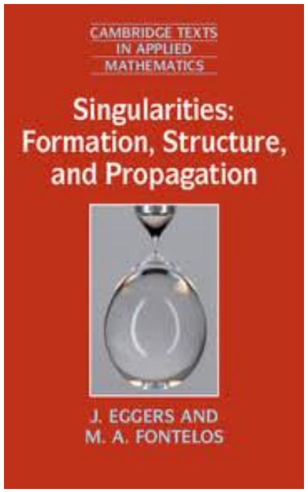
Vortices and Quantum Turbulence

Marco A. Fontelos, CSIC

Bibliography



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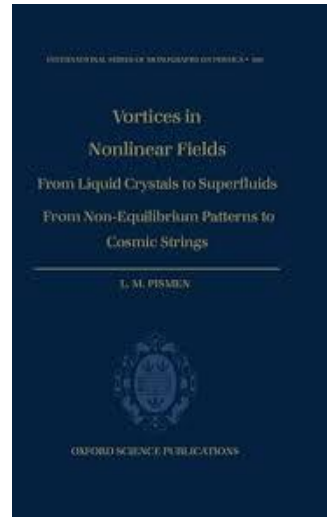

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Quantum turbulence: Theoretical and numerical problems

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Notation:

$$G \to F$$

Gross-Pitaevskii:

$$-i\hbar\phi_t = \frac{\hbar^2}{m}\nabla^2\phi + a(T)\phi - |b(T)|\,\phi\,|\phi|^2$$

(Superfluid, no external magnetic field)
Introduce

$$\phi = \sqrt{\frac{a(T)}{|b(T)|}}u$$

and

$$\kappa = \frac{\hbar}{m}, \quad l^2 = \frac{\hbar^2}{ma(T)}$$

to obtain the following form

$$-i\kappa^{-1}u_t = \nabla^2 u + \frac{1}{l^2}u(1-|u|^2)$$

l dimensions of length, $l^2\kappa^{-1}$ dimensions of time.

Outline

- Vortices: definitions and general considerations
- Vortices in 2D: Disipative vs Hamiltonian dynamics, Ginzburg-Landau and Gross-Pitaevskii
- Vortex filaments
- Singularities and turbulence

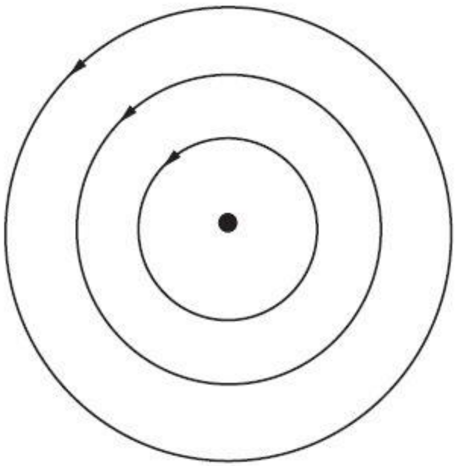


Figure 13.1 The flow lines in a vortex.

Vortex as a singularity

Circulation → Γ

$$\mathbf{v} = \frac{\Gamma}{2\pi}\frac{\mathbf{e}_\theta}{r},$$

$$-i\kappa^{-1}u_t = \nabla^2 u + l^{-2}(1-|u|^2)u,$$

$$\kappa = \hbar/m$$

$$u = \sqrt{\varrho}e^{i\vartheta/2\kappa},$$ Madelung transform

$\varrho = |u|^2$ is the superfluid *density*, and $\vartheta = 2\kappa\theta$ is the *velocity potential*,

the superfluid velocity $\mathbf{v} = \nabla\vartheta$.

Continuity equation

$$\varrho_t + \nabla\cdot(\varrho\mathbf{v}) = 0,$$

Bernoulli equation

$$\vartheta_t + \tfrac{1}{2}|\mathbf{v}|^2 + p = 0,$$

p is *pressure* introduced via the equation of state

$$p = c^2\left[(\varrho - 1) - l^2\varrho^{-1/2}\nabla^2\varrho^{1/2}\right],$$

quantum pressure,

$c = \sqrt{2}\kappa/l$ is the *speed of sound*

Euler equation: $\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0,$

Vorticity: $\boldsymbol{\omega} = \nabla \times \mathbf{v}$

$$\boldsymbol{\omega}_t + \mathbf{v} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{v} = 0$$

In the plane

$$\boldsymbol{\omega}_t + \mathbf{v} \cdot \nabla \boldsymbol{\omega} = 0$$

(Ordinary fluid)

$$\boldsymbol{\omega}(\mathbf{x}, t) = \sum_{i=1}^{N} \Gamma_i \delta(\mathbf{x} - \mathbf{x}_i(t))$$

