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A comprehensive approach for discrete resilience of complex networks

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ABSTRACT

The research and use of the term resilience in various types of technological, physiological, and socioeconomic systems has become very topical in recent years since this term has been applied in different fields with different meanings and connotations. One of the most common meanings of resilience is related to a positive idea that addresses recovery from failures. This study proposes to establish a theoretical and mathematical framework for discrete resilience that allows different systems to be quantitatively compared from this point of view. Also, a definition and a local view of the concept of resilience applicable to different characteristic measures in the field of complex networks is provided. Furthermore, several computational experiments are presented on the values of this new parameter in different types of synthetic and real-world networks, supplying a new set of conceptual tools for network science research.

The term resilience has been widely applied, thus the literature is spread and oriented to particular applications. As a common characteristic underlying all approaches, resilience appears as the capacity of recovering from failures as well as adapting to a new reality. We focus on the capacity of a system to return to the original situation. The topology of the underlying network, in systems that can be described by discrete models, is crucial for the concept of resilience. However, few papers have taken into account the relationships between the substructures of the network, and less attention has been paid to a formal definition. This study proposes to study the behavior of the network in such recovery processes by using graph topological indices, thereby establishing a theoretical and mathematical framework for a discrete resilience concept.

I. INTRODUCTION AND PRELIMINARIES

The term resilience came from the Latin *resilio* that means "rebound"[1] and has been applied in different fields such as ecology, psychology, risk/disaster management, transportation, engineering, energy, or economy (among others), with different meanings.[2–6] Thus, the literature on the concept of resilience is spread, scattered, and oriented to particular applications. As an underlying characteristic common to all approaches, resilience appears as a positive idea that addresses recovery from failures as well as the ability to adapt to a new reality.

In any case, it should be noted that the concept of resilience has been approached with various general definitions from multiple disciplines.[7] Many of these definitions are similar and are related to concepts such as vulnerability, robustness, flexibility, and failure tolerance. For example, in Ref. 8, we find an approximation to the concept of resilience when is understood as the intrinsic ability of a system to adjust its functionality in the presence of a disturbance and unforeseen changes. Other approaches to this concept can be found in Ref. 9, where resilience is defined as the "ability of a system to maintain its functions and structure in the face of internal and external changes and to degrade gracefully when it must," and in[10] where it is defined as the "ability of the system to withstand a major disruption within acceptable degradation parameters and to recover with adequate time and reasonable costs and risks." We can find new nuances in the definition of the resilience of a system in Ref. 11, where it includes both the negative impact that disturbance causes in a system and which is measured by the difference between the expected and the disturbed level of performance, as well as the amount of resources employed to recover the disturbed system. Finally, we think that it is important to mention the definition given in Ref. 12 by the American Society of Mechanical Engineers where resilience is defined as the ability of a system to withstand external and internal disturbances without interruption of the performance of the system function or, if the function is disconnected, to fully recover the function quickly.

In systems that can be described by discrete models, the underlying network topology is crucial to understand the idea of resilience, meaning as the ability of the system to return to the original position or to an equilibrium point. However, few works have taken into account the relationships between the different substructures of the network, and up to the present time, it has not been achieved to establish a formal definition of resilience suitable for networked systems. Following Ref. 13, there is a general question that arises naturally: "how can one quantify resilience?" It is remarkable that many authors have studied system resilience.[1,4,14–21] The ideas underlying these types of studies give alternative meanings to resilience that reflect different points of view. Thus, the concept of resilience comprises meanings ranging from stable or equilibrium states of a system to the capacity of a system to maintain its function when it suffers an incident. In any case, as a common characteristic underlying all approaches, resilience appears as a positive idea that addresses the recovery from failures regardless of their origin as well as the adaptation to new conditions, contemplating, among other elements, the amount of perturbations that a system can absorb and remaining the same state or the capacity to "manage the change" by adapting to a new reality.[2–6,22,23] Other approaches, such as that developed in Ref. 20, identify two main types of resilience in systems: static resilience and multistage resilience, depending on either on the maintenance of the existing functionality developed by the system after being perturbed by failures or attacks, or on the speed with which the state of the system returns to historically normal levels.

We will focus on the capacity of a system to return to the original position or to an equilibrium point thus we are interested in the rate at which a system returns to the state before the disruption or perturbation. Moreover, we take into account the damage produced to the system to measure the rapidity of the recovery. In systems than can be described by discrete models, the topology of the underlying network is crucial for the concept of resilience. However, few papers have taken into account the relationships between the substructures of the network and less attention has been paid to a formal definition of resilience.

An example of the potential application of this work can be found in both the analysis and design of transportation networks when a disruption happens.[24,25] Then, passenger or freight flow through stretches, sections of streets or roads, is interrupted. In this case, often the recovery of the normal functioning is gradual.

The main purpose of this paper is twofold: first to define a functional that let measure the resilience of a system from different viewpoints, in accordance with the performance function of interest; second, to show how the introduced concept of resilience behaves in different kinds of networks.

In order to get insight into the dependency of the resilience of the basic structure of the system, we assume the following assumptions:

- We suppose that the system can be represented by an unweighted and undirected graph.
- Since the functionality of a network relies on the interactions between their components, in our case the nodes of the network, we assume that the disturbances are produced in links.
- We consider that the recovery of the system takes place in discrete instants of time. Without loss of generality, we assume a constant step between consecutive times.
- The recovery process consists on adding a link, each time, from the set of damage edges.

