 13 + 20 + 27 + 34$.

Remark 3. *Given an integer* $n > 0$ *and* $t \in E$, *the divisors* $d \in Div_t(n)$ *that produce representations of* n *as a sum of positive integers in an AP with a common even difference* t *satisfy* $d \leq \sqrt{2n/t}$.

Proof. The difference between this remark and Remark 2 is that, here, we could have a divisor $d = \sqrt{2n/t}$. □

Example 4. *Express the number* 200 *in all possible ways as a sum of an AP of positive terms whose difference is* $t = 16$.

Solution. As $t \in E$, *by Lemma 1 and the previous remark, every usual divisor* d *of* 200, $d \leq \sqrt{2 \cdot 200/16} = 5$, *produces a representation. These can be seen in Table 3.*

Table 3. Solution to Example 4.

d	$200 \oslash_{16} d$	Representation	d	$200 \oslash_{16} d$	Representation
1	200	200	4	26	$26 + 42 + 58 + 74$
2	92	$92 + 108$	5	8	$8 + 24 + 40 + 56 + 72$

2.2.1. The Case $t = 1$

Now, Sylvester's theorem appears as a consequence of Corollary 2 and Remark 2.

Theorem 1 (Sylvester)**.** *Every integer* n, *not a power of* 2, *is a sum of two or more consecutive positive integers. Moreover, the number of such representations is equal to the number of distinct odd divisors of* n *that exceed* 1.

Proof. By Corollary 2, we have $2 \cdot \tau_O(n)$ possible representations, but, by Remark 2, the number of elements $d \in \text{Div}_1(n)$ with $d < \sqrt{2n}$ are half (if $d \in \text{Div}_1(n)$ such that $d < \sqrt{2n}$ and $2n = d \cdot a$, then $a \in \text{Div}_1(n)$ and $a > \sqrt{2n}$). Hence, we will have $\tau_O(n)$ representations with positive integers. □

Example 5. *Express the number* 50 *as a sum of positive and consecutive integers.*

Solution. We will write the divisors of 50 *on* $\mathcal{Z}_1$ *by pairs. By Remark 2, the first row produces representations only.* $D_1(50) = \left\{ \begin{matrix} 1 & 4 & 5 \\ 100 & 25 & 20 \end{matrix} \right\}$. *Then, we have* 3 *representations (the number of odd divisors of* 50*), two of them with more than one term:* $d = 4$, $50 \oslash_1 4 = 11$, $50 = 11 + 12 + 13 + 14$; $d = 5$, $50 \oslash_1 5 = 8$, $50 = 8 + 9 + 10 + 11 + 12$.

Let us now study unique representations as sums of positive integers in AP. For example, some of these results appear in [6,7]. With our approach to the problem we can better understand these representations.

Corollary 5. *A power of two* $n = 2^s$, $s \in \mathbb{Z}$, $s > 0$ *cannot be represented as a sum of two or more positive integers in an AP whose common difference is* $t \in O$.

Proof. By Lemma 1, $\text{Div}_O(2^s) = \{1, 2^{s+1}\}$. Hence, we will have the trivial representation $2^s = 2^s$ and another starting with the negative number $n \oslash_t 2^{s+1}$. □

Corollary 6. *Let $n = 2^s \cdot p$, where p is an odd prime and s is a non-negative integer, and let $t \in O$. If $p < 2^{s+1}/t$, there exists a unique representation of n as a sum of p numbers in an AP with a common difference t. If $2^{s+1} < p/t$, the unique representation will have 2^{s+1} parts.*

Proof. The divisors of $n = 2^s \cdot p$ on $\mathcal{Z}_t$ are $\mathrm{Div}_t(n) = \{1, p, 2^{s+1}, 2^{s+1} \cdot p\}$. Hence, by Remark 2, the divisor $d = p$ will produce a representation if $p < 2^{s+1}/t$, or the divisor $d = 2^{s+1}$ will produce a representation if $2^{s+1} < p/t$. □

Example 6. *We show that 496 and 248 are uniquely representable as sums of two or more consecutive positive integers.*

Solution. $496 = 2^4 \cdot 31$. *By the previous corollary,* $2^5 > 31$, *hence the solution will be produced by the divisor* $d = 31$: $496 \oslash_1 31 = 1$, *hence* $496 = 1 + 2 + \ldots + 30 + 31 = 496$.

$248 = 2^3 \cdot 31$. *As* $2^4 < 31$, *the unique representation will be produced by the divisor* 2^{3+1}: $248 \oslash_1 2^4 = 8$, *hence* $248 = 8 + 9 + \ldots + 22 + 23$.

2.2.2. The Case $t = 2$

Corollary 7. *Every positive integer $n > 1$, not a prime, is a sum of two or more positive integers in APs whose common difference is 2. Moreover, the number of such representations is equal to $\lceil \frac{\tau(n)}{2} \rceil$.*

Proof. By Remark 3, $d \leq \sqrt{n}$. As $\mathrm{Div}_2(n)$ are the usual divisors of n, denoted by $\mathrm{Div}(n)$, if n is a perfect square, we will have $(\tau(n) + 1)/2$ divisors that produce partitions; and if n is not a perfect square, we will have $\tau(n)/2$ possibilities. Hence, the result follows. □

Example 7. *Express the number 36 in all possible ways as partitions in an AP whose difference is 2.*

Solution. By the previous corollary, each usual divisor $d \leq \sqrt{36} = 6$ *produces a partition. These are presented in Table 4.*

Table 4. Solution to Example 7.

d	$36 \oslash_2 d$	Representation	d	$36 \oslash_2 d$	Representation
1	36	36	4	6	$6 + 8 + 10 + 12$
2	17	$17 + 19$	6	1	$1 + 3 + 5 + 7 + 9 + 11$
3	10	$10 + 12 + 14$			

Let us now study unique representations in this case.