Biot-Savart (2D)

$$\mathbf{v}(\mathbf{x}, t) = \frac{1}{4\pi} \int \frac{\boldsymbol{\omega}(\mathbf{x}', t) \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^2} d\mathbf{x}'$$

$$\frac{d\mathbf{x}_i(t)}{dt} = \mathbf{v}(\mathbf{x}_i, t) = \sum_{j \neq i}^{N} \Gamma_j \frac{(\mathbf{x}_i - \mathbf{x}_j)_\perp}{|\mathbf{x}_i - \mathbf{x}_j|^2}$$

$$u_t = -2i\,\delta\mathcal{E}/\delta\overline{u}.$$

Gross-Pitaevskii

$$\mathcal{E} = \tfrac{1}{2}\int \mathcal{H}dx, \quad \mathcal{H} = \nabla\overline{u}\cdot\nabla u + \tfrac{1}{2}(1-|u|^2)^2.$$

Ginzburg-Landau energy (Hamiltonian)

$$-iu_t = \nabla^2 u + (1-|u|^2)u,$$

$$\mathcal{L}(u) = \tfrac{i}{2}(u\,\overline{u}_t - \overline{u}\,u_t) + \nabla u\cdot\nabla\overline{u} + \tfrac{1}{2}(1-|u|^2)^2.$$

Lagrangian

$$S = \int \mathcal{L}(u)d^2\mathbf{x}\,dt$$

$$u = \rho e^{i\theta},$$

$$\theta_t = \rho^{-1}\nabla^2\rho + 1 - |\nabla\theta|^2 - \rho^2,$$

$$-\rho_t = \rho\nabla^2\theta + 2\nabla\rho\cdot\nabla\theta.$$

Ginzburg-Landau equation (dissipative)

$$\frac{\partial \Psi}{\partial t} = \epsilon^2 \Delta \Psi + \Psi - |\Psi|^2 \Psi;$$

$$F_\epsilon[\Psi] = \int \left(\frac{1}{2} |\nabla \Psi|^2 + \frac{1}{4\epsilon^2}(1 - |\Psi|^2)^2 \right) d^D x.$$

$$\begin{aligned} &F_\epsilon[\Psi + \delta\Psi] \\ &= \int \left(\frac{1}{2} \nabla(\Psi + \delta\Psi) \cdot \nabla(\Psi^* + \delta\Psi^*) \right. \\ &\quad \left. + \frac{1}{4\epsilon^2}(1 - (\Psi + \delta\Psi)(\Psi^* + \delta\Psi^*))^2 \right) d^D x \\ &= F_\epsilon[\Psi] + \int \left(\nabla\Psi \cdot \nabla(\delta\Psi^*) - \frac{1}{\epsilon^2} \delta\Psi^* \Psi (1 - |\Psi|^2) \right) d^D x + O\left(|\delta\Psi|^2\right) \\ &= F_\epsilon[\Psi] + \int \left(-\Delta\Psi - \frac{1}{\epsilon^2} \Psi (1 - |\Psi|^2) \right) \delta\Psi^* d^D x + O\left(|\delta\Psi|^2\right). \end{aligned}$$

$$-\epsilon^2 \frac{\delta F_\epsilon[\Psi]}{\delta \Psi^*} = \epsilon^2 \Delta \Psi + \Psi(1 - |\Psi|^2) = \frac{\partial \Psi}{\partial t},$$

$$\frac{dF_\epsilon[\Psi]}{dt} = -\epsilon^2 \int_{\mathbb{R}^D} \left| \frac{\delta F_\epsilon[\Psi]}{\delta \Psi^*} \right|^2 d^D x.$$

Gross-Pitaevskii equation (Hamiltonian)

$$i\frac{\partial \Psi}{\partial t} = \epsilon^2 \Delta \Psi + \Psi - |\Psi|^2 \Psi;$$

$$F_\epsilon[\Psi] = \int \left(\frac{1}{2} |\nabla \Psi|^2 + \frac{1}{4\epsilon^2}(1 - |\Psi|^2)^2 \right) d^D x.$$

$$\begin{aligned}
&F_\epsilon[\Psi + \delta\Psi] \\
&\quad = \int \left(\frac{1}{2} \nabla (\Psi + \delta\Psi) \cdot \nabla (\Psi^* + \delta\Psi^*) \right. \\
&\qquad \left. + \frac{1}{4\epsilon^2}(1 - (\Psi + \delta\Psi)(\Psi^* + \delta\Psi^*))^2 \right) d^D x \\
&\quad = F_\epsilon[\Psi] + \int \left(\nabla\Psi \cdot \nabla(\delta\Psi^*) - \frac{1}{\epsilon^2} \delta\Psi^* \Psi (1 - |\Psi|^2) \right) d^D x + O\left(|\delta\Psi|^2 \right) \\
&\quad = F_\epsilon[\Psi] + \int \left(-\Delta\Psi - \frac{1}{\epsilon^2} \Psi (1 - |\Psi|^2) \right) \delta\Psi^* d^D x + O\left(|\delta\Psi|^2 \right).
\end{aligned}$$

$$-\epsilon^2 \frac{\delta F_\epsilon[\Psi]}{\delta \Psi^*} = \epsilon^2 \Delta \Psi + \Psi (1 - |\Psi|^2) = \frac{\partial \Psi}{\partial t} i,$$

