

Differential-escort transformations and the monotonicity of the LMC-Rényi complexity measure

D. Puertas-Centeno

Departamento de Física Atómica, Molecular y Nuclear, and
Instituto Carlos I de Física Teórica y Computacional,
Universidad de Granada, Granada 18071, Spain

December 6, 2018

Escort distributions have been shown to be very useful in a great variety of fields ranging from information theory, nonextensive statistical mechanics till coding theory, chaos and multifractals. In this work we give the notion and the properties of a novel type of escort density, the *differential-escort densities*, which have various advantages with respect to the standard ones. We highlight the behavior of the differential Shannon, Rényi and Tsallis entropies of these distributions. Then, we illustrate their utility to prove the monotonicity property of the LMC-Rényi complexity measure and to study the behavior of general distributions in the two extreme cases of minimal and very high LMC-Rényi complexity. Finally, this transformation allows us to obtain the Tsallis q-exponential densities as the differential-escort transformation of the exponential density.

1 Introduction

The study of chaotic and complex systems have needed the development of mathematical tools able to capture the fundamental statistical properties of the system. Escort distributions have been introduced in statistical physics for the characterization of multifractal systems [1]. These distributions $\{\tilde{p}_i\}$

conform a one-parameter class of transformations of an original probability distribution $\{p_i\}$ according to $\tilde{p}_i = \frac{p_i^q}{\sum_{i=1}^N p_i^q}$, with $q \in \mathbb{R}$.

This idea previously appeared in relation to the Rényi-entropy-based coding theorem [2, 3] and Rényi-entropy-based fractal dimensions [4]. The mathematical properties of the discrete escort distributions have been widely studied [5, 6, 7, 8]. This concept can be easily extended to the continuous case. Given a real variable $x \in \mathbb{R}$ and a probability distribution $\rho(x)$, such that $\int_{\mathbb{R}} \rho(x) dx = 1$, one has the escort distribution [9] defined as

$$E_q[\rho](x) \equiv \tilde{\rho}(x) = \frac{[\rho(x)]^q}{\int_{\mathbb{R}} [\rho(t)]^q dt}, \quad (1)$$

on the assumption that $\int_{\mathbb{R}} \rho(x)^q dx < \infty$. Note that the parameter q plays a focus role to highlight different regions of $\rho(x)$. These distributions play a relevant role in coding problems, non-equilibrium statistical mechanics [10, 11] and electronic structure [12, 13, 14]. A particular example is the q -exponential distribution

$$e_q(x) \propto (1 + (q - 1)|x|)^{\frac{1}{1-q}} \quad (2)$$

which maximizes the differential Rényi entropy

$$R_q[\rho] = \frac{1}{1 - q} \log \left(\int_{\mathbb{R}} [\rho(x)]^q dx \right), \quad (3)$$

and the differential Tsallis entropy

$$T_q[\rho] = \frac{1}{1 - q} \left(1 - \int_{\mathbb{R}} [\rho(x)]^q dx \right), \quad (4)$$

subject to average-constraints governed by its escort distribution. Of course, in the limit $q \rightarrow 1$ the original distribution is recovered in Eq. (1), the exponential distribution is also recovered in Eq. (2) and the differential Shannon entropy

$$S[\rho] = \lim_{q \rightarrow 1} R_q[\rho] = \lim_{q \rightarrow 1} T_q[\rho] = - \int_{\mathbb{R}} \rho(t) \log[\rho(t)] dt \quad (5)$$

is respectively recovered in Eqs (3) and (4).

The aim of this work is to introduce the notion of *differential-escort transformation*, \mathfrak{E}_α , and to study its basic mathematical properties (probability

invariance, composition rule, scaling property,...). Then, we highlight the strongly regular behavior of the differential Shannon, Rényi and Tsallis entropies under this transformation, observing that the entropic parameter naturally rescales similarly to the rescaling behavior recently found by Korbel [15] for the non-additivity parameter in Tsallis thermostatics [16]. This behavior is related to the rescaling of the relative fluctuations of a system with a finite number of particles, and plays a relevant role in the deformed calculus developed by Borges [17] as it is discussed by Korbel himself. Moreover we also note that the q -exponential distribution is just the differential-escort transformation of the standard exponential distribution; so, differently to what happens to the the standard escort transformation of an exponential distribution which is another exponential. In fact, we show that the differential-escort transformation changes the behavior of the distribution tail in a deeper and more interesting manner than the standard escort transformation, which allows us to propose a possible characterization of power-law-decaying probability densities through Lemma 1. On the other hand, we carry out a study of the behaviour of the LMC-Rényi complexity measure [18, 19] of the transformed densities. Actually, the notion of *differential-escort density* allows us to solve the monotonicity problem of the LMC-Rényi complexity measure recently posed by Rudnicki et al [20]. This fact, in turn, permits us to define the notions of *low-complexity* and *high-complexity* probability densities in the LMC-Rényi sense, as well as we characterize the entropic behavior of these probability densities.

The structure of this work is the following: In section 2 the differential-escort transformation \mathfrak{E}_α is defined and its basic mathematical properties are given. In section 3 the entropic properties of the differential-escort densities are discussed. In section 4 the LMC-Rényi complexity of the differential-escort densities is studied and the monotonicity property of this measure is proven. In section 5 the entropic and complexity behavior of a general probability density when it is deformed until to the low and high complexity limits is studied. Then, in section 6 this transformation is applied to distributions of exponential, q -exponential and general power-law decaying. Finally, in section 7 some conclusions and open problems are given.

2 Differential-escort transformation

In this section we give the notion and properties of the differential-escort transformation. Let us advance that the basic difference with the standard escort transformations is the normalization process. Indeed, while an escort density is normalized according to (1), in the differential-escort case the normalization is achieved through a variable change which imposes the conservation of the probability in any differential interval of the support, as we will see later.

The notion

Let us consider a probability density $\rho(x)$, $x \in \Lambda \subseteq \mathbb{R}$, normalized, so that $\int_{\Lambda} \rho(x) dx = 1$; and let us denote $\mathcal{D}(\mathbb{R})$ for the set of any distribution ρ on any subset of \mathbb{R} .

Definition 1. Let $\alpha \in \mathbb{R}$, and $\rho \in \mathcal{D}(\mathbb{R})$ be a probability density with a connected support ¹ Λ . We define the transformation $\mathfrak{E}_{\alpha} : \mathcal{D}(\mathbb{R}) \rightarrow \mathcal{D}(\mathbb{R})$ as:

$$\mathfrak{E}_{\alpha}[\rho](y) \equiv [\rho(x(y))]^{\alpha}, \quad (6)$$

where $y = y(x)$ is a bijection defined by:

$$\frac{dy}{dx} = [\rho(x)]^{1-\alpha}, \quad y(x_0) = x_0, \quad x_0 \in \Lambda. \quad (7)$$

The support Λ_{α} of the transformed density $\mathfrak{E}_{\alpha}[\rho]$ is given as $\Lambda_{\alpha} = y(\Lambda) = \{y \in \mathbb{R} \mid y = y(x), x \in \Lambda\}$. To make an easier reading we will denote $\rho_{\alpha}(y) \equiv \mathfrak{E}_{\alpha}[\rho](y)$, and generally we will take $x_0 = 0$, and $y(x) = \int_0^x [\rho(t)]^{1-\alpha} dt$.