Most of the content of these assumptions can be relaxed which is out of the scope of this paper.

The structure of the paper is as follows. Section II describes the methodology that has been used in this work and provides a framework for the new structural parameter in complex networks that we introduce. We define a discrete index named *local resilience*, and we present the first results on some types of graphs. Section III is devoted to several computational experiences, where results on synthetic and real networks are shown, analyzed, and discussed. Finally, Sec. IV is dedicated to present some conclusions.

II. METHODOLOGY

Let us consider a complex network modeled by a finite graph $G = (V, E)$. For a vertex $i \in V$, let $N(i)$ be the set of edges incident to i and $\delta_i = |N(i)|$, that is, the degree of such a vertex in graph G. In this work, we consider disruptions on the edges of a single vertex i that disconnect such a vertex from its neighbors but the vertex itself does not disappear, it only loses its incident edges. Let us denote by $G_i = (V, E \setminus N(i))$ the resulting graph from G when the incident edges to vertex i have been removed. Thus, we define the **local performance** of the vertex $i \in V$ in the graph G regarding the measure m as

$$P^i_m(G) = \frac{m(G_i)}{m(G)}. \tag{1}$$

Different topological indices of graphs can be used as a local measure function to approach the local performance of a node in a complex network, depending on our purpose. In this work, we focus on the *(global) efficiency*[26] of G, denoted by $E(G)$, as the local performance function. Let $d(i, j)$ be the distance between two vertices i and j of graph G, that is, the length of a shortest path between these two nodes in G. Then, the *efficiency* of G is defined as

$$E(G) = \frac{1}{n(n-1)} \sum_{i \neq j \in V} \frac{1}{d(i,j)},$$

which is simply the average of the inverses of the distances over all pairs of vertices of G. Thus, in this case, the local performance of a vertex $i \in V$ in the graph G regarding the efficiency $E(G)$, following

Eq. (1), is

$$P^i_E(G) = \frac{E(G_i)}{E(G)}.$$

Observe that the local performance defined by the efficiency measure is a normalized value within the interval $[0,1]$, due to the monotonicity of the efficiency index. That is, the efficiency parameter satisfies the property that $E(G') \leq E(G)$, for any G' obtained by removing some edges from G.

In order to study the behavior of the defined concept, let us consider the recovery process in a network when the damage is located in a single node, which means that node loses all its incident edges, as we mentioned above. Then, we study the evolution of the local performance value in such a node when just one edge is added in each step, by retrieving the edges one by one along all the recovery process.

First, let us introduce some general notation. Let G be a graph and i a vertex in G where the disruption is yield. Let $N(i) = \{e_1, \ldots, e_{\delta_i}\}$ be the set of edges incident to i. In the following, for clarity in notation, we identify the edge e_j with its label j. The different orders to retrieve the edges incident in node i correspond to each permutation of the set of edges $N(i)$. Then, for a given permutation $\sigma = \left(k_1, \ldots, k_{\delta_i}\right)$, we denote by $P^i_{m,k_j}(G)$ the local performance of vertex i when edge k_j is retrieved after the $j-1$ previous edges have been recovered, for $j = 1, \ldots, \delta_i$. Thus, we have a sequence of values for the local performance of the vertex i described as

$$\left(P^i_{m,k_1}(G), \ldots, P^i_{m,k_{\delta_i}}(G)\right). \tag{2}$$

Let us denote by $t(k_j)$ the instant of time when the edge k_j is recovered. Then, $t(k'_0)$ represents the moment of disruption exactly and $t(k_0)$ represents the moment when the recovery process starts. Let $P^i_{m,k'_0}(G)$ be the local performance value when the vertex i is completely disconnected from graph G, that is the minimum local performance in such a node exactly in the moment of disruption, in the instant $t(k'_0)$. We assume that the recovery process of the system does not start at the same instant that the disruption yields, i.e., there exists a reaction time of the system or disrupted state of the complex network, which we have represented by the interval $[t(k'_0), t(k_0)]$. On the left plot of Fig. 1 are represented the local performance values along all the recovery process for a given node i, where the disturbance yields. We mentioned that $P^i_{m,k_j}(G)$, represents the local performance of vertex i when edge k_j is retrieved after the $j-1$ previous edges have been retrieved, which produces the list (2), for $j = 1, \ldots, \delta_i$. Then, for a given permutation σ, we consider

$$\frac{1}{\delta_i + 1}\left(P^i_{m,k'_0}(G) + \sum_{j=0}^{\delta_i - 1} P^i_{m,k_j}(G)\right), \tag{3}$$

which is the sum of the local performance values, along all the recovery process in node i, divided by the number of edges plus one, which represents the number of periods of the recovery process, from the disrupted state to the last retrieved edge.