Corollary 8. *A usual prime p cannot be represented as a sum of two or more positive integers in APs with common difference $t \in E$.*

Corollary 9. *Let p and q be usual primes ($p < q$). Then, p^2, $p \cdot q$, and p^3 have unique nontrivial representations as sums of positive integers in APs with common difference 2.*

Proof. In the three cases, p is the unique nontrivial divisor such that $p \leq \sqrt{n}$. □

We can extend Corollary 9 to arithmetic progressions with even differences.

Corollary 10. *Let p and q be usual primes and $t \in E$ such that $p \leq 2q/t$. Then, the number $n = p \cdot q$ has a unique representation as a sum of p positive terms of an AP whose difference is t.*

Proof. We have only to apply Remark 3 to the usual divisors of $n = p \cdot q$. □

Example 8. *Analyze the representations of the number 4369 as the sum of an AP of positive integers with common difference $t = 30$.*

Solution. As $4369 = 17 \cdot 257$ *and* $17 < 2 \cdot 257/30$*, by the previous corollary, we will have a unique representation with* 17 *terms. Hence,* $4369 \oslash_{30} 17 = 17$ *and* $4369 = 17 + 47 + 77 + \ldots + 437 + 467 + 497$.

To conclude this section, we revisit the basic properties of $\mathcal{Z}_t$ by formulating a result that clarifies the first extension of Sylvester's theorem.

We are going to define the prime numbers on $\mathcal{Z}_t$. Based on Lemma 1, an integer $a > 1$ always has two divisors on $\mathcal{Z}_t$. Indeed, if $t \in E$, 1 and a are divisors of a on $\mathcal{Z}_t$ and if $t \in O$, then 1 and $2a$ are divisors of a on $\mathcal{Z}_t$. Thus, we can write the following definition.

Definition 4. *An integer* $p > 1$ *is called a prime on* $\mathcal{Z}_t$ *if it has only two divisors on* $\mathcal{Z}_t$*. An integer greater than* 1 *that is not a prime on* $\mathcal{Z}_t$ *is termed a composite on* $\mathcal{Z}_t$*.*

We use Lemma 1 to characterize the primes on $\mathcal{Z}_t$.

Theorem 2. *Let* $t \in \mathbb{Z}$*. The primes on* $\mathcal{Z}_t$ *are:*

1. *The usual primes if* $t \in E$*.*
2. *The powers of two if* $t \in O$*.*

Proof. 1. Let $t \in E$. By Lemma 1, if $a > 1$, then $d \mid a \Leftrightarrow d \mid_t a$ and the result holds.

2. Let $t \in O$. If $a > 1$ is not a power of 2, then $a = 2^s \cdot b$, b is odd and $s \in \{0, 1, 2, \ldots\}$. By Lemma 1, $b \mid_t a$. In conclusion, 1, $2a$, and b are divisors of a on $\mathcal{Z}_t$; hence, a is not prime on $\mathcal{Z}_t$.

If $a > 1$ is a power of two, then $a = 2^s$, $s \in \{1, 2, \ldots\}$. By Lemma 1, the divisors of a on $\mathcal{Z}_t$ are the usual divisors of $2a$ except the even usual divisors of a. Hence, the divisors of a on $\mathcal{Z}_t$ are $\{1,\ 2 \cdot 2^s\}$; thus, a is prime on $\mathcal{Z}_t$. $\square$

As we can see, Theorem 1 and Corollaries 5, 7 and 8 are encompassed by Theorem 2.

3. Second Extension

With the same idea as in the previous section, we will define a new product between two integers, but now, the difference between consecutive addends will form the arithmetic progression $(t_1 + t_2 \cdot i)_{i \geq 0}$.

Definition 5. *Given* $m, t_1, t_2 \in \mathbb{Z}$*, for all positive integers* n*, we define the following product operation between two numbers:*

$$m \odot n = m + (m + t_1) + (m + 2t_1 + t_2) + (m + 3t_1 + 3t_2) + \ldots + (m + (n-1)t_1 + \frac{(n-2)(n-1)t_2}{2}).$$

Note that the differences between consecutive addends in parentheses form the arithmetic progression $(t_1 + x \cdot t_2)_{x \geq 0}$.

By adding the right-hand side of the previous expression we obtain

$$m \odot n = m \cdot n + \frac{(n-1) \cdot n \cdot t_1}{2} + \frac{(n-2) \cdot (n-1) \cdot n \cdot t_2}{6}. \tag{5}$$

Furthermore, similar to Definition 3 and Equation (3), we deduce the new quotient:

$$a \oslash b = \frac{a}{b} - (b-1) \cdot (\frac{t_1}{2} + \frac{(b-2) \cdot t_2}{6}). \tag{6}$$

Notation 2. *Let* $t_1, t_2 \in \mathbb{Z}$*. We use the notation* $\mathcal{Z}_{t_1,t_2} = \{\mathbb{Z}, +, \odot, <\}$ *to infer that we are working with the set of integers, the usual order, the addition, and the new multiplication operation.*

The definitions of divisors and primes on $\mathcal{Z}_{t_1,t_2}$ are analogous to Definitions 2 and 4, respectively. Furthermore, similar to Corollary 1 we can identify the set of divisors as follows.