We remark that this definition is valid for any $\alpha \in \mathbb{R}$, contrary with the standard escort distribution for which the parameter α is restricted by the condition $\int_{\Lambda} [\rho(x)]^{\alpha} dx < \infty$ as already indicated in Eq. (1). This extension is possible since the support of a differential-escort density Λ_{α} does not remain invariant contrary to the standard escort case. As we will see later, for any probability density ρ , the operation (1) defines a transformed density ρ_{α} for any $\alpha \in \mathbb{R}$.

¹We assume that the support is connected for easier reading. The definition could be easily extended to any distribution without disturbing its properties.

Let us also point out that the selection of x_0 only implies a translation. In addition, when $\alpha = 1$, one has that the operation \mathfrak{E}_α corresponds to the identity, i.e., $\mathfrak{E}_1[\rho] = \rho$; and when $\alpha = 0$, the operation \mathfrak{E}_α transforms ρ to an uniform distribution with an unitary support, concretely

$$\mathfrak{E}_0[\rho](x) = \begin{cases} 1, & x \in [x_0 - p_-, x_0 + p_+] \\ 0, & \text{otherwise} \end{cases},$$

where $p_- = Prob[x < x_0]$ and $p_+ = Prob[x > x_0]$.

The basic properties

In the following we will give some basic properties of the differential-escort transformation.

Property 1 is the most characteristic property of this transformation which consists in a strong probability invariance far beyond the mere conservation of the norm of the *standard escort* case.

Property 1. *Probability invariance*

Let $\alpha \in \mathbb{R}$ and ρ be a probability density with a connected support Λ . Then, for any pair of points $x_1, x_2 \in \Lambda$ and respectively $y_1 = y(x_1)$ and $y_2 = y(x_2)$ the identity

$$\int_{x_1}^{x_2} \rho(x) dx = \int_{y_1}^{y_2} \rho_\alpha(y) dy, \quad (8)$$

or equivalently

$$Prob[x \in [x_1, x_2]] = Prob[y \in [y_1, y_2]], \quad (9)$$

is fulfilled.

Proof. This property follows straightforwardly from (1) since

$$\int_{y_1}^{y_2} \rho_\alpha(y) dy = \int_{x_1}^{x_2} [\rho(x)]^\alpha \frac{dy}{dx} dx = \int_{x_1}^{x_2} [\rho(x)]^\alpha [\rho(x)]^{1-\alpha} dx = \int_{x_1}^{x_2} \rho(x) dx.$$

□

This property makes a deep difference with escort distributions. While for the latter ones the conservation of the norm is imposed dividing by a real number as indicated in (1), for the differential-escort distributions it naturally holds as a consequence of property 1 since $\int_{\Lambda_\alpha} \rho_\alpha(y) dy = \int_\Lambda \rho(x) dx = 1$.

Moreover, a similar property is fulfilled by a relevant transformation between auxiliary and physical probability densities in the context of quantum gravity [21].

Property 2. Composition law

Let $\alpha, \alpha' \in \mathbb{R}$, then

$$\mathfrak{E}_\alpha[\mathfrak{E}_{\alpha'}[\rho]] = \mathfrak{E}_{\alpha'}[\mathfrak{E}_\alpha[\rho]] = \mathfrak{E}_{\alpha\alpha'}[\rho] \quad (10)$$

holds.

Proof. By definition, $\mathfrak{E}_\alpha[\rho(x)](y) \equiv \rho_\alpha(y) = [\rho(x)]^\alpha$, where $dy = [\rho(x)]^{1-\alpha}dx$. Moreover, one has $\mathfrak{E}_\gamma[\mathfrak{E}_\alpha[\rho(x)]](z) = \mathfrak{E}_\gamma[\rho_\alpha(y)](z) = [\rho_\alpha(y)]^\gamma = [\rho(x)]^{\alpha\gamma}$ where $dz = [\rho_\alpha(y)]^{1-\gamma}dy$, so that one has $dz = [\rho(x)]^{\alpha(1-\gamma)}[\rho(x)]^{1-\alpha}dx = [\rho(x)]^{1-\alpha\gamma}dx$. \square

This property is similar to the one of the standard escort transformations, but the latter one holds in a more restrictive sense by taking into account that the standard escort transformations are not typically well defined for any $\alpha \in \mathbb{R}$. On the other hand this property allows us to find the inverse element of the differential-escort transformation

$$[\mathfrak{E}_\alpha]^{-1} = \mathfrak{E}_{\alpha^{-1}}, \quad \alpha \neq 0, \quad (11)$$

what allows us to say that $\mathfrak{E}_{\alpha \neq 0}[\mathcal{D}(\mathbb{R})] = \mathcal{D}(\mathbb{R})$.

Let us finally give the composition rule between the differential-escort and the scaling transformations. For example, in the standard escort case defined in Eq. (1), the composition rule with the scaling transformation is given by $E_\alpha[\rho^{(a)}] = E_\alpha[\rho]^{(a)}$, where $\rho^{(a)}$ denotes the scaling transformed distribution $\rho^{(a)}(x) = a\rho(ax)$, $a > 0$. As stated in the following property, in the composition law for the differential-escort case the power operation is also inherited by the scaling parameter a .

Property 3. Scaling Property

Let $a \in \mathbb{R}_+$, $\alpha \in \mathbb{R}$, ρ be a probability distribution, and the scaling transformed distribution $\rho^{(a)}(x) = a\rho(ax)$. Then, it holds

$$\mathfrak{E}_\alpha[\rho^{(a)}] = \mathfrak{E}_\alpha[\rho]^{(a^\alpha)}. \quad (12)$$

Proof. From the hypotheses of this statement one has the associated differential-escort distribution $\rho_\alpha(y) = \rho(x)^\alpha$, where $y(x) = \int_0^x [\rho(t)]^{1-\alpha} dt$, or equivalently $y = \int_0^{x(y)} [\rho(t)]^{1-\alpha} dt$. Later, we consider the differential-escort distribution of the scaling transformed, $(\rho^{(a)})_\alpha(z) = [\rho^{(a)}(x)]^\alpha$, with $z(x) = \int_0^x [\rho^{(a)}(t)]^{1-\alpha} dt$, so we have:

$$(\rho^{(a)})_\alpha(z) = a^\alpha [\rho(ax(z))]^\alpha, \quad z(x) = a^{-\alpha} \int_0^{ax} [\rho(t)]^{1-\alpha} dt.$$

Then, we can write $a^\alpha z = \int_0^{ax(z)} [\rho(t)]^{1-\alpha} dt$. On the other hand, taking into account that $y = \int_0^{x(y)} [\rho(t)]^{1-\alpha} dt$, we have that $ax(z) = x(a^\alpha z)$ and finally

$$(\rho^{(a)})_\alpha(z) = a^\alpha [\rho(x(a^\alpha z))]^\alpha = a^\alpha \rho_\alpha(a^\alpha z) = \rho_\alpha^{(a^\alpha)}(z).$$

□

3 The entropic properties

The functional ingredients of differential Rényi and Tsallis entropies (3), (4) are the entropic moments of the probability distribution ρ ,

$$W_q[\rho] = \int_{\mathbb{R}} [\rho(x)]^q dx. \quad (13)$$

In this section we will study the behavior of these entropy-like functionals for the differential-escort distributions, finding that it is much simpler than the corresponding one for the standard escort case. Interestingly, the rescaling

$$q_\alpha = 1 + \alpha(q - 1), \quad (14)$$

for the parameter q , so much relevant in deformed algebra [17] and Tsallis thermostatics [15], naturally appears in the entropic moment W_q of the differential-escort distributions as shown in the next property.