↑

$$\frac{dF_\epsilon[\Psi]}{dt} = -\epsilon^2 \int_{\mathbb{R}^D} \left| \frac{\delta F_\epsilon[\Psi]}{\delta \Psi^*} \right|^2 d^D x.$$

0

$$F_\epsilon \simeq \pi \int_1^{R/\epsilon} \frac{n^2}{\xi} \rho^2 d\xi \simeq \pi n^2 \ln \frac{R}{\epsilon}. \tag{13.101}$$

The logarithmically diverging contribution (13.101) coming from the neighborhood of each vortex is called the *self-energy*.

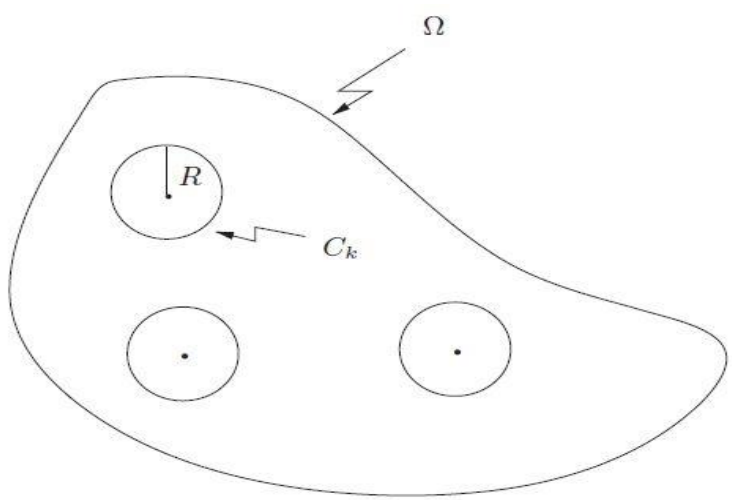


Figure 13.19 Vortices in a domain Ω. The free energy is separated into the self-energy contribution (13.101) coming from a disk of radius R around each vortex and the contribution coming from the exterior of the disks.

$$\Psi(r,\theta) = \rho\left(\frac{r}{\epsilon}\right)e^{i\varphi(r,\theta)}.$$

$$\begin{aligned}
F_{\text{int}} &= \int_{\cup C_k}\left(\frac{1}{2}|\nabla\rho|^2 + \frac{1}{2}\rho^2|\nabla\varphi|^2 + \frac{1}{4\epsilon^2}(1-\rho^2)^2\right)d^2x \\
&= \sum_k \int_{C_k}\left(\frac{1}{2}|\nabla\rho|^2 + \frac{1}{2}\rho^2|\nabla\varphi|^2 + \frac{1}{4\epsilon^2}(1-\rho^2)^2\right)d^2x \\
&\simeq \sum_k \pi n_k^2 \ln\frac{R}{\epsilon}.
\end{aligned}$$

$$\begin{aligned}
F_{\text{ext}} &= \int_{\Omega\setminus\cup C_k}\left(\frac{1}{2}|\nabla\rho|^2 + \frac{1}{2}\rho^2|\nabla\varphi|^2 + \frac{1}{4\epsilon^2}(1-\rho^2)^2\right)d^2x \\
&\simeq \int_{\Omega\setminus\cup C_k}\frac{1}{2}\rho^2|\nabla\varphi|^2\,d^2x \simeq \int_{\Omega\setminus\cup C_k}\frac{1}{2}|\nabla\varphi|^2\,d^2x.
\end{aligned}$$

$$\Delta\varphi = 0$$

$$F_\epsilon[\Psi] = \pi \left(\sum_{i=1}^{N} n_i^2\right) \ln \frac{1}{\epsilon} + W(a_1, a_2, \ldots, a_N) + O(\epsilon),$$

$$W(a_1, a_2, \ldots, a_k) = -\pi \sum_{i \neq j} n_i n_j \ln \left|a_i - a_j\right| + B(g, \partial\Omega),$$

$$W(a_1, a_2, \ldots, a_N) = -\pi \sum_{j \neq k} n_j n_k \ln \left|a_j - a_k\right|$$

$$v_i = \frac{da_i}{dt}.$$

$(a_1, a_2, \ldots, a_N)$ to the new locations $(a_1, a_2, \ldots, a_N) + (\delta a_1, \delta a_2, \ldots, \delta a_N)$ produces a change in energy