Corollary 11. *An integer $d > 0$ is a divisor of n on $\mathcal{Z}_{t_1,t_2}$, denoted by $d \,|_{t_1,t_2}\, n$, if and only if $n \oslash d$ is an integer.*

The divisors of an integer n, denoted by $\mathrm{Div}_{t_1,t_2}(n)$, are studied in the following lemmas. As we will see, there will be six possible sets of divisors depending on whether t_1 is even or odd and depending on whether $t_2 \equiv 1, 2, 0 \pmod 3$. We will denote these sets as $\mathrm{Div}_{\mathrm{E},1}(n)$, $\mathrm{Div}_{\mathrm{E},2}(n)$, $\mathrm{Div}_{\mathrm{E},3}(n)$, $\mathrm{Div}_{\mathrm{O},1}(n)$, $\mathrm{Div}_{\mathrm{O},2}(n)$, and $\mathrm{Div}_{\mathrm{O},3}(n)$. For instance, $\mathrm{Div}_{\mathrm{E},2}(n)$ means that t_1 is an even number and $t_2 \equiv 2 \pmod 3$.

Each of these six cases will produce an extension of Sylvester's theorem. We will focus on the first case and state the results of the other cases by providing solved examples for each.

3.1. Case $t_1 \in E$ and $t_2 \equiv 1 \pmod 3$

Lemma 2. *Let $t_1 \in E$ and $t_2 \equiv 1 \pmod 3$. The elements of the set $Div_{t_1,t_2}(n)$ satisfy the following conditions:*

(a) If $d \in Div(a)$, then $d \,|_{t_1,t_2}\, a \iff 3 \nmid d$.

(b) If $d \in \big(Div(3a) \setminus Div(a)\big)$, then $d \,|_{t_1,t_2}\, a \iff 3 \mid d$ and $\frac{3a}{d} \equiv 1 \pmod 3$.

Proof. Let $\mathrm{Div}(a)$ be the set of usual divisors of a. Then, $\mathrm{Div}_{t_1,t_2}(n) \subseteq \mathrm{Div}(6a)$. This is because: $d \mid a$ on $\mathcal{Z}_{t_1,t_2} \Leftrightarrow b \odot d = a \Leftrightarrow 6a/d = 6b + 3(d-1)t_1 + (d-2)(d-1)t_2 \in \mathbb{Z}$.

(a) Suppose $d \mid a$. By Equation (6) and Corollary 10, it is easy to observe that

$$d \,|_{t_1,t_2}\, a \Leftrightarrow \frac{(d-1)(d-2)t_2}{6} \in \mathbb{Z} \Leftrightarrow 3 \nmid d.$$

Indeed, if $3 \nmid d$, then $(d-1)(d-2)$ is a multiple of 6 and $(d-1)(d-2)t_2/6$ is an integer for all values of t_2. If $3 \mid d$, then $(d-1)(d-2)$ is not divisible by 3 and hence not divisible by 6 for all values $t_2 \equiv 1 \pmod 3$.

We claim that if $d \mid 2a$ and $d \nmid a$, then $d \nmid_{t_1,t_2} a$.

To see this, note that $d \mid 2a$ and $d \nmid a \Rightarrow \exists h \in \mathbb{Z}$ such that $2a = dh \Rightarrow a/d = h/2$. By hypothesis $d \nmid a$, so h is odd. Then,

$$a \oslash d = \frac{h}{2} - (d-1) \cdot \Big(\frac{t_1}{2} + \frac{(d-2) \cdot t_2}{6}\Big).$$

Therefore,

$$a \oslash d \in \mathbb{Z} \Leftrightarrow \frac{1}{2} - \frac{(d-1)(d-2)t_2}{6} \in \mathbb{Z}. \tag{7}$$

But if $d \equiv 1, 2 \pmod 3$, then $(d-1)(d-2)t_2/6$ is an integer, say e_1, for all values of t_2. Based on (7), $1/2 - e_1 \notin \mathbb{Z} \Leftrightarrow a \oslash d \notin \mathbb{Z}$ and if $d \equiv 0 \pmod 3$, then $(d-1)$ or $(d-2)$ is even and the product of the other and t_2 is not divisible by 3. So, $\exists e_2 \in \mathbb{Z}$ such that $(d-1)(d-2)t_2/6 = e_2/3$. Based on (7), $1/2 - e_2/3 \notin \mathbb{Z} \Leftrightarrow a \oslash d \notin \mathbb{Z}$.

Thus, we have proved that if $d \mid 2a$ and $d \nmid a$, then $d \nmid_{t_1,t_2} a$. This implies that $\mathrm{Div}_{t_1,t_2}(a) \subseteq \mathrm{Div}(3a)$, and we have to examine only the following point (b) to exhaust all possibilities of the case $t_1 \in \mathrm{E}$, $t_2 \equiv 1 \pmod 3$.

(b) Suppose $d \mid 3a$ and $d \nmid a$. Then, $\exists h \in \mathbb{Z}$ such that $3a = dh \Rightarrow a/d = h/3$ and $h \equiv 1$ or $2 \pmod 3$. Similar to the proof of part (a), we deduce

$$a \oslash d \in \mathbb{Z} \Leftrightarrow \frac{h}{3} - \frac{(d-1)(d-2)t_2}{6} \in \mathbb{Z}.$$

Now, if $d \equiv 1, 2 \pmod 3$, then $(d-1)(d-2)t_2/6$ is an integer, say e, for all t_2. Hence, $\frac{h}{3} - e \notin \mathbb{Z}$ and if $d \equiv 0 \pmod 3$, say $d = 3d_1$, then $(d-1)$ or $(d-2)$ is even:

In the first case, $(d-1)/2 \equiv 1 \pmod 3$. Let $(d-1)/2 = 3d_2 + 1$ and let $t_2 = 3t_2^* + 1$ for some $d_2, t_2^* \in \mathbb{Z}$.

$$a \oslash d \in \mathbb{Z} \Leftrightarrow \frac{h}{3} - \frac{(3d_2+1)(3d_1-2)(3t_2^*+1)}{3} \in \mathbb{Z} \Leftrightarrow \frac{h+2}{3} \in \mathbb{Z}.$$

Then $h \equiv 1 \pmod 3$. In the second case, $(d-2)$ is even, and we reach the same conclusion as before, that is, $a \oslash d \in \mathbb{Z} \Leftrightarrow h \equiv 1 \pmod 3$. Therefore, the result follows. □

Example 9. *Express the number* 336 *in all possible ways as a sum of an integer sequence* $(a_n)_{n>0}$ *whose differences* $a_{n+1} - a_n$*, form the arithmetic progression* $\{0, 1, 2, 3, \dots\}$*.*

Solution. We have to find the divisors of 336 *on* $\mathcal{Z}_{0,1}$*.*

- *By Point (a) of Lemma 2, we have to eliminate multiples of* 3 *from the usual divisors of* 336*:* $\{1, 2, 4, 7, 8, 14, 16, 28, 56, 112, \} \subseteq Div_{0,1}(336)$*.*
- $Div(3 \cdot 336) \setminus Div(336) = \{9, 18, 36, 63, 72, 126, 144, 252, 504, 1008\}$*. By Point (b) of Lemma 2,* $\{9, 36, 63, 144, 252, 1008\} \subseteq Div_{0,1}(336)$*. For instance,* 18 *is not a divisor of* 336 *on* $\mathcal{Z}_{0,1}$ *because* $3 \cdot 336/18 = 56 \equiv 2 \pmod 3$*.*

Hence, $Div_{0,1}(336) = \{1, 2, 4, 7, 8, 9, 14, 16, 28, 36, 56, 63, 112, 144, 252, 1008\}$ *and the solutions are presented in Table 5.*

Table 5. Solution to Example 9.

d	$336 \oslash d$	Representation	d	$336 \oslash d$	Representation
1	336	336	28	-105	$-105 - 105 - \dots + 246$
2	168	$168 + 168$	36	-189	$-189 - 189 - \dots + 406$
4	83	$83 + 83 + 84 + 86$	56	-489	$-489 - 489 - \dots + 996$
7	43	$43 + 43 + \dots + 53 + 58$	63	-625	$-625 - \dots + 1266$
8	35	$35 + 35 + \dots + 50 + 56$	112	-2032	$-2032 - \dots + 4073$
9	28	$28 + 28 + \dots + 49 + 56$	144	-3382	$-3382 - \dots + 6671$
14	-2	$-2 - 2 - \dots + 64 + 76$	252	-10457	$-10457 - \dots + 20918$
16	-14	$-14 - 14 - \dots + 77 + 91$	1008	-168840	$-168840 - \dots + 337681$

In this example, the number of even divisors is three times that of the odd divisors. Is this a coincidence? We now show that it is not.

Although the theory we are developing is elementary, here we can already see the power of the approach. The next two corollaries would have been challenging to prove without this focus on the problem.

Corollary 12. *Let* $n = 2^a 3^b p_1^{e_1} \cdots p_r^{e_r}$ *be the prime factorization of* n*, where* $p_1, \dots, p_r$ *are distinct primes congruent to* $1 \pmod 3$ *and* a *is even. The number of representations of* n *as the sum of integer sequences* $(a_i)_{i\geq 0}$ *in which the differences between consecutive terms* $(a_{i+1} - a_i)_{i \geq 0}$ *form an AP whose first term is an even number with common difference congruent to* $1 \pmod 3$ *is given by* $(3a/2 + 2) \cdot (e_1 + 1) \cdots (e_r + 1)$*.*

Proof. By point (a) of Lemma 2, we have $a \cdot \prod_{i=1}^{r}(e_i + 1)$ even and $\prod_{i=1}^{r}(e_i + 1)$ odd divisors of n on $\mathcal{Z}_{t_1,t_2}$, where $t_1 \in E$, $t_2 \equiv 1 \pmod 3$. Since the product of two or more primes of the form $1 \pmod 3$ is of the same form and $2^s \equiv 1 \pmod 3$ if and only if s is even, point (b) implies that we have $(a/2) \cdot \prod_{i=1}^{r}(e_i + 1)$ new even divisors on $\mathcal{Z}_{t_1,t_2}$ and $\prod_{i=1}^{r}(e_i + 1)$ odd divisors. In total we have $(3a/2) \cdot \prod_{i=1}^{r}(e_i + 1)$ even divisors and $2 \cdot \prod_{i=1}^{r}(e_i + 1)$ odd divisors on $\mathcal{Z}_{t_1,t_2}$, and the result follows. □

Corollary 13. *Under the hypotheses Corollary 12, if* $a = 4$*, then the number of even divisors on* $\mathcal{Z}_{t_1,t_2}$*,* $t_1 \in E$*,* $t_2 \equiv 1 \pmod 3$ *will be three times the number of odd divisors on* $\mathcal{Z}_{t_1,t_2}$*.*

From Example 9, $336 = 2^4 \cdot 3 \cdot 7$ and then, we have $(3 \cdot 4/2) \cdot (1+1) = 12$ representations with an even number of terms and $2 \cdot (1+1) = 4$ representations with an odd number of terms.

We could develop similar results to this one for each lemma that we will study in this section. This objective will not be ours; therefore, we will only present the following corollary.