Property 4. Rescaling of the entropic moments

Let ρ be a probability distribution, $a \in \mathbb{R}$ and $\alpha \in \mathbb{R}$. Then the entropic moments $W_q[\rho]$ transform as

$$W_q[\rho_\alpha] = W_{q_\alpha}[\rho], \quad (15)$$

where q_α is given in (14).

Proof. If $W_q[\rho_\alpha] < \infty$, then

$$W_q[\rho_\alpha] = \int_{\mathbb{R}} [\rho_\alpha(y)]^q dy = \int_{\mathbb{R}} [\rho(x)]^{\alpha q} [\rho(x)]^{1-\alpha} dx = W_{1+(q-1)\alpha}[\rho].$$

In case that $W_q[\rho_\alpha] = \infty$, we consider the following equality between finite integrals

$$\int_{y_1}^{y_2} [\rho_\alpha(y)]^q dy = \int_{x_1}^{x_2} [\rho(x)]^{q\alpha} dx$$

for any $x_1, x_2 \in \Lambda$ and $y_{1,2} = y(x_{1,2})$. So, one has

$$\frac{W_q[\rho_\alpha]}{W_{q\alpha}[\rho]} = \lim_{(x_1, x_2) \rightarrow (x_m, x_M)} \frac{\int_{y_1}^{y_2} [\rho_\alpha(y)]^q dy}{\int_{x_1}^{x_2} [\rho(x)]^{q\alpha} dx} = \lim_{(x_1, x_2) \rightarrow (x_m, x_M)} 1 = 1.$$

□

For completeness, note that when $q = 1$, then $q_\alpha = 1$ and both $W_1[\rho_\alpha] = W_1[\rho] = 1$ as one expects.

The rescaling behavior in this property is automatically inherited by the differential Shannon, Rényi and Tsallis entropies, as pointed out in the next property.

Property 5. Entropies transformations

Let $q, \alpha \in \mathbb{R}$ and ρ be a probability distribution. Then, the differential Shannon, Rényi and Tsallis entropies of the differential-escort distributions transform as

$$\frac{S[\rho_\alpha]}{S[\rho]} = \frac{R_q[\rho_\alpha]}{R_{q\alpha}[\rho]} = \frac{T_q[\rho_\alpha]}{T_{q\alpha}[\rho]} = \alpha. \quad (16)$$

Proof. Taking into account that $\frac{1-q_\alpha}{1-q} = \alpha$, the equality for the Rényi and Tsallis entropies trivially follows from property 4.

The Shannon case could be simply understood as the limit case $q \rightarrow 1$, however, for the sake of illustration, we give the pretty simple and nice natural proof:

$$S[\rho_\alpha] = - \int_{\Lambda_\alpha} \rho_\alpha(y) \log[\rho_\alpha(y)] dy = - \int_{\Lambda} \rho(x) \log[\rho(x)^\alpha] dx = \alpha S[\rho].$$

□

Finally, as a direct consequence of the Jensen inequality we can assert that the Rényi entropy R_q of the transformed density ρ_α is a concave function of α when $q > 1$ and convex when $q < 1$. Just as property (5) claims, Shannon entropy has a linear behavior with the deformation parameter α . This behavior is given in the following property.

Property 6. *Let ρ be a non-uniform probability density, then the Rényi differential entropy of the differential-escort distribution fulfills the following identity:*

$$\operatorname{sgn}\left(\frac{\partial^2 R_q[\rho_\alpha]}{\partial \alpha^2}\right) = \operatorname{sgn}(1 - q). \quad (17)$$

So, $R_q[\rho_\alpha]$ is concave with α for $q > 1$ and convex for $q < 1$. When ρ is an uniform density, then one has $\frac{\partial^2 R_q[\rho_\alpha]}{\partial \alpha^2} = 0$.

Proof. One can easily compute

$$\frac{\partial^2 R_q[\rho_\alpha]}{\partial \alpha^2} = \frac{1 - q}{\left(\int_\Lambda \rho^{q\alpha}\right)^2} \left[\int_\Lambda \rho^{q\alpha} \log^2 \rho \int_\Lambda \rho^{q\alpha} - \left(\int_\Lambda \rho^{q\alpha} \log \rho\right)^2 \right],$$

for any probability density. On the other hand, due to Jensen's inequality one has

$$\left(\frac{\int_\Lambda \rho^{q\alpha} \log \rho}{\int_\Lambda \rho^{q\alpha}}\right)^2 \leq \frac{\int_\Lambda \rho^{q\alpha} \log^2 \rho}{\int_\Lambda \rho^{q\alpha}},$$

where the equality holds if and only if ρ is an uniform probability density. So, for non-uniform probability densities, it is straightforward to have that $\operatorname{sgn}\left(\frac{\partial^2 R_q[\rho_\alpha]}{\partial \alpha^2}\right) = \operatorname{sgn}(1 - q)$. \square

4 The LMC-Rényi Monotonicity

The concept of monotonicity of a complexity measure was recently presented in [20] and proven for the Fisher-Shannon and Crámer-Rao complexity measures. In this section we analyse the behavior of the LMC-Rényi complexity measure under the differential-escort transformation, and then we show its monotonicity property. Let us first recall that the LMC-Rényi complexity measure is defined [19, 22, 23] as

$$C_{p,q}[\rho] = e^{R_p[\rho] - R_q[\rho]}, \quad p < q. \quad (18)$$

Note that the case $(p \rightarrow 1, q = 2)$ corresponds to the plain LMC complexity measure [24]

$$C_{1,2}[\rho] = D[\rho]e^{S[\rho]}, \quad (19)$$

which quantifies the combined balance of the average height of $\rho(x)$ (also called disequilibrium $D[\rho] = e^{R_2[\rho]}$), and its total spreading. This measure has been related with the *degree of multifractality* of the distribution [25] and widely applied in various contexts from electronic systems to seismic events [22, 26, 27]. It satisfies interesting mathematical properties, such as invariance under scaling and translation transformations, invariance under replication and has a lower bound [22] which is achieved by the uniform densities. Obviously, this complexity measure inherits the regularity of the previous section which together with property 5 allows us to write