$$\delta F_\epsilon[\Psi] = \sum_i \nabla_{a_i} W(a_1, a_2, \ldots, a_N) \cdot \delta a_i + O(\epsilon). \qquad (13.114)$$

$$\delta\Psi = \sum_i \nabla\Psi_i \cdot \delta a_i,$$

$$\left\langle \frac{\partial \Psi}{\partial t}, \delta\Psi \right\rangle = \epsilon^2 \delta F_\epsilon \left[\Psi\right],$$

$$\left\langle \frac{\partial \Psi}{\partial t}, \delta\Psi \right\rangle = \int \left(\sum_i v_i \cdot \nabla \Psi_i \right) \left(\sum_j \nabla \Psi_j^* \cdot \delta a_j \right) d^2x,$$

$$\int \nabla \Psi_i \cdot \nabla \Psi_j^* d^2x = 2\pi \delta_{ij} n_i^2 \ln(1/\epsilon) + O(1).$$

The result is zero if $i \neq j$ since the vortices do not overlap. Hence we conclude that

$$\left\langle \frac{\partial \Psi}{\partial t}, \delta\Psi \right\rangle = \sum_i 2\pi n_i^2 \ln(1/\epsilon) v_i \cdot \delta a_i + O(1). \tag{13.118}$$

Inserting (13.118) and (13.114) into (13.116), we obtain

$$\frac{|\ln \epsilon|}{\epsilon^2} v_i = -\frac{1}{2\pi n_i^2} \nabla_{a_i} W(a_1, a_2, \ldots, a_N), \tag{13.119}$$

$$
\begin{aligned}
\nabla_{a_i} W &= -\pi \sum_{j \neq k} n_j n_k \left(\frac{\delta_{ji}(a_j - a_k)}{\left|a_j - a_k\right|^2} + \frac{\delta_{ki}(a_k - a_j)}{\left|a_j - a_k\right|^2} \right) \\
&= -2\pi \sum_j n_i n_j \frac{a_i - a_j}{\left|a_i - a_j\right|^2}.
\end{aligned} \tag{13.120}
$$

It follows that the motion of each vortex is governed by (Ginzburg-Landau)

$$
\frac{da_i}{dT} = \frac{1}{n_i} \sum_{j \neq i} \frac{n_j (a_i - a_j)}{\left|a_i - a_j\right|^2}, \tag{13.121}
$$

with a new time scale

$$
T = \frac{\epsilon^2}{|\ln \epsilon|} t.
$$

Gross-Pitaevskii

$$i\frac{\partial\Psi}{\partial t} = \epsilon^2\Delta\Psi + \Psi - |\Psi|^2\,\Psi$$

$$H(\Psi) = \epsilon^2 \int \left(\frac{1}{2}\,|\nabla\Psi|^2 + \frac{1}{4\epsilon^2}(1-|\Psi|^2)^2\right) d^2x;$$

$$i\frac{\partial\Psi}{\partial t} = -\frac{\delta H(\Psi)}{\delta\Psi^*};$$

$$\left\langle i\frac{\partial\Psi}{\partial t}, \delta\Psi\right\rangle = \sum_i \nabla_{a_i} W(a_1, a_2, \ldots, a_N)\cdot\delta a_i\,.$$

$$\boxed{\frac{da_i^{\perp}}{dT} = -\sum_{j\neq i}\frac{n_j(a_i-a_j)}{\left|a_i-a_j\right|^2},}$$

with $T = \epsilon^2 t$.

Gross-Pitaeveskii also supports wave-type solutions

$$\varrho_t + \nabla \cdot (\varrho \mathbf{v}) = 0,$$

Bernoulli equation

$$\vartheta_t + \tfrac{1}{2}|\mathbf{v}|^2 + p = 0,$$

$$\vartheta_{tt} - c^2 \nabla^2 \vartheta = 0.$$

Kelvin (sound) waves

$$c = \sqrt{dp/d\varrho} = \sqrt{2}\kappa/l$$

- Interaction of sound waves and vortices
- Radiation by a moving vortex
- Vortex singularities as a source of sound

Vortex filaments

$$\boldsymbol{\omega}(\mathbf{x}, 0) = \frac{\Gamma}{2\pi} \delta_G \mathbf{t}(s, 0)$$

$$\mathbf{v}(\mathbf{r}) = \frac{\Gamma}{4\pi} \int \frac{\mathbf{t}(s') \times [\mathbf{r} - \mathbf{r}(s')]}{|\mathbf{r} - \mathbf{r}(s')|^3} ds'.$$

$$\mathbf{t} = \frac{\partial \mathbf{r}}{\partial s} \equiv \mathbf{r}_s. \tag{13.18}$$

The rate of change of **t** is a measure of the curvature $\kappa(s)$, and the resulting vector is normal to the curve:

$$\mathbf{t}_s = \kappa \mathbf{n}. \tag{13.19}$$

This means that **n** and the binormal

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} \tag{13.20}$$

span the plane perpendicular to the local direction of the filament (see Fig. 13.8).

