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DECOMPOSITIONS OF PERIODIC MATRICES INTO A SUM OF SPECIAL MATRICES*

PETER DANCHEV†, ESTHER GARCÍA‡, AND MIGUEL GÓMEZ LOZANO§

Abstract. We study the problem of when a periodic square matrix of order $n \times n$ over an arbitrary field $\mathbb{F}$ is decomposable into the sum of a square-zero matrix and a torsion matrix and show that this decomposition can always be obtained for matrices of rank at least $\frac{n}{2}$ when $\mathbb{F}$ is either a field of prime characteristic, or the field of rational numbers, or an algebraically closed field of zero characteristic. We also provide a counterexample to such a decomposition when $\mathbb{F}$ equals the field of the real numbers.

Key words. Periodic, Idempotent, Torsion, Nilpotent, Characteristic polynomial.

AMS subject classifications. 15A21, 15A24, 16U60.

1. Introduction.

The decomposition of matrices over an arbitrary field into the sum of some crucial elements, like nilpotent elements, idempotent elements, potent elements, and units, was in the focus of many researchers for a long time (see, e.g., [1], [2], [3], [4], [5], [11], [12], [13], and [14] and the bibliography cited therewith).

A square matrix A is called *periodic* if $A^m = A^n$ for certain $m > n \geq 1$; when $n = 1$, the matrix A is called *m-potent* or just, for short, *potent*. Likewise, when $m = 2$ and $n = 1$, A is known as an *idempotent* matrix. A major type of potent matrices is the so-called *torsion* matrices: invertible matrices A with the property that $A^s = \mathrm{Id}$ for some integer $s \geq 1$, where Id stands for the identity matrix. We also say that A is a *nilpotent* matrix if there exists some integer $k > 1$ such that $A^k = 0$; if k is the smallest number with this property, we call it *index of nilpotence*. In the case where $A^2 = 0$, the matrix A is called *square-zero matrix*. It is obvious that both nilpotent and potent matrices are always periodic.

In [9], we investigated the situation in which a square matrix of arbitrary size $n \geq 2$ over an arbitrary field could be expressed as the sum of an invertible matrix and a nilpotent matrix N with N^k=0 for some fixed $k \geq 1$. It was shown there that this is always possible when the rank of the matrix is no less than $\frac{n}{k}$.

On the same vein, in [8] we showed that a nilpotent matrix of size n over a division ring can be expressed as the sum of a torsion matrix and a square-zero matrix if, and only if, its rank is at least $\frac{n}{2}$. The key idea in [8] was to show that the torsion matrices satisfied certain characteristic polynomials. Extending that idea, in [10], we studied when block matrices could be modified by adding a square-zero matrix to obtain prescribed characteristic polynomials. We shall freely use this fact in the sequel in order to decompose a periodic matrix into some special elements.

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†Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria (danchev@math.bas.bg).

‡Departamento de Matemática Aplicada, Ciencia e Ingeniería de Materiales y Tecnología Electrónica, Universidad Rey Juan Carlos, 28933 Móstoles (Madrid), Spain (esther.garcia@urjc.es).

§Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, 29071 Málaga, Spain (miggl@uma.es).

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Our main achievements, presented in detail below, which motivated writing of the present paper are Proposition 2.2 and its Corollary 2.3 as well as Theorem 2.8, respectively, in which we continue our study in-depth of the special matrix decompositions as indicated above.

Throughout the text of the current article, we denote by $\mathbb{M}_n(R)$ the matrix ring consisting of all square matrices of size $n \times n$, where $n \geq 1$ is a natural number, over a ring R. As usual, the letters $\mathbb{F}$ and Δ are reserved to designate a field and a division ring, respectively.

2. Decomposing periodic matrices.

We start our work with the following preliminary comments.