Observe that the value $P^i_{m,k_0}(G)$ is contained in every sum of $P^i_{m,k_j}(G)$, from $j = 1$ to $\delta_i - 1$. That is, each term $P^i_{m,k_j}(G)$ can be decomposed into two parts, the first being $P^i_{m,k_0}(G)$ and the second the real improvement in the local performance for adding a new edge. Then, in order to compute just the improvement of the local performance in vertex i, from the beginning of the recovery process exactly, that is, from the first to the last recovered edge, let us adjust the expression (3) to the following (see the left plot in Fig. 1):

$$\frac{1}{\delta_i + 1}\sum_{j=0}^{\delta_i}\left(P^i_{m,k_j}(G) - P^i_{m,k_0}(G)\right). \tag{4}$$

Finally, we define the local resilience in vertex i regarding to the measure m under the recovery process σ as

$$R^i_{m,\sigma}(G) = \frac{1}{\delta_i + 1}\sum_{j=0}^{\delta_i}\left(\frac{P^i_{m,k_j}(G) - P^i_{m,k_0}(G)}{1 - P^i_{m,k_0}(G)}\right), \tag{5}$$

which is a normalized value within the interval $[0,1]$, obtained from the expression (4) (see the right plot in Fig. 1).

Observe that Eq. (5) establishes a general definition for discrete resilience. It is flexible due to its possibility to consider any local measure function m. In this paper, we focus on the efficiency $E = E(G)$. Hence, the exact resilience definition that we consider in this work for examples and computational experiments is

$$R^i_{E,\sigma}(G) = \frac{1}{\delta_i + 1}\sum_{j=0}^{\delta_i}\left(\frac{P^i_{E,k_j}(G) - P^i_{E,k_0}(G)}{1 - P^i_{E,k_0}(G)}\right). \tag{6}$$

In the process where only one edge is recovered in each period of time, two different and antipodal strategies have been considered in this study:

- **Smart recovering**, when it is recovered the edge which provides the highest value of the local resilience as possible in each moment of the recovery process, from the first to the last edge. That is, in each step of the recovery process after a disturbance in a node, among all the links that can be recovered at such a moment, to select the edge whose recovery provides the best local performance value as possible.
- **Worst-case recovering**, when it is recovered the edge which provides the lowest value of the local resilience as possible in each moment of the recovery process, from the first to the last edge. That is, in each step of the recovery process after a disturbance in a node, among all the links that can be recovered at such a moment, to select the edge whose recovery provides the worst local performance value as possible.

Observe that is a Greedy algorithm, due to the locally optimal choice is taking at each stage, according to the recovery strategy. This is a linear process vs a factorial process, if we consider the permutation of the edges to recover which provides the global optimal choice. Let us assume that the order of the sequence

$$\left(P^i_{m,k_1}(G), \ldots, P^i_{m,k_{\delta_i}}(G)\right),$$

corresponds to one of the two described strategies. Denote by $R^i_{m,S}(G)$ and $R^i_{m,W}(G)$ to the local resilience in node i regarding the measure m when smart and worse-case recovering have been applied, respectively, according to Eq. (5).

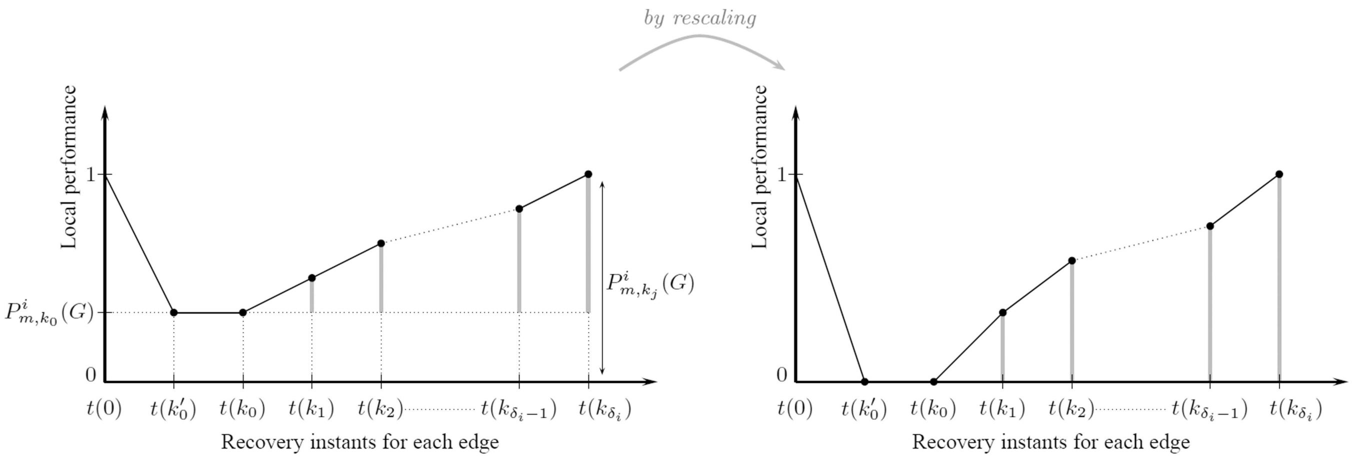

FIG. 1. Description of the values $P^i_{m,k_j}(G)$, for permutation $\sigma = (k_1, \ldots, k_{\delta_i})$, and their rescaling.

Let us present the first results on two classes of graphs. To carry out the study, the networks have been subjected to a single point failure, understood as the elimination of the adjacent edges of the vertex and the edges are retrieved one by one, from among all the possible edges to recover. The compared networks are denoted by P, which is a path on 14 vertices with a 5-complete graph in one of the end-nodes, and by S, which is a star on 7 vertices with a 3-complete graph in each end-node (see Fig. 2). Observe that both graphs have the same number of nodes and the same number of links, that is, 19 nodes and 24 links.