We present the following corollary in analogy to Corollary 12.

Corollary 14. *Let $n = 2^a 3^b p_1^{e_1} \cdots p_r^{e_r}$ be the prime factorization of n, where $p_1, \ldots, p_r$ are distinct primes congruent to $1 \pmod 3$ and a is even. The number of representations of n as the sum of integer sequences $(a_i)_{i \geq 0}$ in which the differences between consecutive terms $(a_{i+1} - a_i)_{i \geq 0}$ form an AP whose first term is an even number with common difference congruent to $1 \pmod 3$ is two times the odd divisors of n that are not multiples of 3.*

Proof. By point (a) of Lemma 2, there are $\prod_{i=1}^{r}(e_i + 1)$ distinct odd divisors. By point (b) of Lemma 2, there are also $\prod_{i=1}^{r}(e_i + 1)$ divisors on $\mathcal{Z}_{t_1,t_2}$, $t_1 \in E$, $t_2 \equiv 1 \pmod 3$. □

The following theorem can be a consequence of Lemma 2 or the last corollary.

Theorem 3. *Let $t_1 \in E$, $t_2 \equiv 1 \pmod 3$. The set of primes on $\mathcal{Z}_{t_1,t_2}$ is $\{3^s : s \in \mathbb{Z}, s > 0\}$.*

Proof. If $a > 1$ and $a \notin \{3^s : s \in \mathbb{Z}, s > 0\}$, then $a = 3^{s_1} \cdot b$, $s_1 \geq 0$, $b > 1$ and $3 \nmid b$. By Lemma 2, 1, b, and $3a$ are divisors of a on $\mathcal{Z}_{t_1,t_2}$. Hence, a is not prime on $\mathcal{Z}_{t_1,t_2}$.

Finally, if $a = 3^s$, by Lemma 2, the unique divisors of a on $\mathcal{Z}_{t_1,t_2}$ are 1 and $3a$. Hence, a is prime on $\mathcal{Z}_{t_1,t_2}$. □

It follows that if $t_1 \in E$ and $t_2 \equiv 1 \pmod 3$, then a power of three, say 3^s, has only two representations on $\mathcal{Z}_{t_1,t_2}$. The first is trivial and the second begins with the negative term $3^s \oslash 3^{s+1}$.

Now, we turn to partitions under these conditions. Similar to (4), if d is a divisor of n on $\mathcal{Z}_{t_1,t_2}$, $t_1, t_2 \in \mathbb{Z}$, the first term of the representation will be $n \oslash d$. Hence,

$$\frac{n}{d} - (d-1) \cdot \left(\frac{t_1}{2} + \frac{(d-2) \cdot t_2}{6}\right) > 0 \Leftrightarrow t_1 < \frac{2n}{d(d-1)} - \frac{(d-2) \cdot t_2}{3}. \tag{8}$$

With (8), taking $t_1 = 0$ and $t_2 = 1$, we obtain a bound for the divisors on $\mathcal{Z}_{t_1,t_2}$, $t_1 \in E$, $t_2 \equiv 1 \pmod 3$ that produce representations with a positive first term.

Remark 4. *Let $t_1 \in E$ and $t_2 \equiv 1 \pmod 3$. The divisors d of n on $\mathcal{Z}_{t_1,t_2}$ such that $n \oslash d > 0$ satisfy $d \leq \lfloor \sqrt[3]{6n} \rfloor + 1$.*

Proof. In order to find a bound, we take $t_1 = 0$, $t_2 = 1$ and use (8). We will not have a positive first term if

$$\frac{2n}{d(d-1)} - \frac{(d-2)}{3} \leq 0 \Leftrightarrow \frac{6n}{d(d-1)(d-2)} \leq 1. \tag{9}$$

If $d - 2 \geq \sqrt[3]{6n}$, then there will be no representation with a positive first term. Hence, $d < \sqrt[3]{6n} + 2$. Now, the case $d = \lfloor \sqrt[3]{6n} \rfloor + 2$ is ruled out because $d(d-1)(d-2) \geq 6n$ (we would have the product of three consecutive numbers, two of them $\geq n$). So, $d \leq \lfloor \sqrt[3]{6n} \rfloor + 1$. □

We can verify this result with Example 9. The divisors d that produce representations with a first positive term verify $d \leq \lfloor \sqrt[3]{6 \cdot 336} \rfloor + 1 = 13$.

We conclude this section by studying some results on unique representations.

Corollary 15. *Let $t_1 \in E$ and $t_2 \equiv 1 \pmod 3$ and let $n = 3^s \cdot p$, where $s \geq 0$ and $p \equiv 2 \pmod 3$. Then, n has a unique representation as a sum of an integer sequence whose differences between consecutive terms form the arithmetic progression $(t_1 + t_2 \cdot i)_{i \geq 0}$ with more than one term and less than $3n$ terms.*

Proof. By Lemma 2, the divisors of n on $\mathcal{Z}_{t_1,t_2}$ are $\{1, p, 3 \cdot n\}$ and the result follows. □

Example 10. *We illustrate Corollary 15 with $n = 891$ and differences between consecutive terms $a_{i+1} - a_i = 2 + 4 \cdot (i-1)$, $i \geq 1$.*

Since $891 = 3^4 \cdot 11$, the divisors of 891 on $\mathcal{Z}_{2,4}$ are $\{1, 11, 2673\}$. The solution is produced by the divisor $d = 11$: $891 \oslash 11 = 891/11 - 10 \cdot (2/2 + 9 \cdot 4/6) = 11$, $891 = 11 + 13 + 19 + 29 + 43 + 61 + 83 + 109 + 139 + 173 + 211$.