Remark 2.1. *A periodic matrix A with $A^m = A^n$ for certain integers $m > n \geq 1$ clearly satisfies a polynomial of the form $x^n(x^{m-n} - 1)$. Therefore, the elementary divisors of such matrix are of the form x, x^k (with $k \geq 2$) and/or divisors of the polynomial $x^{m-n} - 1$; thus, this means that the matrix A is similar to a matrix of the form*

$$\left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & N_{k_1} & 0 & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 & 0 \\ \hline 0 & 0 & 0 & N_{k_s} & 0 \\ \hline 0 & 0 & 0 & 0 & T \end{array}\right),$$

where 0 in the left-upper block corresponds to the elementary divisors of the form x, the elements $N_{k_1}, \ldots, N_{k_s}$ correspond to the elementary divisors of the form x^{k_i} (with $k_i \geq 2$), and T contains in its main diagonal the (invertible) companion matrices associated to factors of $x^{m-n} - 1$ (so, one inspects that T is a torsion matrix). Notice that, over fields of characteristic zero, these factors of $x^{m-n} - 1$ are irreducible polynomials.

Thanks to this decomposition into special blocks, with no too many efforts we can express each periodic matrix as the sum of an idempotent matrix and a torsion matrix. Concretely, the following is true:

Proposition 2.2. *Every periodic matrix over a field is the sum of an idempotent matrix and a torsion matrix.*

Proof. As explained in Remark 2.1 quoted above, any periodic matrix A with $A^m = A^n$ for certain integers $m > n \geq 1$ is similar to a matrix of the form

$$\left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & N_1 & 0 & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 & 0 \\ \hline 0 & 0 & 0 & N_s & 0 \\ \hline 0 & 0 & 0 & 0 & T \end{array}\right).$$

Now, we can do the following:

- Express the left-upper block as $(0) = (\mathrm{Id}) + (-\mathrm{Id})$ (i.e., idempotent + torsion).

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- Decompose each nilpotent component N_{k_i} as

$$N_{k_i} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \vdots & & & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{idempotent}} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & \ddots & & \vdots \\ 0 & 0 & 1 & -1 \end{pmatrix}}_{\text{torsion of index } k_i+1},$$

  where the last matrix is torsion, because it is the companion matrix of the polynomial $x^{k_i} + x^{k_i-1} + \cdots + 1$, which is a factor of the polynomial $x^{k_i+1} - 1$.
- Write $T = (0) + T$ (i.e., idempotent + torsion).

This concludes the proof after all. ∎

As a direct consequence, we yield the following.

COROLLARY 2.3. *Every nilpotent matrix over a division ring is the sum of an idempotent matrix and a torsion matrix.*

Proof. Knowing the classical fact that each nilpotent matrix over a division ring Δ is similar to its Jordan canonical form J, which is indeed a matrix over the field $\mathrm{Z}(\Delta)$, that is, the center of Δ, we then may apply Proposition 2.2 to obtain the wanted decomposition. ∎

In this aspect, an important query which immediately arises is of whether or not Proposition 2.2 could be expanded to division rings, i.e., whether or not Corollary 2.3 remains valid for periodic matrices. However, even for potent matrices over division rings, the situation seems to be rather complicated.

Inspired by this decomposition and trying to generalize our work [8], where we decomposed nilpotent matrices of order $n \times n$ and rank at least $\frac{n}{2}$ as the sum of a torsion matrix and a square-zero matrix, we pose the following question.

Question: *Is any periodic matrix representable as the sum of a torsion matrix and a square-zero matrix?*

The rest of the paper will be devoted to the examination of this problem. To this target, we will make use of the description of periodic matrices in Remark 2.1 alluded to above and of our key result from [8].

LEMMA 2.4. *[8, Proposition 2.1] If a nilpotent block N_k of size $k \geq 2$ is followed by s blocks corresponding to elementary divisors of the form x such that $0 \leq s \leq k-2$, then the resulting matrix can be decomposed into the sum of a torsion matrix and a square-zero matrix.*

The following technicality deals with the critical situation when k blocks corresponding to elementary divisors of the form x are combined with a torsion block of order $k \times k$. It is rather straightforward, and so we omit its proof.