To complete the geometrical description of a curve we need to specify the derivatives of the basis vectors **t**, **n**, and **b**. One can show that in addition to (13.19) (see for example [62]) one has

$$\mathbf{n}_s = -\kappa \mathbf{t} + \tau \mathbf{b}, \quad \text{and} \quad \mathbf{b}_s = -\tau \mathbf{n}. \tag{13.21}$$

The second scalar function, $\tau(s)$, is called the torsion of the curve. It describes how **b** winds itself around the centerline. The three equations in (13.19) and (13.21) are known as the *Frenet–Serret formulas*.

$$I_{1/2} = \frac{\Gamma}{4\pi} \int_{|s-s'| \lesseqgtr \epsilon} \frac{\mathbf{t}(s') \times [\mathbf{r} - \mathbf{r}(s')]}{|\mathbf{r} - \mathbf{r}(s')|^3} ds',$$

$$\mathbf{r}(s) = \mathbf{r}(s') + \frac{d\mathbf{r}}{ds}|(s')(s-s') + \frac{1}{2}\frac{d^2\mathbf{r}}{ds^2}(s')(s-s')^2 + O((s-s')^3)$$

$$= \mathbf{r}(s') + \mathbf{t}(s')(s-s') + \frac{1}{2}\kappa\mathbf{n}(s')(s-s')^2 + O((s-s')^3).$$

$$\left|\mathbf{r} - \mathbf{r}(s')\right|^3 = \left(\delta^2 + |s-s'|^2\right)^{3/2} + O(|s-s'|^4).$$

$$-\frac{1}{2}\frac{\kappa\,|s-s'|^2\,\mathbf{b}}{\left(\delta^2 + |s-s'|^2\right)^{3/2}} + \text{lower-order terms.}$$

$$I_1 \simeq -\frac{\Gamma\kappa}{8\pi}\int_{s-\epsilon}^{s+\epsilon}\frac{(s-s')^2 ds'}{\left[\delta^2+(s-s')^2\right]^{3/2}}\mathbf{b} \simeq \frac{\Gamma\kappa\ln(\epsilon/\delta)}{4\pi}\mathbf{b}.$$

$$I_2 \simeq -\frac{\Gamma\kappa}{8\pi}\left(\int_{s+\epsilon}^{s+s_0}\frac{ds}{|s-s'|}+\int_{s-s_0}^{s-\epsilon}\frac{ds}{|s-s'|}\right)\mathbf{b} \simeq -\frac{\Gamma\kappa\ln\epsilon}{4\pi}\mathbf{b}.$$

$$\mathbf{v} = -\frac{\Gamma\kappa\ln\delta}{4\pi}\mathbf{b},$$

$$\frac{\partial\mathbf{r}}{\partial t} = \frac{\Gamma\ln\delta^{-1}}{4\pi}\mathbf{r}_s \times \mathbf{r}_{ss}$$

Binormal flow equation (da Rios 1905)

$$\kappa(s,t') \;=\; e^{i\frac{\Gamma}{R^2}\sqrt{n^4-n^2}t'}\cos\left(\frac{ns}{R}\right)$$

Example 13.3 (Stability of a filament) The binormal flow equation (13.28) for a vortex filament has a stationary solution corresponding to a straight filament $\mathbf{r}_s = s\mathbf{e}_3$. Now consider small perturbations of the form

$$\mathbf{r}(s,t) = \mathbf{r}_0 + \epsilon \mathbf{g}(s,t), \tag{13.29}$$

with $\mathbf{g}$ perpendicular to $\mathbf{e}_3$: $\mathbf{g}(s,t) = g_1\mathbf{e}_1 + g_2\mathbf{e}_2$. It follows that

$$\mathbf{r}_s \times \mathbf{r}_{ss} = \epsilon\left(\mathbf{e}_3 + \epsilon\mathbf{g}_s\right) \times \mathbf{g}_{ss} = \epsilon\mathbf{e}_3 \times \mathbf{g}_{ss} + O(\epsilon^2),$$

so that, at linear order,

$$\mathbf{g}_t = a\left(\mathbf{e}_3 \times \mathbf{g}_{ss}\right), \tag{13.30}$$

where $a = \Gamma \ln \delta^{-1}/(4\pi)$.