We leave it to the interested reader to obtain corollaries similar to the foregoing ones, given that the set of terms with four divisors on $\mathcal{Z}_{t_1,t_2}$, $t_1 \in E$, $t_2 \equiv 1 \pmod 3$ is formed by numbers of the form $n = 3^s \cdot p$, $s \geq 0$, $p \equiv 1 \pmod 3$. The divisors would be $\{1, 3^{s+1}, p, 3n\}$.

The remaining sections are similar, therefore, we will state the results without proofs (similar to those presented in this section). We will only provide examples in some cases.

3.2. Case $t_1 \in E$ and $t_2 \equiv 2 \pmod 3$

Lemma 3. *Let $t_1 \in E$ and $t_2 \equiv 2 \pmod 3$. The elements of the set $Div_{t_1,t_2}(n)$ satisfy the following conditions:*

(a) If $d \in Div(a)$, then $d \mid_{t_1,t_2} a \Leftrightarrow 3 \nmid d$.

(b) If $d \in \big(Div(3a) \setminus Div(a)\big)$, then $d \mid_{t_1,t_2} a \Leftrightarrow 3 \mid d$ and $\frac{3a}{d} \equiv 2 \pmod 3$.

From Lemma 3, a power of 3 only has the divisor $d = 1$ on $\mathcal{Z}_{t_1,t_2}$. This proves the next statement.

Corollary 16. *Let $t_1 \in E$ and $t_2 \equiv 2 \pmod 3$. A power of three cannot be represented as a sum of two or more terms of an integer sequence $(a_i)_{i \geq 1}$ verifying that the difference between consecutive terms forms the arithmetic progression $(t_1 + t_2 \cdot n)_{n \geq 0}$.*

Example 11. *Express the number 50 in all possible ways as a sum of an integer sequence $(a_n)_{n>0}$ whose differences $a_{n+1} - a_n$, form the arithmetic progression $\{0, 2, 4, 6, \ldots\}$.*

Solution. By Lemma 3, $Div_{0,2}(50) = \{1, 2, 3, 5, 10, 25, 30, 50, 75\}$. The solution can be seen in Table 6.

Table 6. Solution to Example 11.

d	$50 \oslash d$	Representation	d	$50 \oslash d$	Representation
1	50	50	25	-182	$-182 - 182 - \ldots + 324 + 370$
2	25	$25 + 25$	30	-269	$-269 - 269 - \ldots + 487 + 543$
3	16	$16 + 16 + 18$	50	-783	$-783 - 783 - \ldots + 1569$
5	6	$6 + 6 + 8 + 12 + 18$	75	-1800	$-1800 - \ldots + 3602$
10	-19	$-19 - 19 - \ldots + 37 + 53$			

Similar to Remark 4, we can calculate a bound for positive representations:

Remark 5. *Let $t_1 \in E$ and $t_2 \equiv 2 \pmod 3$. The divisors d of n on $\mathcal{Z}_{t_1,t_2}$ such that $n \oslash d > 0$ verify that $d \leq \lfloor \sqrt[3]{3n} \rfloor + 1$.*

For instance, in the previous example, $\lfloor \sqrt[3]{3 \cdot 50} \rfloor + 1 = 6$. Hence, the divisors of 50 on $\mathcal{Z}_{0,2}$ greater than 6 will not produce partitions.

Now, we continue with the results of unique representation.

Theorem 4. *Let $t_1 \in E$, $t_2 \equiv 2$ (mod 3). The set of primes on $\mathcal{Z}_{t_1,t_2}$ is $\{3^{s-1} \cdot p : s \in \mathbb{N}, p$ usual prime, $p \equiv 1$ (mod 3)$\}$. That is, the unique divisors of $n = 3^{s-1} \cdot p$ on $\mathcal{Z}_{t_1,t_2}$ will be 1 and p.*

Example 12. *Consider $n = 189$ with consecutive differences between terms $a_{i+1} - a_i = 5 \cdot (i-1)$, $i \geq 1$.*

Then, $189 = 3^3 \cdot 7$ implies that the divisors of 189 on $\mathcal{Z}_{0,5}$ are $\{1, 7\}$. The unique non trivial solution is: $d = 7$, $189 \oslash 7 = 2$, $189 = 2 + 2 + 7 + 17 + 32 + 52 + 77$.

In order to finish this case, we consider sets of numbers with three or four divisors.

Corollary 17. *Let $t_1 \in E$ and $t_2 \equiv 2$ (mod 3). The set of numbers with three divisors on $\mathcal{Z}_{t_1,t_2}$ is $\{3^{s-1} \cdot p : s \in \mathbb{N}, p$ usual prime, $p \equiv 2$ (mod 3)$\}$. The sets of numbers with four divisors are $\{3^{s-1} \cdot p \cdot q : s \in \mathbb{N}, p$ and q usual primes, $p, q \equiv 1$ (mod 3)$\}$ and $\{3^{s-1} \cdot p^3 : s \in \mathbb{N}$, p usual prime, $p \equiv 1$ (mod 3)$\}$.*

Proof. If $n = 3^{s-1} \cdot p$, $p \equiv 2$ (mod 3), the divisors on $\mathcal{Z}_{t_1,t_2}$ will be $\{1, p, 3^s\}$. If $n = 3^{s-1} \cdot p \cdot q$, $p, q \equiv 2$ (mod 3), the divisors on $\mathcal{Z}_{t_1,t_2}$ will be $\{1, p, q, p \cdot q\}$. If $n = 3^{s-1} \cdot p^3$, $p \equiv 1$ (mod 3), the divisors on $\mathcal{Z}_{t_1,t_2}$ will be $\{1, p, p^2, p^3\}$. $\square$

3.3. Case $t_1 \in E$ and $t_2 \equiv 0$ (mod 3)

This case is one of the most accessible; therefore, we will only state the results.