LEMMA 2.5. *If $A \in \mathbb{M}_{2k}(\mathbb{F})$ is a block matrix of the form*

$$A = \left(\begin{array}{c|c} \mathbf{0}_{k,k} & \mathbf{0}_{k,k} \\ \hline \mathbf{0}_{k,k} & T \end{array}\right) \in \mathbb{M}_n(\mathbb{F}),$$

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for some $T \in \mathbb{M}_k(\mathbb{F})$*, then*

$$N = \left(\begin{array}{c|c} -T & -T \\ \hline T & T \end{array}\right),$$

is a square-zero matrix and

$$(A - N)^6 = \left(\begin{array}{c|c} T^6 & \mathbf{0}_{k,k} \\ \hline \mathbf{0}_{k,k} & T^6 \end{array}\right).$$

In particular, if T *is torsion, the matrix* A *decomposes as the sum of a torsion matrix and a square-zero matrix.*

The following result, which was proven in [10], will also be very pivotal when combining blocks of 0's with torsion companion matrices:

Lemma 2.6. *[10, Theorem 2.3] Let* $\mathbb{F}$ *be a field, let* $n, k \in \mathbb{N}$ *with* $k < n - k$*, and consider the block matrix*

$$A = \left(\begin{array}{c|c} \mathbf{0}_{k,k} & \mathbf{0}_{k,n-k} \\ \hline \mathbf{0}_{n-k,k} & A_{22} \end{array}\right) \in \mathbb{M}_n(\mathbb{F}),$$

consisting of k *rows and columns of zeros, and an invertible non-derogative matrix* A_{22}*. Then, for any monic polynomial* $q(x)$ *of degree* n *whose trace coincides with the trace of* A*, there exists a square-zero matrix* N *such that the characteristic polynomial of* $A + N$ *coincides exactly with* $q(x)$*.*

The following technical statement will be useful when dealing the periodic matrices over $\mathbb{Q}$.

Lemma 2.7. *If* $p(x) \in \mathbb{Q}[x]$ *is a cyclotomic polynomial of degree* $n - k \geq 2$*,* $k < n - k$*, and*

$$A = \left(\begin{array}{c|c} \mathbf{0}_{k,k} & \mathbf{0}_{k,n-k} \\ \hline \mathbf{0}_{n-k,k} & C(p(x)) \end{array}\right) \in \mathbb{M}_n(\mathbb{Q}),$$

where $C(p(x))$ *denotes the companion matrix of the polynomial* $p(x)$*, then there exists a square-zero matrix* N *such that* $A + N$ *is a torsion matrix.*

Proof. Recall that the trace of every cyclotomic polynomial in $\mathbb{Q}[x]$ is always 0, 1 or -1 (just by considering the Möbius function).

(i) If the trace of $p(x)$ is 0, Lemma 2.6 implies that there exists a square-zero matrix N such that the characteristic polynomial of $A + N$ is $x^n - 1$; in particular, $(A + N)^n = \mathrm{Id}$.

(ii) If the trace of $p(x)$ is 1, Lemma 2.6 leads to a square-zero matrix N such that the characteristic polynomial of $A + N$ is $x^n - x^{n-1} + x^{n-2} - \cdots + (-1)^n$; in particular, if n is even, we have $(A + N)^{2(n+1)} = \mathrm{Id}$, and if n is odd, we have $(A + N)^{n+1} = \mathrm{Id}$.

(iii) If the trace of $p(x)$ is -1, by Lemma 2.6 there exists a square-zero matrix N such that the characteristic polynomial of $A + N$ is $x^n + x^{n-1} + x^{n-2} + \cdots + 1$; in particular, $(A + N)^{n+1} = \mathrm{Id}$.

This finishes the proof. ∎

Now, we are able to prove the main assertion of this paper which generalizes our recent result in [8] (compare with [9] as well) and gives a satisfactory necessary and sufficient condition for a square matrix over certain fields to be a sum of a torsion matrix and a square-zero matrix.

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Theorem 2.8. *Let $\mathbb{F}$ be either the field of the rational numbers, or an algebraically closed field of characteristic zero, or a field of prime characteristic. Then, every periodic matrix $A \in \mathbb{M}_l(\mathbb{F})$ is decomposable as the sum of a torsion matrix and a square-zero matrix if, and only if, the rank of A is at least $\frac{l}{2}$.*

Proof. Since all torsion matrices have full rank, and the rank of a square-zero matrix is at most $\frac{l}{2}$, the necessary condition is obvious.