Property 7. *Let $p < q$ and $\alpha \in \mathbb{R}$. Then, the LMC-Rényi complexity of the probability distribution ρ transforms as*

$$C_{p,q}[\rho_\alpha] = (C_{p,q}[\rho])^\alpha. \quad (20)$$

Moreover a straightforward application of the Jensen inequality allows one to find

Property 8. *Let $p < q$. Then, the LMC-Rényi complexity of the probability distribution ρ is bounded as*

$$C_{p,q}[\rho] \geq 1, \quad (21)$$

and the equality trivially holds when ρ belongs to the class Ξ of uniform distributions:

$$\Xi = \{\chi^{(a)}(x - x_0) \mid a > 0, x_0 \in \mathbb{R}\}, \quad \chi^{(a)}(x) = \begin{cases} a^{-1}, & x \in [0, a] \\ 0, & \text{otherwise} \end{cases}. \quad (22)$$

So, the LMC-Rényi complexity measure is universally bounded, and the family of minimizing densities is given by the class of uniform densities Ξ . Actually, this class remains invariant under differential-escort transformations. In fact, restricting us to Ξ , the transformation \mathfrak{E}_α just corresponds with a scaling change.

Property 9. Uniformity transformations

Let $\alpha \in \mathbb{R}$. Then,

$$\rho \in \Xi \iff \mathfrak{E}_\alpha[\rho] \in \Xi, \quad \alpha \neq 0. \quad (23)$$

Particularly, one has that $\mathfrak{E}_\alpha[\chi^{(a)}] = \chi^{(a^\alpha)}$.

Proof. For any real α , one has

$$[\chi^{(a)}(x)]^\alpha = a^{-\alpha}, \quad \forall x \in [0, a];$$

and $dy = a^{\alpha-1}dx$ from Eqs. (6) and (7), respectively. Then, with $y(0) = 0$ one has that $y(x)$ obeys the linear relation $y(x) = a^{\alpha-1}x$. And by taking into account that $y([0, a]) = [0, a^\alpha]$ one obtains

$$\mathfrak{E}_\alpha[\chi^{(a)}](y) = [\chi^{(a)}(x(y))]^\alpha = a^{-\alpha}, \quad \forall y \in [0, a^\alpha];$$

or equivalently $\mathfrak{E}_\alpha[\chi^{(a)}] = \chi^{(a^\alpha)}$. \square

Let us now show that the LMC-Rényi complexity measure is monotone with respect to the class of differential-escort transformations $\{\mathfrak{E}_\alpha\}_{\alpha \in [0,1]}$ in the Rudnicki et al sense; this means that $C[\mathfrak{E}_\alpha[\rho]] \leq C[\rho]$ for any density ρ . We will see that this inequality is a direct consequence of the concavity of the Rényi entropy 6 with respect to the parameter of the deformation α .

First we observe that

$$\frac{\partial^2 R_q[\rho_\alpha]}{\partial q \partial \alpha} = \frac{-\alpha}{1-q} \frac{\partial^2 R_q[\rho_\alpha]}{\partial \alpha^2}, \quad (24)$$

which together with property 6 gives

$$\text{sgn} \left(\frac{\partial^2 R_q[\rho_\alpha]}{\partial q \partial \alpha} \right) = -\text{sgn}(\alpha), \quad \rho \notin \Xi. \quad (25)$$

Then, if we consider the derivative with respect to α we have that $\frac{\partial C_{p,q}[\rho_\alpha]}{\partial \alpha} = C_{p,q}[\rho_\alpha] \left(\frac{\partial R_p[\rho_\alpha]}{\partial \alpha} - \frac{\partial R_q[\rho_\alpha]}{\partial \alpha} \right)$, and so taking into account that $p < q$ one has

$$\text{sgn} \left(\frac{\partial C_{p,q}[\rho_\alpha]}{\partial \alpha} \right) = -\text{sgn} \left(\frac{\partial^2 R_q[\rho_\alpha]}{\partial q \partial \alpha} \right) = \text{sgn}(\alpha), \quad \rho \notin \Xi. \quad (26)$$

So, from Eq. (26) it trivially follows the searched property:

Property 10. *Let $p < q$. Then, the LMC-Rényi complexity of the probability distribution ρ fulfills that*

$$C_{p,q}[\rho_{\alpha'}] \geq C_{p,q}[\rho_\alpha],$$

for any $\alpha' > \alpha > 0$ or $\alpha' < \alpha < 0$. Moreover, if $\rho \notin \Xi$ the equality only holds for $\alpha = 1$ and the minimal value is only obtained when $\alpha = 0$. In the case that $\rho \in \Xi$, then $C_{p,q}[\mathfrak{E}_\alpha[\rho]] = C_{p,q}[\rho] = 1$.

Even more, for $\alpha = 0$ the minimal possible value of the complexity measure is reached, $C_{p,q}[\mathfrak{E}_0[\rho]] = 1$. That is due to, for any ρ one has that $\mathfrak{E}_0[\rho] = \chi^{(1)}$.

The last three properties can be summarized by means of the following theorem:

Theorem 1. *Given the family of uniform distributions Ξ , and the class of transformations \mathfrak{E}_α , then the triplet $(C_{p,q}, \Xi, \mathfrak{E}_\alpha)$ satisfies the monotonicity property of the LMC-Rényi measure of complexity.*

The comparison of this result with the monotonicity property of the Crámer-Rao and Fisher-Shannon complexity measures obtained by Rudnicki et al. [20] allows us to observe that the class of differential-escort operations plays for the LMC-Rényi measure of complexity the same role than the class of convolution-with-the-Gaussian operations in the Crámer-Rao and Fisher-Shannon cases.

5 Low and high complexity limits

In this section we conduct a study of the behavior of the statistical properties of a general density, when deformed in extreme cases $\alpha \sim 0$ and $\alpha \rightarrow +\infty$. To this end, we will first give three statements for the general case that will be useful in the study of the limit cases.

Proposition 1. *Let $\rho(x)$ a bounded density, then the entropic moments $W_q[\rho]$ satisfies*

$$W_q[\rho] < \infty \iff q > q_c[\rho],$$

with $q_c[\rho] < 1$.

Proof. Given a bounded probability density $\rho(x)$ then the proof trivially follows taking into account that Rényi entropy R_q is decreasing in q , and that $R_q[\rho] \geq R_\infty[\rho] = -\log(\rho_{max})$. \square

For example, for an exponential-like decaying density one has $q_c[\rho] = 0$, but for a power-law decaying density as $\mathcal{O}(x^{-\beta})$ then $q_c[\rho] = 1/\beta \in (0, 1)$,

or for any N-piecewise density $q_c[\rho] = -\infty$. On the other hand, it is easy to see that

$$q_c[\rho_\alpha] = 1 - \frac{1 - q_c[\rho]}{\alpha}. \quad (27)$$

Deserves noting that if we take $\alpha_c = 1 - q_c[\rho]$ then $q_c[\rho_{\alpha_c}] = 0$, what means that ρ_{α_c} has an infinite support $W_0[\rho_{\alpha_c}] = W_{q_c}[\rho] = \infty$, but all entropic moments with positive parameter q are finite.