Lemma 4. *Let $t_1 \in E$ and $t_2 \equiv 0$ (mod 3), then $d \mid a \Leftrightarrow d \mid_{k_1,k_2} a$.*

Theorem 5. *Let $t_1 \in E$ and $t_2 \equiv 0$ (mod 3). The set of primes on $\mathcal{Z}_{t_1,t_2}$ is the set of the usual primes.*

Remark 6. *Let $t_1 \in E$ and $t_2 \equiv 0$ (mod 3). The divisors d of n on $\mathcal{Z}_{t_1,t_2}$ such that $n \oslash d$>0 verify that $d \leq \lfloor \sqrt[3]{2n} \rfloor + 1$.*

Finally, we will study and state the remaining results in a single section. We will start with a Lemma similar to Lemmas 2–4. To clarify the lemma, we will present a simple example. After this, we will classify the set of prime numbers in each case. The reader will be able to obtain results of unique representation similar to those already studied. Then, we will state a Remark providing us with an upper bound for the divisors that produce representations with all positive terms. Lastly, we will state a theorem concerning powers of 3.

3.4. Case $t_1 \in O$, $t_2 \equiv 1, 2$, and 0 (mod 3)

Lemma 5. *Let $t_1 \in O$, $t_2 \in \mathbb{Z}$. The divisors of $n \in \mathbb{Z}$ on $\mathcal{Z}_{t_1,t_2}$ satisfy the following conditions:*

1. *$t_2 \equiv 1$ (mod 3):*
   - (a) *If $d \in Div(n)$, then $d \mid_{t_1,t_2} a \Leftrightarrow 2 \nmid d$ and $3 \nmid d$.*
   - (b) *If $d \in \big(Div(2a) \setminus Div(a)\big)$, then $d \mid_{t_1,t_2} a \Leftrightarrow 2 \mid d$ and $3 \nmid d$.*
   - (c) *If $d \in \big(Div(3a) \setminus Div(a)\big)$, then $d \mid_{t_1,t_2} a \Leftrightarrow 2 \nmid d$ and $3 \mid d$ and $\frac{3a}{d} \equiv 1$ (mod 3).*
   - (d) *If $d \in \Big(Div(6a) \setminus \big(Div(3a) \cup Div(2a)\big)\Big)$, then $d \mid_{t_1,t_2} a \Leftrightarrow 2 \mid d$ and $3 \mid d$ and $\frac{6a}{d} \equiv 5$ (mod 6).*
2. *$t_2 \equiv 2$ (mod 3):*
   - (a) *If $d \in Div(a)$, then $d \mid_{t_1,t_2} a \Leftrightarrow 2 \nmid d$ and $3 \nmid d$.*
   - (b) *If $d \in \big(Div(2a) \setminus Div(a)\big)$, then $d \mid_{t_1,t_2} a \Leftrightarrow 2 \mid d$ and $3 \nmid d$.*
   - (c) *If $d \in \big(Div(3a) \setminus Div(a)\big)$, then $d \mid_{t_1,t_2} a \Leftrightarrow 2 \nmid d$ and $3 \mid d$ and $\frac{3a}{d} \equiv 2$ (mod 3).*
   - (d) *If $d \in \Big(Div(6a) \setminus \big(Div(3a) \cup Div(2a)\big)\Big)$, then $d \mid_{t_1,t_2} a \Leftrightarrow 2 \mid d$ and $3 \mid d$ and $\frac{6a}{d} \equiv 1$ (mod 6).*
3. *$t_2 \equiv 0$ (mod 3):*

(a) *If $d \in Div(a)$, then $d \mid_{t_1,t_2} a \Leftrightarrow 2 \nmid d$.*
(b) *If $d \in \big(Div(2a) \setminus Div(a)\big)$, then $d \mid_{t_1,t_2} a \Leftrightarrow 2 \mid d$.*

We can observe the symmetry mentioned in the introduction that occurs in the different sets of divisors. For example, if a divisor d does not yield a representation in Point 1 c), d would indeed yield a representation under the conditions of Point 2 c).

Example 13. *Express the number 20 in all possible ways as a sum of a sequence $(a_n)_{n>0}$ verifying $a_{i+1} - a_i = i + 2$, $i \in \mathbb{N}$.*

Solution. As the first difference is 3, we need to calculate the set $Div_{3,1}(20)$ using Point 1 of Lemma 5.

- *$Div(20) = \{1, 2, 4, 5, 10, 20\}$. By (a), only 1 and 5 are divisors of 20 on $\mathcal{Z}_{3,1}$.*
- *$Div(40) \setminus Div(20) = \{8, 40\}$. By (b), 8 and 40 are divisors of 20 on $\mathcal{Z}_{3,1}$.*
- *$Div(60) \setminus Div(20) = \{3, 6, 12, 15, 30, 60\}$. By (c), the candidates are 3 and 15 but $60/3 = 20 \equiv 2$ (mod 3) and $60/15 = 4 \equiv 1$ (mod 3). Hence, $3 \nmid_{3,1} 20$ and $15 \mid_{3,1} 20$. We reiterate the symmetric nature of the divisors. Although 3 is not a divisor of 20 on $\mathcal{Z}_{3,1}$, while 15 is, we would have the symmetrical case on $\mathcal{Z}_{3,2}$, for example.*
- *$Div(120) \setminus (Div(60) \cup Div(40)) = \{24, 120\}$. By d), $24 \mid_{3,1} 20$ because $120/24 = 5 \equiv 5$ (mod 6) but $120 \nmid_{3,1} 20$ because $120/120 = 1 \equiv 1$ (mod 6).*

Hence, $Div_{3,1}(20) = \{1, 5, 8, 15, 24, 40\}$ and the solution can be seen in Table 7.