The cross products of (13.30) with $\mathbf{e}_2$ and $\mathbf{e}_1$ yield the following equations for the components of $\mathbf{g}$:

$$g_{1,t} = -a g_{2,ss}, \quad g_{2,t} = a g_{1,ss}. \tag{13.31}$$

The solution to (13.31) is of the form

$$g_1 = Ae^{iks+\omega(k)t}, \quad g_2 = Be^{iks+\omega(k)t}, \tag{13.32}$$

which results in the equations

$$A\omega(k) = ak^2B, \quad B\omega(k) = -ak^2A.$$

There exists a nontrivial solution for A and B provided that the determinant vanishes, giving the dispersion relation

$$\omega^2(k) + a^2k^4 = 0. \tag{13.33}$$

Hence $\omega(k) = \pm ak^2 i$, so perturbations will neither grow nor decay in time. This is characteristic of systems with a Hamiltonian structure, where energy is conserved. The nonlinearity of (13.33) points to the presence of dispersive phenomena, where the waves forming an initial wave packet travel with different speeds. The real solutions are of the form

$$g_1(s,t) = \cos(ks \pm ak^2t), \quad g_2(s,t) = \sin(ks \pm ak^2t), \tag{13.34}$$

representing helical propagation of the perturbation $\mathbf{g}$. □

$$\frac{\partial \mathbf{r}}{\partial t} = \frac{\Gamma \ln \delta^{-1}}{4\pi} \mathbf{r}_s \times \mathbf{r}_{ss}$$

$$\mathbf{r}(s,t) = t'^{1/2}\mathbf{G}(\xi), \quad \xi = \frac{s}{t'^{1/2}}, \tag{13.35}$$

where we have put $t' = t_0 - t$. Inserting the similarity form (13.35) for the similarity profile $\mathbf{G}(\xi)$ into (13.28) leads to

$$\frac{1}{2}\mathbf{G}(\xi) - \frac{\xi}{2}\mathbf{G}_\xi(\xi) = \mathbf{G}_\xi(\xi) \times \mathbf{G}_{\xi\xi}(\xi). \tag{13.36}$$

From the matching condition (3.16) we conclude that the behavior of the filament must be linear at infinity, and so

$$\mathbf{r}(s,t) = \mathbf{G}_0^{\pm} s, \quad s \to \pm\infty. \tag{13.37}$$

Given that the problem is rotationally invariant, the only parameter that matters is the angle θ between the directions $\mathbf{G}_0^+$ and $\mathbf{G}_0^-$.

Taking derivatives of (13.36) with respect to ξ and setting $\mathbf{T} = \mathbf{G}_\xi$ we get

$$-\frac{\xi}{2}\mathbf{T}_\xi = \mathbf{T} \times \mathbf{T}_{\xi\xi}. \tag{13.38}$$

We now rewrite (13.38) using the self-similar version $\mathbf{T}_\xi = \bar{\kappa}\mathbf{N}$ and $\mathbf{N}_\xi = -\bar{\kappa}\mathbf{T} + \bar{\tau}\mathbf{B}$ of the Frenet–Serret formulas (13.19), (13.21); $\bar{\kappa}$ and $\bar{\tau}$ are rescaled versions of the curvature and torsion, respectively:

$$\kappa(s,t) = t'^{-1/2}\bar{\kappa}(\xi), \quad \tau(s,t) = t'^{-1/2}\bar{\tau}(\xi). \tag{13.39}$$

Thus we obtain

$$\begin{aligned} -\frac{\xi}{2}\bar{\kappa}\mathbf{N} &= \mathbf{T} \times (\bar{\kappa}_\xi \mathbf{N} - \bar{\kappa}^2\mathbf{T} + \bar{\kappa}\bar{\tau}\mathbf{B}) = \bar{\kappa}_\xi \mathbf{B} + \bar{\kappa}\bar{\tau}\mathbf{T} \times (\mathbf{T} \times \mathbf{N}) \\ &= \bar{\kappa}_\xi \mathbf{B} - \bar{\kappa}\bar{\tau}\mathbf{N}, \end{aligned} \tag{13.40}$$

where we have used the orthonormality conditions in (13.44) below. Scalar-multiplying (13.40) by $\mathbf{B}$ and $\mathbf{N}$, respectively, one finds that

$$\bar{\kappa}(\xi) = a, \quad \bar{\tau}(\xi) = \frac{\xi}{2}, \tag{13.41}$$

$$\mathbf{G}_\xi = \mathbf{T}(\xi), \tag{13.42}$$

where the tangent vector $\mathbf{T}(\xi)$ obeys the Frenet–Serret system (13.13), (13.21). Thus we have

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}_\xi = \begin{pmatrix} 0 & a & 0 \\ -a & 0 & \xi/2 \\ 0 & -\xi/2 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{13.43}$$

together with the conditions

$$|\mathbf{T}| = |\mathbf{N}| = |\mathbf{B}| = 1, \quad \mathbf{T}\cdot\mathbf{N} = \mathbf{T}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{N} = 0. \tag{13.44}$$