On the other hand, is easy to see that the LMC-Rényi complexity measure is not only bounded inferiorly but also superiorly.

Proposition 2. *For any density $\rho \notin \Xi$ and any pair $p < q$ then*

$$1 < C_{p,q}[\rho] < C_{p,\infty}[\rho] = \frac{\rho_{max}}{\langle \rho^{p-1} \rangle^{\frac{1}{p-1}}}, \quad (28)$$

contrary if $\rho \in \Xi$ then $C_{p,q}[\rho] = C_{p,\infty}[\rho] = 1$.

Proof. Note that, if $\rho \notin \Xi$ and $q' > q > p$, then from Eq. (18) and property 8 it follows that $C_{p,q'}[\rho] = C_{p,q}[\rho]C_{q,q'}[\rho] > C_{p,q}[\rho]$. So taking $q' \rightarrow \infty$ one obtains Eq. (28). On the other hand, as is claimed in property 8, if $\rho \in \Xi$ then $C_{p,q}[\rho] = 1, \forall p < q$. \square

Let us now introduce the notion of *entropic cumulant* of order n of a probability density ρ . Note that these entropic cumulants have the same structure than the ordinary cumulants $k_n = \left. \frac{d^n \log \langle e^{px} \rangle}{dp^n} \right|_{p=0}$.

Definition 2. *Let $n \in \mathbb{N}$ and $\rho \in \mathcal{D}(\mathbb{R})$, so the entropic cumulant of order n , $\mathfrak{K}_n[\rho]$, is defined as*

$$\mathfrak{K}_n[\rho] = \left. \frac{d^n \log \langle \rho^{q-1} \rangle}{dq^n} \right|_{q=1}. \quad (29)$$

Particularly,

$$\begin{aligned} \mathfrak{K}_0[\rho] &= 0, \\ \mathfrak{K}_1[\rho] &= -S[\rho] = \langle \log \rho \rangle \\ \mathfrak{K}_2[\rho] &= \langle \log^2 \rho \rangle - \langle \log \rho \rangle^2, \\ \mathfrak{K}_3[\rho] &= \langle \log^3 \rho \rangle - 3\langle \log^2 \rho \rangle \langle \log \rho \rangle + 2\langle \log \rho \rangle^3 \\ &\dots \end{aligned}$$

It is worth to mention that, when ρ is an uniform density, then $\mathfrak{K}_n[\rho] = 0$, $\forall n > 1$; this behavior is similar to the Gaussian probability density with respect to the ordinary cumulants.

This definition and the next property will be useful in the following propositions 3, 6 and 9.

Property 11. *Given any probability density ρ and any $\alpha \in \mathbb{R}$ then*

$$\mathfrak{K}_n[\rho_\alpha] = \alpha^n \mathfrak{K}_n[\rho], \quad n \in \mathbb{N} \quad (30)$$

and then it follows that $\frac{\mathfrak{K}_{n+1}[\rho_\alpha]}{\mathfrak{K}_n[\rho_\alpha]} = \alpha \frac{\mathfrak{K}_{n+1}[\rho]}{\mathfrak{K}_n[\rho]}$.

Proof. From definition 2 one has that

$$\mathfrak{K}_n[\rho_\alpha] = \left. \frac{d^n}{dq^n} \log W_q[\rho_\alpha] \right|_{q=1} = \left. \frac{d^n}{dq^n} \log W_{q_\alpha}[\rho] \right|_{q=1} = \alpha^n \left. \frac{d^n}{dq^n} \log W_q[\rho] \right|_{q=1} = \alpha^n \mathfrak{K}_n[\rho],$$

where we have used property 4 and taken into account that, for all $\alpha \neq 0$ then $q = 1 \iff q_\alpha = 1$. \square

Finally, the third proposition is achieved through the Taylor series of the Rényi entropy $R_q[\rho]$ on its entropic parameter around $q = 1$.

Proposition 3. *Given any probability density ρ , then the associated LMC-Rényi measure can be formally expressed as*

$$C_{p,q}[\rho] = e^{\frac{\mathfrak{K}_2[\rho]}{2}(q-p)} \prod_{n=2}^{\infty} e^{\frac{\mathfrak{K}_{n+1}[\rho]}{(n+1)!} [(q-1)^n - (p-1)^n]},$$

provided that the series is convergent.

Proof. Let us consider the Taylor series of $\log W_q[\rho]$ around $q = 1$,

$$R_q[\rho] = \frac{1}{1-q} \sum_{n=0}^{\infty} \left. \frac{d^n (\log \langle \rho^{q-1} \rangle)}{dq^n} \right|_{q=1} \frac{(q-1)^n}{n!} = - \sum_{n=0}^{\infty} \mathfrak{K}_n[\rho] \frac{(q-1)^{n-1}}{n!},$$

provided that the series is convergent. So, taking into account that $\mathfrak{K}_0 = 0$ one can write $R_q[\rho] = -\mathfrak{K}_1 - \sum_{n=1}^{\infty} \mathfrak{K}_{n+1}[\rho] \frac{(q-1)^n}{(n+1)!}$. So the LMC-Rényi complexity measure can be expressed as

$$C_{p,q}[\rho] = e^{R_p[\rho] - R_q[\rho]} = e^{\sum_{n=1}^{\infty} \frac{\mathfrak{K}_{n+1}[\rho]}{(n+1)!} [(q-1)^n - (p-1)^n]}.$$

\square

Particularly, for the conventional LMC complexity measure one has that

$$C_{1,2}[\rho] \equiv C_{LMC}[\rho] = e^{\frac{\mathfrak{K}_2[\rho]}{2}} e^{\frac{\mathfrak{K}_3[\rho]}{3!}} e^{\frac{\mathfrak{K}_4[\rho]}{4!}} \dots$$

It is specially interesting that, for $p, q \sim 1$ we can write

$$C_{p,q}[\rho] \simeq e^{\frac{\mathfrak{K}_2[\rho]}{2}(q-p)}. \quad (31)$$

Moreover, when $\rho \in \Xi$, then $\mathfrak{K}_2[\rho] = 0$, and so $C_{p,q}[\rho] = e^{\frac{\mathfrak{K}_2[\rho]}{2}(q-p)} = 1$, $\forall p < q$.

Low complexity

Given any probability density ρ (with $C_{p,q}[\rho] < \infty$), and choosing a real number $\alpha \simeq 0$, then following the Theorem 1 one can always consider that ρ_α is a *low complexity density* (in the LMC-sense). First, we note that when $\alpha \rightarrow 0$, Eq. (27) diverges, so

Proposition 4. *Let $\rho(x)$ a bounded and low complexity density, then the critical entropic parameter $q_c[\rho] \ll 0$.*

Proof. Taking $\alpha \rightarrow 0$ in Eq. (27) one obtains $q_c[\rho] \rightarrow -\infty$. \square

On the other hand, the upper bound of the LMC-Rényi measure of a *low complexity density* goes to the unity, in such way that Eq (28) is crushed.

$$1 < C_{p,q}[\rho_\alpha] < C_{p_\alpha,\infty}[\rho]^\alpha, \quad \alpha \simeq 0. \quad (32)$$

So, it follows that

Proposition 5. *For a low complexity density ρ , one has that*

$$1 < C_{p,q}[\rho] < C_{p,\infty}[\rho], \quad (33)$$

but, $C_{p,\infty}[\rho] \simeq 1$.