Figure 3 shows the relationship between a local graph parameter and the local resilience value on the vertices of the two presented graphs, regarding the efficiency measure, that is, $R^i_{E,\sigma}(G)$. The considered local parameters are the degree, the betweenness centrality,[27] the clustering coefficient,[28] and the eigenvector centrality.[29] Let $i \in V$ be any vertex of a graph G on n vertices. The betweenness centrality is the sum of the fraction of all-pairs shortest paths that pass through node i and it is defined as

$$b_i = \frac{1}{n(n-1)} \sum_{j \neq k \in V} \frac{s_{jk}(i)}{s_{jk}},$$

where $s_{jk}(i)$ represents the number of shortest paths from j to k through i and s_{jk} is the number of shortest paths between vertices j and k. The clustering coefficient is the fraction of possible triangles through that node that exists, that is,

$$c_i = \frac{T(i)}{\delta_i(\delta_i - 1)},$$

where $T(i)$ is the number of triangles through node i and δ_i is its degree. The eigenvector centrality computes the centrality for a node based on the centrality of its neighbors and it is computed as follows:

$$x_i = \frac{1}{\lambda} \sum_j a_{ij}\, x_k,$$

with $A = (a_{ij})$ being the adjacency matrix of the graph and $\lambda \neq 0$ is a constant, in a matrix form is $\lambda x = xA$.

Both strategies, smart and worst-case recovery, have been studied and they are represented in the same panel by blue and red dots, respectively (see Fig. 3). That is, for each vertex $i \in V$, the values $R^i_{E,S}(G)$ and $R^i_{E,W}(G)$ have been computed, according to Eq. (6), and represented together with the corresponding classic local parameter.

As a first trivial observation for both graphs, the values for the local resilience under the smart recovering process are greater than the values for the local resilience under the worst-case recovering process in general, as expected. Local parameters and local resilience have been computed for all nodes in both graphs, which have a total number of 19 nodes, and they have grouped themselves naturally according to their resilience value in each case, such that in S, we find three clusters, where some values are equal in the smart and the worst-case recovery strategy, whereas nodes in P have a more heterogeneous behavior.

Although both graphs have the same number of nodes and the same number of links, if we compare them, the local resilience of their nodes presents very different behavior with respect to the other indices, see Fig. 3. For instance, focus on the degree parameter and the smart recovering strategy, if we compare the nodes of high degree (degree 5 in P and degree 6 in S), we find a much higher local resilience value in P than in S (in this case, observe that the red dot is on the blue dot in the degree panel of S), despite they differ just in one unity. Similar considerations can be described for the eigenvector centrality. Respecting the betweenness centrality, in general, observe that in both graphs and for both strategies, greater betweenness centrality values correspond to lower local resilience values, but there are some exceptions in P, as can it be observed in Fig. 3. For the clustering coefficient and for both recovering process, it seems that high values of this parameter correspond to also high values for local resilience, but this behavior is not monotonic, observe that there are some exceptions in P (see Fig. 3) applying the worst-case recovering strategy. Therefore, the local resilience parameter allows to distinguish between two

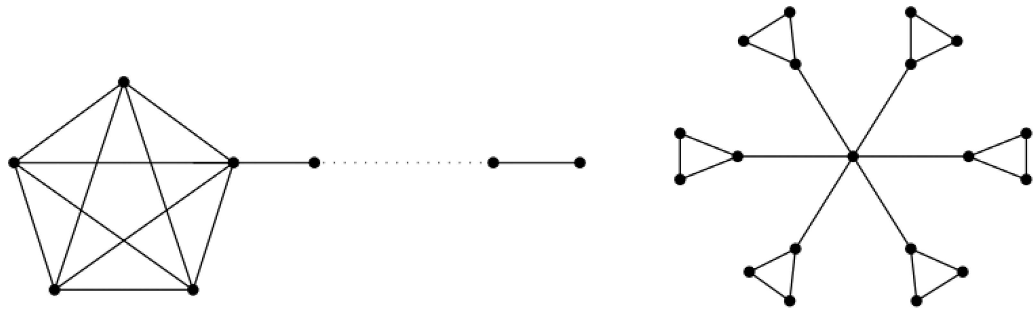

FIG. 2. Graph P (on the left) and graph S (on the right).

networks that have exactly the same number of nodes and links, thanks to the differences in the development of this measure in each of them.

Other assessments can be done with the goal to compare two relevant nodes in the networks. For example, observe that the highest degree node in network P has a higher value of local resilience than the highest degree node in network S. In this case, taking into account that the efficiency is the chosen performance function to define the resilience, can be influenced by the degree (the value of the local structural parameter) of the neighbors of the highest degree nodes in each network. Observe that in network P the neighbors of the highest degree node have higher degree than the neighbors of the highest degree node in network S. Hence, the trend of local resilience with local structural properties also depends on the local structural properties.

III. COMPUTATIONAL EXPERIMENTS: RESULTS AND DISCUSSION

In this section, some computational experiments on synthetic and real networks are presented in order to compare the local resilience measure introduced in the previous section with some classic local parameters, such as the degree, the betweenness centrality, the clustering coefficient, and the eigenvector centrality. In all the cases, the input data have been the network, that is, the set of nodes and links that compose them. Local resilience values have been computed for all nodes of the networks and for the both mentioned strategies, smart and worst-case recovery. Results are shown in different figures.