S. Gutiérrez, J. Rivas, L. Vega, Formation of singularities and self-similar vortex motion under the

$$\mathbf{G}(s) = \mathbf{A}^{\pm}(c_0)\left(s + 2\frac{c_0^2}{s}\right) - 4c_0\frac{\mathbf{n}}{s^2} + O(1/s^3), \qquad s \to \pm\infty;$$

$$\mathbf{T}(s) = \mathbf{A}^{\pm}(c_0) - 2c_0\frac{\mathbf{b}}{s} + O(1/s^2), \qquad s \to \pm\infty;$$

$$(\mathbf{n} - i\mathbf{b})(s) = \mathbf{B}^{\pm}(c_0)e^{is^2/4}e^{ic_0^2 \log s} + O(1/s), \qquad s \to \pm\infty;$$

Feynman's scenario for turbulence

R. P. Feynman, Application of quantum mechanics to liquid helium". II. Progress in Low Temperature Physics. Vol. 1. Amsterdam: North-Holland Publishing Company.

The energy spectrum of turbulence is a mean quantity and is one of the main objects studied in turbulence theory. We write this spectrum as follows

$$E^{(3D)}(\mathbf{k}) = \frac{1}{2} \int_{\mathbb{R}^3} \langle \mathbf{u}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x} + \mathbf{r}) \rangle e^{-i\mathbf{k}\cdot\mathbf{r}} \frac{d\mathbf{r}}{(2\pi)^3}, \tag{2.1}$$

The super-script $(3D)$ refers to the fact that $E^{(3D)}$ represents the kinetic energy density in the 3D $\mathbf{k}$-space, i.e.

$$\frac{1}{2}\langle u^2 \rangle = \int_{\mathbb{R}^3} E^{(3D)}(\mathbf{k}) d\mathbf{k}. \tag{2.2}$$

On the other hand, for isotropic spectra the same information is contained in a 1D spectrum which is obtained from $E^{(3D)}$ by integration over the unit sphere in the 3D $\mathbf{k}$-space. This gives

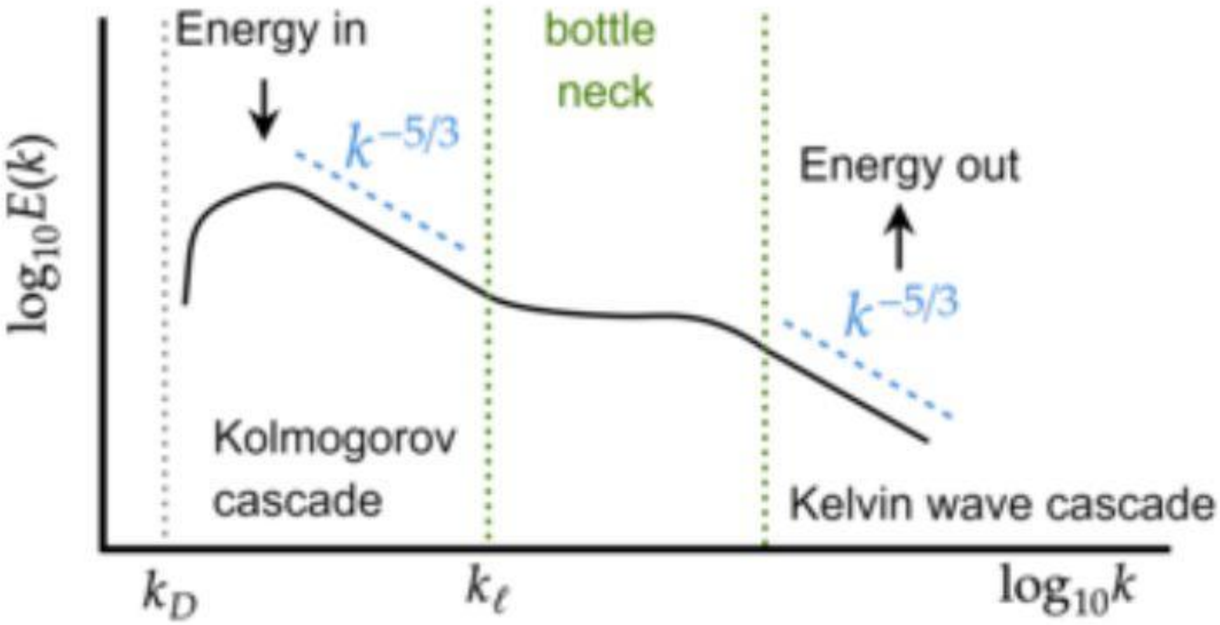
$$E^{(1D)}(k) = 4\pi k^2 E^{(3D)}(k)$$

so that $E^{(1D)}(k)$ represents the energy density over $k = |\mathbf{k}|$,

$$\frac{1}{2}\langle u^2 \rangle = \int_0^{+\infty} E^{(1D)}(k) dk. \tag{2.3}$$

$$E^{(1D)} = C\varepsilon^{2/3}k^{-5/3},$$

Kolmogorov law



Energy in
bottle
neck
$k^{-5/3}$
Energy out
$k^{-5/3}$
Kolmogorov
cascade
Kelvin wave cascade
$\log_{10}E(k)$
k_D
k_ℓ
$\log_{10}k$