Proof. Given a positive $\alpha \simeq 0$, and a probability density ρ such that $C_{p_\alpha,\infty}[\rho] < \infty$, then due to property (7) one has that $C_{p,\infty}[\rho_\alpha] = (C_{p_\alpha,\infty}[\rho])^\alpha \simeq 1$ \square

Finally, taking the Taylor series of $R_q[\rho_\alpha]$ around $\alpha = 0$ one obtains $C_{p,q}[\rho_\alpha] \sim e^{\alpha^2 \mathfrak{K}_2[\rho] \frac{(q-p)}{2}}$, but just taking into account property 11, then $e^{\alpha^2 \mathfrak{K}_2[\rho] \frac{(q-p)}{2}} = e^{\mathfrak{K}_2[\rho_\alpha] \frac{(q-p)}{2}}$. That is to say, we can assure that

Proposition 6. *If ρ is a low complexity density, then for any fixed $p < q \ll \infty$*

$$C_{p,q}[\rho] \simeq e^{\frac{\mathfrak{K}_2[\rho]}{2}(q-p)}. \quad (34)$$

Proof. Given a positive $\alpha \simeq 0$, from proposition 3 and property 11 one has that

$$C_{p,q}[\rho_\alpha] = e^{\frac{\alpha^2 \mathfrak{K}_2[\rho]}{2}(q-p)} \prod_{n=2}^{\infty} e^{\alpha^{n+1} \frac{\mathfrak{K}_{n+1}[\rho]}{(n+1)!} [(q-1)^n - (p-1)^n]} \simeq e^{\frac{\alpha^2 \mathfrak{K}_2[\rho]}{2}(q-p)} = e^{\frac{\mathfrak{K}_2[\rho_\alpha]}{2}(q-p)}.$$

□

In fact, note that taking into account property 11, the lowest *entropic cumulants* $\mathfrak{K}_n[\rho]$ domain for the *low complexity* densities. Moreover, in these cases one typically has that $|\mathfrak{K}_{n+1}[\rho]| < |\mathfrak{K}_n[\rho]|$.

High complexity

In order to explore the high complexity limit, one can take any non-uniform probability density ρ , and a very large $\alpha \gg 1$. So, following Theorem 1 one can claim that ρ_α is a *high complexity* density.

First of all, note that the critical entropic parameter $q_c[\rho]$ of a high complexity density is closed to one (27), that is to say

Proposition 7. *Let $\rho(x)$ a bounded and high complexity density, then the critical entropic parameter $q_c[\rho] \lesssim 1$.*

Proof. Taking $\alpha \gg 1$ in Eq. (27) one obtains $q_c[\rho] \rightarrow 1$. □

On the other hand, the inequality (28) losses the upper bound. So we can assure that

Proposition 8. *For a high complexity density ρ one has that*

$$1 < C_{p,q}[\rho] < C_{p,\infty}[\rho], \quad (35)$$

but, $C_{p,\infty}[\rho] \gg 1$, for any fixed $p \ll \infty$.

Proof. Given a non uniform probability density ρ , with $C_{p,q}[\rho] > 1$, then one obtains $C_{p,\infty}[\rho_\alpha] = (C_{p,q}[\rho])^\alpha \rightarrow \infty$ when $\alpha \rightarrow \infty$. □

Finally, it deserves to note that, although Eq. 31 must be valid for values of the parameters p and q enough close to one, for fixed p and q is possible to find a density enough complex, in such a way that Eq. 31 is not satisfied. In fact $C_{p,q}[\rho_\alpha] = C_{p_\alpha,q_\alpha}[\rho]^\alpha \simeq e^{\frac{\mathfrak{K}_2[\rho]}{2}(q-p)\alpha^2} = e^{\frac{\mathfrak{K}_2[\rho_\alpha]}{2}(q-p)}$, whenever $p_\alpha \simeq 1$ and $q_\alpha \simeq 1$; that is to say $\alpha(p-1) \simeq 0$ and $\alpha(q-1) \simeq 0$.

Moreover, taking into account Eq. (11), for a *high complexity* density the highest order entropic cumulants $\mathfrak{K}_n[\rho]$ will be dominants. All these considerations are summarized in the next proposition.

Proposition 9. *If ρ is a high complexity density, then the domain of parameters p, q for what Eq. (31) remains valid is extremely tiny. In fact, the highest order entropic cumulants $\mathfrak{K}_n[\rho]$ domain the behavior of the LMC-Rényi complexity measure. Actually, one has that $|\mathfrak{K}_{n+1}[\rho]| > |\mathfrak{K}_n[\rho]|$.*

Example

In the following we give an example with numerical values. Note that, due to LMC-Rényi is invariant under replication transformation the number N of different regions does not play a relevant role in the behavior of this complexity measure. So, for our purpose it is enough a simple example with $N = 3$.

We are going to represent an initial distribution with three steps whose heights are $h_1 = \frac{3}{2}, h_2 = 1, h_3 = \frac{1}{2}$ and their weights are $w_1 = w_2 = w_3 = \frac{1}{3}$. In Figure 1 we show the complexity reduction process through the here studied transformation

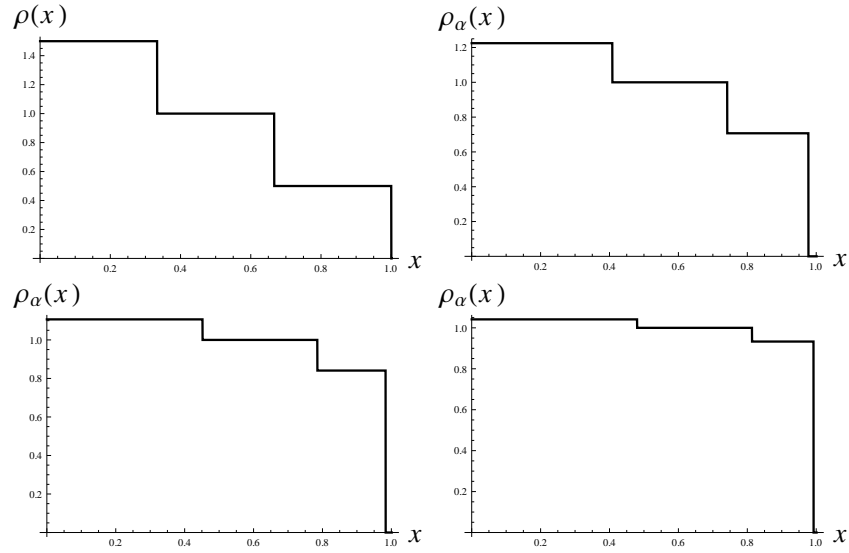


Figure 1: Transformed density $\rho_\alpha(x)$ for different values of the transformation parameter $\alpha = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{10}$.