A. Synthetic networks

Three classic random families of synthetic networks have been selected for testing the relationships between the new local resilience measure with the mentioned structural local parameters. For more interest in the comparative, the three considered networks have the same number of nodes (500) and almost the same number of links (2500 aprox.).

First, we consider Erdős–Rényi random graphs,[30] denoted by $ER(n,p)$, which are unweighted and undirected graphs of $n \in \mathbb{N}$ nodes such that each possible link between two nodes exists in $ER(n,p)$ with prefixed probability $p \in [0,1]$, i.e., given a pair of nodes i,j in $ER(n,p)$, then there exists the link between i and j with probability p. Since the linking strategy in this mode is independent of the nature of the nodes, then $ER(n,p)$ is a homogeneous network and therefore the local properties of each node are quite uniform.[31] However, left panels in Fig. 4 show that local resilience is not correlated with degree, betweenness, clustering coefficient, and eigenvector centrality, in general. The nodes with high local resilience are nodes with low betweenness or low clustering coefficient [see Figs. 4(d) and 4(g)], meanwhile for the degree and the eigenvector centrality are nodes with an intermediate value [see Figs. 4(a) and 4(j)], as can be observed by plotting the relationships between the local resilience and the degree, the betweenness centrality, clustering coefficient, and eigenvector centrality of each node for an Erdős–Rényi random graph $ER(n,p)$ of 500 nodes, 2478 links, and linking probability $p = 0.02$.

Second, we study now Watts–Strogatz random networks.[28] Remember that a Watts–Strogatz random network $WS(n,k,p)$ is an undirected and unweighted network of $n \in \mathbb{N}$ nodes constructed by using a rewiring process, such that we start from a regular ring lattice of n nodes each one connected to $k \leq n$ neighbors and each link is rewired with probability $p \in [0,1]$. Note that random networks exhibit small-world properties, such as short average path lengths and high clustering.[28] Central panels in Fig. 4 show that the local resilience measure is correlated with the betweenness centrality and clustering coefficient [see Figs. 4(e) and 4(h)], but not with the degree and eigenvector centrality [see Figs. 4(b) and 4(k)], where the nodes with high local resilience are those with intermediate values for such structural parameters. In these panels, it is plotted the local resilience and all the structural parameters considered for a Watts–Strogatz random network $WS(n,k,p)$ of 500 nodes, 2500 links, starting regularity $k = 10$, and rewiring probability $p = 0.25$, being clear that local resilience is far from being correlated with the considered structural measures.

Finally, Barabási–Albert $BA(n,m)$ networks are analyzed. These are random undirected and unweighted networks of $n \in \mathbb{N}$ nodes constructed by using a growing process that starts with a fixed network of $m \in \mathbb{N}$ nodes and at each time step a new node is added by linking to $m \in \mathbb{N}$ existing nodes proportionally to its degree (preferential attachment).[32] It is well-known that Barabasi–Albert networks are heterogeneous and present power-law (or scale-free)