Table 7. Solution to Example 13.

d	$20 \oslash d$	Representation	d	$20 \oslash d$	Representation
1	20	20	15	−50	$-50 - 47 - \ldots + 67 + 83$
5	−4	$-4 - 1 + 3 + 8 + 14$	24	−118	$-118 - 115 - \ldots + 179 + 204$
8	−15	$-15 - 12 - \ldots + 18 + 27$	40	−305	$-305 - 302 - \ldots + 512 + 553$

Theorem 6. *Let $t_1 \in O$, $t_2 \in \mathbb{Z}$. The set of primes is:*

1. *$\{2^{2l-1} \cdot 3^{s-1} : s, l \in \mathbb{N}\}$ if $t_2 \equiv 1$ (mod 3).*
2. *All numbers are composite if $t_2 \equiv 2$ (mod 3).*
3. *$\{2^l : l \in \mathbb{N}\}$ if $t_2 \equiv 0$ (mod 3).*

Proof. The proof is similar to those of previous theorems. We have to use Lemma 5. We comment only that if $t_2 \equiv 2$ (mod 3), then all numbers $a > 1$ have at least three divisors: if $a = 2^{l-1} \cdot h$, h odd, then 1, 2^l and $6a$ are divisors of a. Hence, a is not prime. It can be proved that the set of numbers with exactly three divisors is the following: $\{2^{2l-2} \cdot 3^{s-1} > 1 : s, l \in \mathbb{N}\}$. □

Remark 7. *Let $t_1 \in O$, $t_2 \in \mathbb{Z}$. The divisors d of n on $\mathcal{Z}_{t_1,t_2}$ such that $n \oslash d>0$ verify:*

1. *If $t_2 \equiv 1$ (mod 3), then $d \leq \lfloor \sqrt[3]{6n} \rfloor$.*
2. *If $t_2 \equiv 2$ (mod 3), then $d \leq \lfloor \sqrt[3]{3n} \rfloor + 1$.*
3. *If $t_2 \equiv 0$ (mod 3), then $d \leq \lfloor \sqrt[3]{2n} \rfloor + 1$.*

Table 8 shows the numbers with 3 and 4 divisors on $\mathcal{Z}_{t_1,t_2}$, $t_1 \in$ O. With this information and the previous remark, we can obtain results of unique representation easily. For Table 8, we will consider that $s, l \in \mathbb{N}$, and p, q are usual primes with $p \equiv 1$ (mod 6).

Table 8. Numbers with 3 and 4 divisors on $\mathcal{Z}_{t_1,t_2}$ ($t_1 \in \mathrm{O}$).

t_2	Numb. with 3 Div.	Divisors	Numb. with 4 Div.	Divisors
1 (mod 3)	$2^{2l-2} \cdot 3^{s-1}$	$1, 2^{2l-1}, 3^s$	$2^{2l-1} \cdot 3^{s-1} \cdot p$	$1, 2^{2l}, p, 2^{2l} \cdot p$
2 (mod 3)	$2^{2l-2} \cdot 3^{s-1}$	$1, 2^{2l-1}, 2^{2l-1} \cdot 3^s$	$2^{2l-1} \cdot 3^{s-1}$	$1, 2^{2l}, 3^s, 2^{2l} \cdot 3^s$
0 (mod 3)	$\varnothing$	$\varnothing$	$2^{l-1} \cdot q$	$1, 2^l, q, 2^l \cdot q$

Example 14. *We show that $n = 39548$ has a unique partition such that the differences between consecutive parts form the arithmetic progression $\{1, 4, 7, \ldots\}$.*

Solution. $n = 39548 = 2^2 \cdot 9887$, then the divisors of n on $\mathcal{Z}_{1,3}$ are 1, 2^3, 9887, and $2n$. As $\lfloor \sqrt[3]{2n} \rfloor + 1 = 43$, then the solution is produced by the divisor $d = 2^3$. $39548 \oslash 8 = 4919$, $39548 = 4919 + 4920 + 4924 + 4931 + 4941 + 4954 + 4970 + 4989$.

Our last result is the following theorem concerning the powers of 3. We emphasize again that although the methods used are elementary, this result would be difficult to discover and prove without the approach employed in this article.

Theorem 7. *A power of 3 cannot be represented as a sum of three or more parts of an integer sequence $(a_i)_{i \geq 1}$ such that the differences between consecutive parts form an arithmetic progression whose common difference is 1 or 2 (mod 3).*

Proof. $D_{\mathrm{E},1}(3^s) = \{1, 3^{s+1}\}$, $D_{\mathrm{E},2}(3^s) = \{1\}$, $D_{\mathrm{O},1}(3^s) = \{1, 2, 3^{s+1}\}$, $D_{\mathrm{O},2}(3^s) = \{1, 2, 2 \cdot 3^{s+1}\}$. Using Remarks 4, 5, and 7, the result follows. □

As a consequence of this theorem, we conclude the paper with a corollary similar to Sylvester's original theorem.

Corollary 18. *A power of 3 cannot be represented as a sum of three or more terms of a positive integer sequence $(a_i)_{i \geq 1}$ such that the differences between consecutive terms are consecutive integers.*

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