Wave turbulence (or weak turbulence)

$$i\frac{\partial\psi}{\partial t} = \omega(k)\psi + g'\int d\mathbf{k}_1\int d\mathbf{k}_2\int d\mathbf{k}_3\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3)$$
$$\times V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\psi(\mathbf{k}_1)\psi(\mathbf{k}_2)\psi^*(\mathbf{k}_3), \quad (11)$$

where $\omega(k) = k^2/2$ and $V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = 1$. Such a four-wave weakly interacting system in the WT regime must be described by the following equation for the wave action[2] $n(\mathbf{k}) = |\psi(\mathbf{k})|^2$:

$$\frac{\partial n(\mathbf{k})}{\partial t} = 4\pi g'^2\int d\mathbf{k}_1\int d\mathbf{k}_2\int d\mathbf{k}_3\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3)$$
$$\times\delta(\omega(\mathbf{k})-\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)+\omega(\mathbf{k}_3))$$
$$\times n(\mathbf{k})n(\mathbf{k}_1)n(\mathbf{k}_2)n(\mathbf{k}_3)\left[\frac{1}{n(\mathbf{k})}+\frac{1}{n(\mathbf{k}_3)}-\frac{1}{n(\mathbf{k}_1)}-\frac{1}{n(\mathbf{k}_2)}\right]. \quad (12)$$

Using Zakharov transformation on the k variables,[131] two power law solutions of the type $n \sim k^\nu$ are possible in this system with

$$\nu_E = -2\beta/3 - d, \quad (13)$$

$$\nu_N = -2\beta/3 - d + \alpha/3, \quad (14)$$

which correspond to energy (ν_E) and particle (ν_N) cascades. Here, d is the spatial dimension of the system, while α and β are the degrees of homogeneity of ω and V, i.e., $\omega(\lambda k) = \lambda^\alpha \omega(k)$ and $V(\lambda \mathbf{k}_1, \lambda \mathbf{k}_2, \lambda \mathbf{k}_3, \lambda \mathbf{k}_4) = \lambda^\beta V(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$. In the case of 3D GPE, we have $\alpha = 2$, $\beta = 0$, and $d = 3$. These parameters lead to the energy and wave action cascade power law spectra, which are $\nu_E = -3$ and $\nu_N = -7/3$, respectively.

Kelvin waves and the decay of quantum superfluid turbulence

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We present a comprehensive statistical study of free decay of the quantized vortex tangle in superfluid ^{4}He at low and ultra-low temperatures, $0 \leqslant T \leqslant 1.1\,$K. Using high resolution vortex filament simulations with full Biot-Savart vortex dynamics, we show that for ultra-low temperatures $T \lesssim 0.5\,$K, when the mutual friction parameters $\alpha \simeq \alpha' < 10^{-5}$, the vortex reconnections excite Kelvin waves with wave lengths λ of the order of the inter-vortex distance ℓ. These excitations cascade down to the resolution scale $\Delta\xi$ which in our simulations is of the order $\Delta\xi \sim \ell/100$. At this scale the Kelvin waves are numerically damped by a line-smoothing procedure, that is supposed to mimic the dissipation of Kelvin waves by phonon and roton emission at the scale of the vortex core. We show that the Kelvin waves cascade is statistically important: the shortest available Kelvin waves at the end of the cascade determine the mean vortex line curvature S, giving $S \gtrsim 30/\ell$ and play major role in the tangle decay at ultra-low temperatures below 0.6 K. The found dependence of ℓS on the resolution scale $\Delta\xi$ agrees with the L'vov-Nazarenko energy spectrum of weakly-interacting Kelvin waves, $E_{\rm LN} \propto k^{-5/3}$ rather than the spectrum $E_{\rm LN} \propto k^{-1}$, suggested by Vinen for turbulence of Kelvin waves with large amplitudes. We also show that already at $T = 0.8\,$K, when α and α' are still very low, $\alpha \simeq \alpha' < 10^{-3}$, the Kelvin wave cascade is fully damped, vortex lines are very smooth, $S \simeq 2/\ell$ and the tangle decay is predominantly caused by the mutual friction.

Venues for research

- Understanding superflow in connection with
- moving bodies: vortex generation
- Vortex filaments equations
- The reconnection problem is totally open
- (both in normal and super fluids)
- Wave turbulence equations are not totally
- justified as limit of GP
- Uncertain exponent in turbulent cascade of
- Kelvin waves