It is interesting to give the values of the LMC complexity for these distributions, $C_{LMC}[\rho_\alpha] \simeq 1.06923, 1.01818, 1.00468, 1.00076$ for $\alpha = 1, 0.5, 0.25, 0.1$ respectively. In Figure 2 we represent the complexity increasing process of this probability density

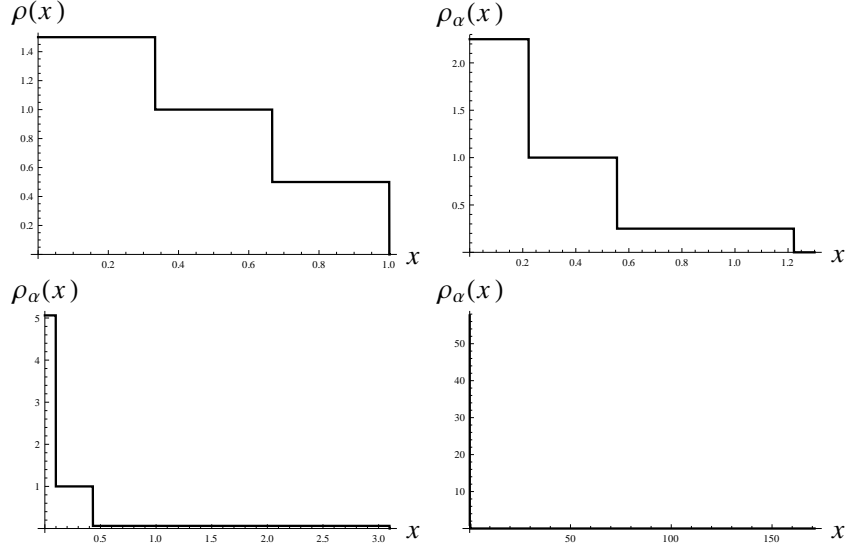


Figure 2: Transformed density $\rho_\alpha(x)$ for different values of the transformation parameter $\alpha = 1, 2, 4, 10$.

Note that, in the case $\alpha = 10$, one has that $(w_1)_\alpha \simeq 0.008$ and $(h_1)_\alpha \simeq 57$, $(w_2)_\alpha \simeq 0.03$ and $(h_2)_\alpha = 1$ and finally $(w_3)_\alpha \simeq 170$ and $(h_3)_\alpha \simeq 0.001$. So, in this case the graphic representation is really difficult to be performance. For the sake of illustration, we give the case $\alpha = 100$, for which $(w_1)_\alpha \simeq 10^{-18}$ and $(h_1)_\alpha \simeq 4 \times 10^{17}$, $(w_2)_\alpha \simeq 0.03$ and $(h_2)_\alpha = 1$ and finally $(w_3)_\alpha \simeq 2 \times 10^{29}$ and $(h_3)_\alpha \simeq 7 \times 10^{-31}$, what seems to be near to impossible to be graphically represented with accuracy (even using a logarithmic scale in both axes) while still being a 3-piecewise density. The values of the LMC complexity for these densities are $C_{LMC}[\rho_\alpha] \simeq 1.06923, 1.25988, 2.02809, 12.1843, 3 \times 10^{13}$ for $\alpha = 1, 2, 4, 10, 100$ respectively.

6 q-exponential and power-law decaying densities

The exponential and q-exponential distributions are fundamental tools in the extensive and non-extensive formalisms [28]. They can be obtained by maximizing the differential Rényi and Tsallis entropies with a suitable constraint[16], or by maximizing differential Shannon entropy with some tail constraints [29]. In this section we will study the q-exponential distribution

in the framework of the differential-escort transformations which will be able to naturally relate it to the exponential one.

The exponential function $\mathcal{E}(x) = e^{-x}$, with $x \in [0, \infty)$, is recovered taking the limit $q \rightarrow 1$ in the family of q-exponential functions defined by

$$e_q(x) = (1 + (1 - q)x)_+^{\frac{1}{1-q}}, \quad (36)$$

where $(t)_+ = \max\{t, 0\}$. Tsallis introduced [16] the q-exponential probability densities which are proportional to $e_q(-x)$. For convenience, we denote the q-exponential densities as

$$\mathcal{E}_q(y) \equiv e_q\left(-\frac{y}{2-q}\right). \quad (37)$$

Note that when $q \in (1, 2)$ the support is non compact and the tail of the probability density decays as a heavy-tailed distribution; in contrast when $q < 1$, the support is compact.

It is worth to realize that the standard escort transformation of a q-exponential density is another q-exponential; indeed,

$$E_\alpha[\mathcal{E}_q] = \mathcal{E}_{q'}, \quad q' = 1 + \frac{q-1}{\alpha}. \quad (38)$$

Note that, if $q = 1$ then $q' = 1$; that is to say, the escort transformation of an exponential distribution is another exponential distribution. On the other hand, if $q > 1$ the support of \mathcal{E}_q is not compact and so necessarily $\alpha > q - 1 > 0$ for the sake of satisfying the convergence condition given in (1); and in consequence, when $q \in (1, 2)$ necessarily $q' \in (1, 2)$. Finally, when $q < 1$ one has that $q' < 1$ for any $\alpha > 0$. In other words, the escort transformation E_α keep unchanged the three regions of the parameter q ($q < 1, q = 1, q > 1$); this behavior is expected since the standard escort transformation keep the support invariant.

This behavior is totally different for the differential-escort transformation, which indeed changes the length of the support. In fact, it transforms not only a q-exponential distribution in another one, but also: given any initial value of the parameter $q < 2$, any other parameter $q' < 2$ can be obtained through the use of \mathfrak{E}_α with $\alpha \neq 1$, as we shall see below.

From definition 1, given any α one has that

$$\mathfrak{E}_\alpha[\mathcal{E}](y) = e^{-\alpha x(y)}, \quad (39)$$

with

$$y(x) = \int_0^x e^{(\alpha-1)t} dt = \frac{1}{\alpha-1} (e^{(\alpha-1)x} - 1), \quad \alpha \neq 1; \quad (40)$$

and so one easily obtains

$$x(y) = \frac{1}{\alpha-1} \log(1 + (\alpha-1)y). \quad (41)$$

So, inserting (41) in (39) we have that

$$\mathfrak{E}_\alpha[\mathcal{E}](y) = (1 + (\alpha-1)y)^{\frac{\alpha}{1-\alpha}}, \quad (42)$$

where, from Eq. (40), $y \in [0, \infty]$ for $\alpha > 1$ and $y \in [0, \frac{1}{1-\alpha}]$ when $\alpha < 1$. In fact, we can rewrite Eq. (42) as

$$\mathfrak{E}_\alpha[\mathcal{E}](y) = e^{\frac{2\alpha-1}{\alpha}(-\alpha y)}. \quad (43)$$