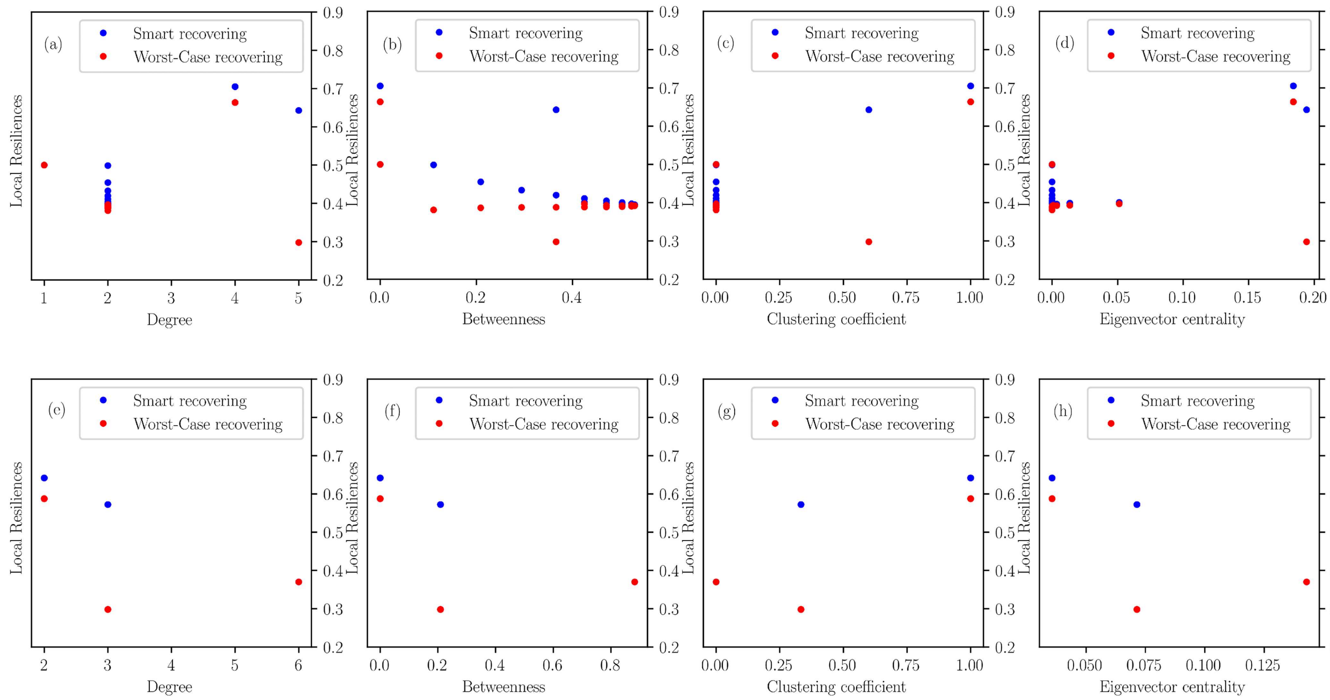

FIG. 3. Local structural parameters vs local resilience in the graphs *P* [panels (a)–(d)] and *S* [panels (e)–(h)].

degree distribution, so the structural role of each node in such networks are quite variable.[32] Right panels in Fig. 4 show that local resilience is strongly inversely correlated with most of the structural local parameters considered for a Barabási–Albert graph $BA_{n,m}$ of 500 nodes, 2475 links, and $m = 5$. Observe that, in general, high values of the local resilience correspond to nodes with low values of the structural parameters, but there are also some exceptions. For instance, there are vertices with an intermediate degree value with a high resilience or there are vertices with high clustering coefficient with an intermediate resilience [see Figs. 4(c) and 4(i)].

It is remarkable the fact that it seems that there is some correlation between the (highest) value of local resilience with some (higher) structural local parameters and therefore local resilience strongly depends on the network structure. This fact suggests that there is a clear parallelism between the correlations observed above and the sound results obtained for the robustness and stability for bistable[22] and multistable[23] dynamics in a series of relevant papers.

The local resilience parameter can be also used for comparing the networks between them, when it is appropriated. The three generated networks in this computational experiment have the same number of nodes (500 nodes) and, almost, the same number of links (2500 aprox.), which suggests to mention some comparisons. In general, for all the classic local parameters and for both recovery strategies, the values that achieve the local resilience are higher in the Watts–Strogatz network than in the Erdős–Rényi or Barabási–Albert networks. This fact could be related to the small-world properties of the network or maybe to other some underlying structural properties. Attending to the degree of the nodes, vertices with an intermediate degree in Erdős–Rényi and Watts–Strogatz network are those that attain higher local resilience values, in front of the vertices of minimum degree in Barabási–Albert network, which are those that achieve higher local resilience values, which can be relationed with the heterogeneity of nodes in Barabási–Albert network, vs a more homogeneity of nodes in Erdős–Rényi and Watts–Strogatz networks. Another outstanding fact is the low correlations between the structural parameters and the local resilience, where we can find it, slightly, only in the betweenness centrality and clustering coefficient of the Watts–Strogatz network.

In summary, this local resilience definition is a new quantitative structural measure which is not usually correlated to the considered classical measures and permits to establish interesting comparisons between networks.

B. Real networks

Local resiliences, understood as the local resilience under the smart and worst-case recovery strategies, are compared with the structural parameters of previous sections.

Results are presented for a couple of real technological networks: on the one hand, public transportation system (metro) of Madrid (Spain)[33,34] that can be modeled as a complex network of 234 nodes representing each metro station and each link is a direct connection between such stations and, on the other hand, the USA

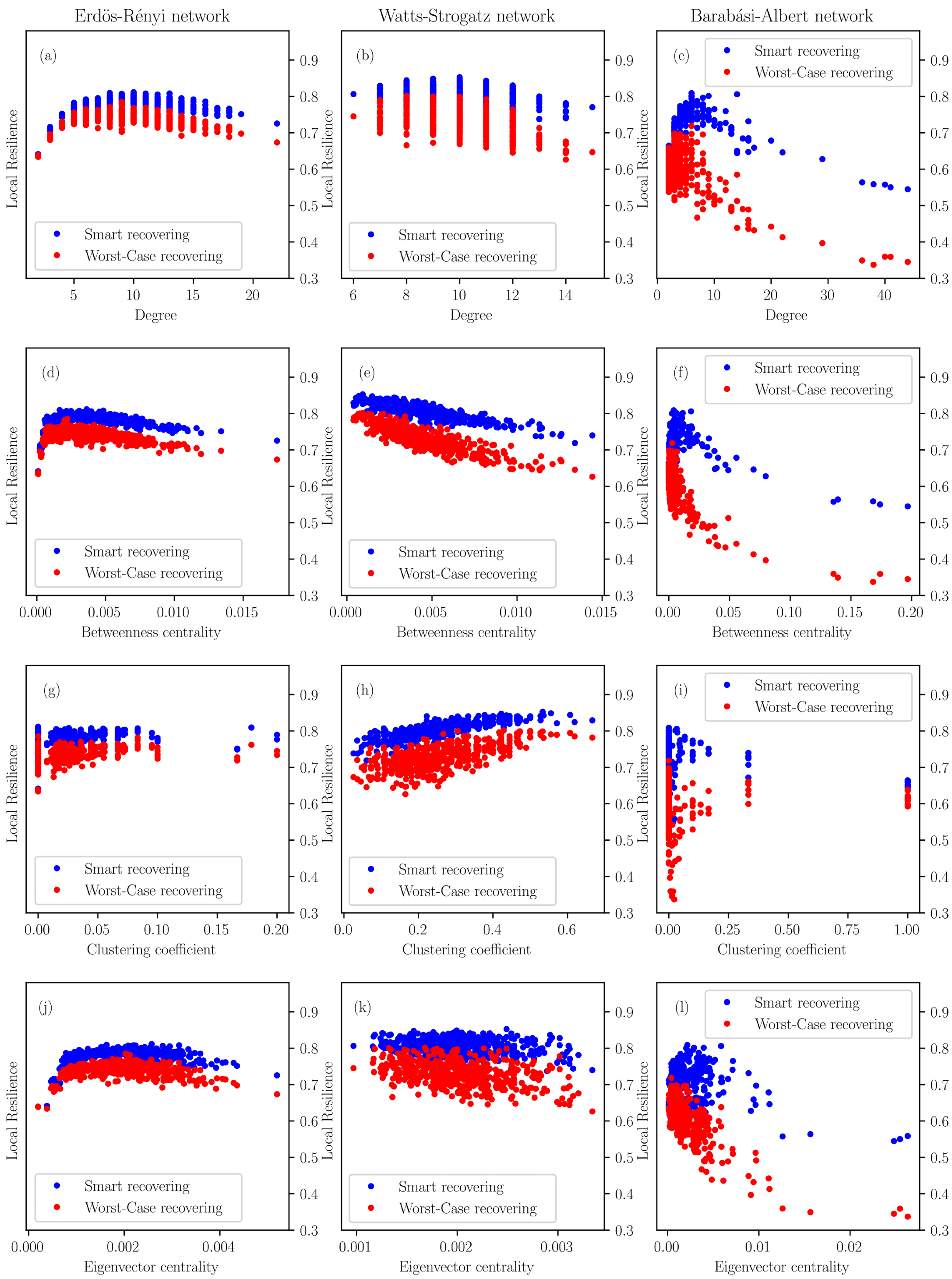

FIG. 4. Local structural parameters vs local resilience in a Erdős–Rényi network of 500 nodes, 2478 links and linking probability $p = 0.02$ [panels (a), (d), (g), and (j)], in a Watts–Strogatz network of 500 nodes, 2500 links, starting regularity $k = 10$ and rewiring probability $p = 0.25$ [panels (b), (e), (h), and (k)] and in a Barabási–Albert network of 500 nodes, 2475 links, and $m = 5$ [panels (c), (f), (i), and (l)].

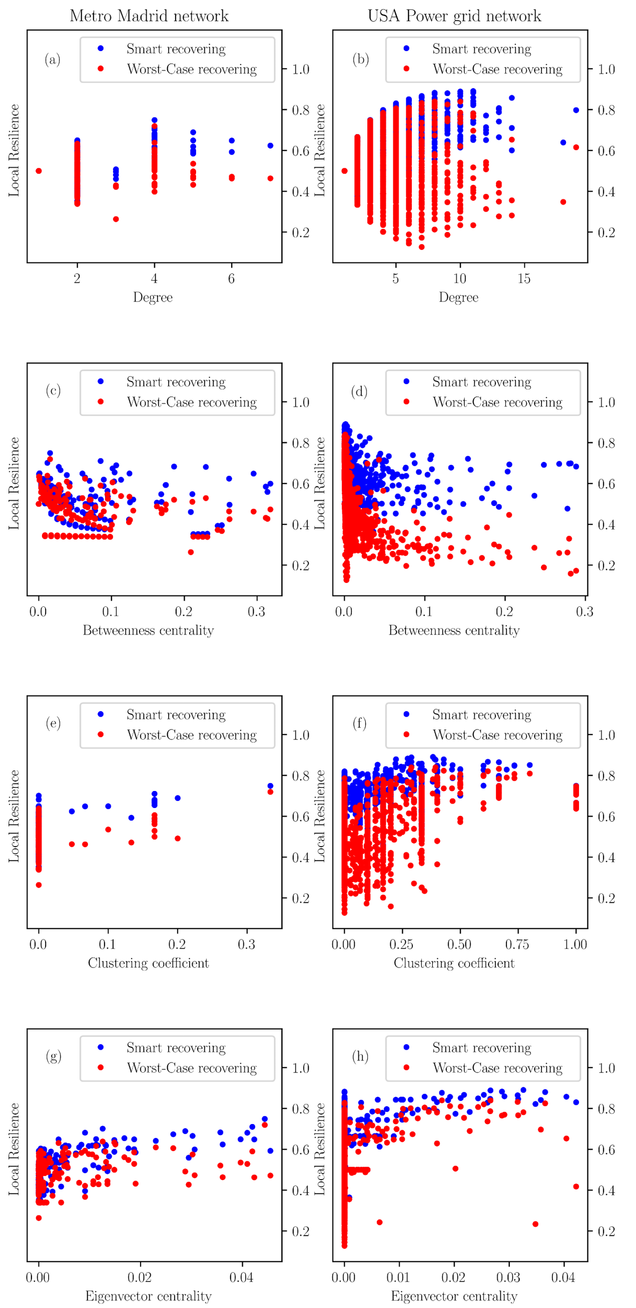

FIG. 5. Local structural parameters vs local resilience in the Madrid metro of 234 nodes [panels (a), (c), (e), and (g)] and in the USA power grid of 4941 nodes [panels (b), (d), (f), and (h)].

power grid network[28,34] which is an undirected and unweighted network representing the Western States Power Grid of the United States, such that each of its 4941 nodes represent one transform or power relay point and two nodes are connected if a power line runs between them.