Or equivalently, choosing $\alpha = \frac{1}{2-q}$ and using the notation introduced in Eq. (37), one can write

$$\mathfrak{E}_{\frac{1}{2-q}}[\mathcal{E}] = \mathcal{E}_q. \quad (44)$$

On the other hand, taking into account that $\mathfrak{E}_1[\rho] = \rho$ and considering the composition property 2 and Eq. (44) one obtains the identities

$$\mathfrak{E}_{2-q}[\mathcal{E}_q] = \mathfrak{E}_{2-q}[\mathfrak{E}_{\frac{1}{2-q}}[\mathcal{E}]] = \mathfrak{E}_{\frac{2-q}{2-q}}[\mathcal{E}] = \mathcal{E}, \quad \forall q < 2. \quad (45)$$

From which, taking any couple $q, \tilde{q} < 2$ one can write the following relation between q-exponential densities

$$\mathfrak{E}_{2-q}[\mathcal{E}_q] = \mathfrak{E}_{2-\tilde{q}}[\mathcal{E}_{\tilde{q}}], \quad (46)$$

or equivalently, using again the composition property,

$$\mathfrak{E}_{\frac{2-q}{2-\tilde{q}}}[\mathcal{E}_q] = \mathcal{E}_{\tilde{q}}, \quad (47)$$

or as well $\mathfrak{E}_\alpha[\mathcal{E}_q] = \mathcal{E}_{\bar{q}}$ with

$$\bar{q} = 2 + \frac{q-2}{\alpha}. \quad (48)$$

Thus, as we have previously anticipated, starting with any $q < 2$ we can obtain any other value $\bar{q} < 2$. In particular, when $\bar{q} > 1$ it occurs that

$\alpha > 2 - q$, when $\bar{q} = 1$ one has $\alpha = 2 - q$, and when $\bar{q} < 1$ it happens that $\alpha < 2 - q$. Note that the value $\alpha = 2 - q$ plays a critical role. Finally, it is worth mentioning that, when $\alpha > 0$ then $\bar{q} < 2$, but taking $\alpha < 0$ one obtains $\bar{q} > 2$ which normally is not considered, however note that these densities are correctly defined and they satisfy the normalization condition $\int_{\Lambda_\alpha} \rho_\alpha(y) = 1$.

These results are a little bit extended in the next lemma:

Lemma 1. *Let $\rho(x) > 0, \forall x \in [0, \infty)$, be a probability density, such that the tail of $\rho(x)$ decreases as $\mathcal{O}(x^{-\beta})$, $\beta > 1$. Then, for $\alpha > \alpha_c = \frac{\beta-1}{\beta}$, the tail of the transformed distribution $\rho_\alpha(y)$ decreases as $\mathcal{O}\left(y^{\frac{-\beta\alpha}{1-\beta(1-\alpha)}}\right)$. On the other hand, for $\alpha < \alpha_c$, the distribution ρ_α has a compact support. Finally, when $\alpha = \alpha_c$ the support is non-compact and there is an exponential decay.*

Proof. Let $\alpha \in \mathbb{R}$.

Given $x \gg 1$, the original density fulfilled $\rho(x) \sim x^{-\beta}$. On the other hand the variable change is defined as $y(x) = \int_0^x [\rho(t)]^{1-\alpha} dt$. Then, the length of the support of ρ_α is given by $W_0[\rho_\alpha] = W_{1-\alpha}[\rho] = \int_0^\infty \rho(x)^{1-\alpha} dx \sim \int_{a>0}^\infty x^{-\beta(1-\alpha)}$. So, it is clear that the support of ρ_α is compact iff $\alpha < \frac{\beta-1}{\beta}$, and in the case $\alpha \geq \frac{\beta-1}{\beta}$ we have that $\lim_{x \rightarrow \infty} y(x) \rightarrow \infty$.

In the case $\alpha \geq \frac{\beta-1}{\beta}$, one can suppose $x \gg 1$, and so $\rho_\alpha(y) \propto [x(y)]^{-\beta\alpha}$, and in the other hand $\frac{dy}{dx} = \rho(x)^{1-\alpha} \propto x^{-\beta(1-\alpha)}$.

Note that when $\alpha = \alpha_c = \frac{\beta-1}{\beta}$, then $-\beta(1-\alpha) = -1$, and so $y(x) \propto \ln x$, or equivalently $x(y) \propto e^y$. In this case we have that $\rho_\alpha(y) \propto [x(y)]^{-\beta\alpha} \propto e^{-(\beta-1)y}$.

Finally, when $\alpha > \frac{\beta-1}{\beta}$, so $y(x) \propto x^{1-\beta(1-\alpha)}$; i.e, when $x, y \gg 1$ we have that $x(y) \propto y^{\frac{1}{1-\beta(1-\alpha)}}$. Thus, $\rho_\alpha(y) \propto y^{\frac{-\beta\alpha}{1-\beta(1-\alpha)}}$. \square

It is interesting to note that under the conditions of Lemma 1, and in the high complexity limit, all the expected values become to be infinite, as well as the respective entropic moments W_q when $q < 1$. This is in concordance with the proposition 7, which states that, the entropic moments of the density are not well defined in the high complexity limit.

It is known that any distribution is characterized by its standard moments, provided that they exist. However, power-law-decaying probability densities does not fully satisfy this condition. In order to tackle this problem, Tsallis et al. [30] purposed to use escort mean values. This make sense, taking into account that the escort density has more well defined moments

than the original ones by choosing adequately the escort parameter. However, note that all escort transformation of a heavy tailed density remains being a heavy tailed, that is to say, a dense set of moments (with real parameter) remains always infinite. Contrary, as stated by Lemma 1, through the differential-escort density, we can always find a probability density which all its real moments correctly defined, at least for power-law-decaying probability densities. For these reasons, the characterization via *differential escort* densities seems to be more accurate than via escort ones.

7 Conclusions

In this paper we have presented the concept of differential-escort transformation of an univariate probability density. Its basic mathematical properties as composition and strong probability invariance have been studied. Then, we have shown the regular behavior of the differential Shannon, Rényi and Tsallis entropies for the differential-escort distributions. Moreover, the convex behavior of the Rényi entropy with respect to the differential-escort operation has been the keystone in the proof of the monotonicity property of the LMC-Rényi complexity measure. Note that the differential-escort operation allows to define equivalence classes of probability densities where exists a total order with respect to their LMC-Rényi complexity. Later we have analyzed the statistical properties of a general probability density when it is deformed to both extreme complexity cases, the low and high complexity limits. Finally, we have studied the behavior of the exponential and q-exponential densities, showing not only the stability of the q-exponential family, but also the existence of a critical value of the deformation parameter for what the behavior of the tail, if any, dramatically changes to an exponential one.

Interestingly, the action of this operation over a probability density allows for a clear interpretation of the probability conservation. Indeed, the conservation of the probability in any region of the transformed-space is clear by construction, what has a clear mass conservation interpretation.

On the other hand, the simplicity of the differential-escort transformations together with the general character of the presented results seem to indicate that this way of thinking would deserve to be explored from a more general point of view. Let us advance for example that the use of a differential-escort-based methodology has allowed for a huge generalization of the Stam inequality [31].

Acknowledgement

I am very grateful to Prof. J. S. Dehesa for useful discussions.

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