As can be observed in Fig. 5, the local resilience in relation to the structural parameters considered (degree, betweenness centrality, clustering coefficient, and eigenvector centrality) reaches slightly higher values for the USA power grid network compared to those obtained for the Madrid subway network, both for the smart and worst-case recovering, despite the fact that the number of nodes of the systems is very different, being the U.S. power grid a more heterogeneous network. For both networks, we observe that the values of the local resilience are more clustered in the cases of the degree and the clustering coefficient [see Figs. 5(a), 5(b), 5(e), and 5(f)], than in the other parameters [see Figs. 5(c), 5(d), 5(g), and 5(h)].

The best local resilience values are obtained, in general, and for all the parameters, under the smart recovering process, which is a expected behavior. However, for small values of the betweenness centrality parameter, we can observe how there are vertices, in both networks, which achieve high local resilience values despite following the worst-case recovering process [see Figs. 5(c) and 5(d)], which can indicate that for these nodes there is not a great difference between the smart or the worst-case recovery. Regarding the degree, we notice that the vertices of higher local resilience value, in general, are those with an intermediate degree, and not those with maximum or minimum degree [see Figs. 5(a) and 5(b)]. For the clustering coefficient and the eigenvector centrality, most vertices with low values of these parameters correspond to high values of the local resilience, but it is remarkable that there are also vertices with high values for clustering coefficient and the eigenvector centrality which present high values of the local resilience, or even the highest value of the local resilience in the case of the Madrid subway [see Figs. 5(e) and 5(g)].

IV. CONCLUSIONS

This study provides a new theoretical and mathematical setting for the concept of resilience in complex networks. A quantitative definition of resilience located in a node of a network with respect to a generic measure has been introduced. The efficiency has been the selected local performance function for this work, but the local resilience definition can be adapted to other measures of interest, depending of our study goal, which makes it a flexible and malleable definition. This local concept permits to study the recovery capacity of a network after a disturbance in a particular node. The disruption is based on removing all the edges incident to such vertex and the recovery process on adding an edge in each period of time according to a strategy. The assumption of adding just an edge in each instant or period of time can be easily relaxed, as well as other considerations. These other variants can be the object of study for futures works.

Two recovering strategies have been considered: the one that provides the highest value of local resilience in each step of the recovery process (smart recovering) and the one that produces the smallest value of local resilience in each step of the recovery process (worst-case recovering). Four local and classic parameters have been compared with this new definition of local resilience: degree, betweenness centrality, clustering coefficient, and eigenvector centrality. Several computational experiments have been presented in graphs and synthetic networks, as well as in two real networks, from which general and interesting conclusions can been extracted.

From the results obtained on the two considered graphs (P and S), we observe that the local resilience parameter can be an interesting tool to compare networks with exactly the same number of nodes or/and links, due to the different local resilience behavior observed. This fact reflects the importance of the structural properties underlying on graphs, and not only general parameters as the number of nodes or links.

The computations carried out in this work on the synthetic networks show that this new parameter is not directly correlated to other classic local indices, as could be expected in some cases. The values of the local resilience in the three considered families, Erdős–Rényi, Barabási–Albert and Watts–Strogatz networks, have different behavior vs the classic local parameters, and there is no correlation in most of the comparisons. One interesting question could be what structural properties of these networks produces correlation between the parameters in some families but not in others, despite the heterogeneity on its nodes in some cases.

Finally, the experiments on real networks have been on the Madrid subway network and the USA power grid network. These are networks of different nature, with different number of nodes and links, but some interesting observations can be outstanding. For instance, in both networks, the nodes with highest values of the local resilience are those with an intermediate degree in the network and not those with an extremal degree. This fact highlights that these networks share some structural properties which produces that the best nodes with respect to this new local resilience parameter are not the nodes with maximum or minimum degree. Regarding the classic local parameters related to the importance or relevance of a node in the network, as betweenness centrality, clustering coefficient and eigenvector centrality, the general behavior is that the least relevant nodes, which have a low value of these parameters, are those with higher local resilience values, however, there are other numerous nodes which are the most important in the network, with a high value of these parameters, and also have high levels of local resilience. Hence, low values of the parameters are not correlated with low values of the resilience and neither high values of the parameters are correlated with high values of the resilience. This fact suggests that the resilience can be used to distinguish between nodes which have similar values for some classic local parameter, for instance, betweenness centrality (or any other). While for the betweenness centrality all of them can seem similar, maybe the node with the highest local resilience value between them can be most relevant in the network than the others. Hence, the local resilience can be an interesting differential tool.

Therefore, the local resilience parameter introduced in this paper is a clear new tool to measure the resilience of a network from a local sight. This definition is very general, which can be applied to any performance measure on a discrete context, by allowing to compare the resilience of different networks between them or to compare a same system from different points of view.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Rocío M. Casablanca: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Supervision (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal). **Regino Criado:** Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Supervision (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal). **Juan A. Mesa:** Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Supervision (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal). **Miguel Romance:** Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Supervision (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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