Suppose now that $A \in \mathbb{M}_l(\mathbb{F})$ verifies $A^m = A^n$ for some integers $m > n \geq 1$ and that the rank of A is at least $\frac{l}{2}$; following Remark 2.1 stated above, the matrix A is similar to a matrix of the form

$$\left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & N_{k_1} & 0 & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 & 0 \\ \hline 0 & 0 & 0 & N_{k_s} & 0 \\ \hline 0 & 0 & 0 & 0 & T \end{array}\right);$$

and it must be that: (1) when $\mathbb{F} = \mathbb{Q}$, the elementary divisors which are factors of $x^{m-n} - 1$ are (different) cyclotomic polynomials; (2) when $\mathbb{F}$ is algebraically closed of characteristic zero, all of the factors of $x^{m-n} - 1$ are (different) degree-one polynomials of the form $x - \lambda$, where λ is a root of the unity.

Furthermore, since the rank of A is greater than or equal to $\frac{l}{2}$, we can distribute the 0-blocks of order one in the following way:

(a) Follow each N_{k_i} of size $k_i \geq 2$ by s_i-blocks corresponding to elementary divisors of the form x, where $0 \leq s_i \leq k_i - 2$; then, decompose N_{k_i} and the following s_i Jordan blocks of the form (0) into a torsion matrix and a square-zero matrix applying Lemma 2.4.

(b) Depending on which field we are dealing with, we distribute the remaining 0-blocks of order one in the following ways:

- (b.1) When $\mathbb{F} = \mathbb{Q}$, follow each companion matrix of size m_i in the main diagonal of T (associated to a cyclotomic polynomial) by t_i-blocks corresponding to elementary divisors of the form x, where $0 \leq t_i \leq m_i$; when $t_i = m_i$ decompose into a torsion matrix and a square-zero matrix utilizing Lemma 2.5, and when $t_i < m_i$ use Lemma 2.7 to decompose into a torsion matrix and a square-zero matrix;
- (b.2) When $\mathbb{F}$ is an algebraically closed field, follow each companion matrix associated to $x - \lambda$ (λ being a root of the unity) by $0 \leq s \leq 1$ blocks corresponding to elementary divisors of the form x; when $s = 1$ decompose into a torsion matrix and a square-zero matrix employing Lemma 2.5, and when $s = 0$ just take $N = (0)$;
- (b.3) When the characteristic of $\mathbb{F}$ is a prime p, suppose that the whole block T has size $t \geq 0$ and follow it by r-blocks corresponding to elementary divisors of the form x, where $0 \leq r \leq t$, and let us call it T_1; since the minimal polynomial of T_1 is algebraic over the prime field $\mathbb{F}_p$, because it divides $x^r(x^{m-n} - 1)$, the characteristic polynomial of T_1 is also algebraic over $\mathbb{F}_p$ and, therefore, we can exploit [8, Theorem 1.8] to decompose T_1 as the sum of a torsion matrix and a square-zero matrix.

This finishes the proof after all. ∎

The next construction shows that the above result is *not* true in general for arbitrary fields.

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Example 2.9. *When working with non-algebraically closed fields of characteristic zero other than $\mathbb{Q}$, e.g., the field $\mathbb{R}$ consisting of all real numbers, the decomposition of a periodic matrix into a torsion matrix and a square-zero matrix does not always hold. For instance, if we consider the matrix*

$$A = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & -1 \\ 0 & 1 & -\sqrt{2} \end{array}\right) \in \mathbb{M}_3(\mathbb{R}),$$

we can easily check that A is periodic, because $A^9 = A$ holds; nevertheless, there does not exist a square-zero matrix $N \in \mathbb{M}_3(\mathbb{R})$ such that $A + N$ is torsion: indeed, otherwise, the characteristic polynomial of $A + N$ would be of the form

$$p(x) = x^3 + \sqrt{2}x^2 + ax + b \in \mathbb{R}[x],$$

for some $a, b \in \mathbb{R}$ and, moreover, such polynomial should divide $x^n - 1$ for some $n \geq 3$; its three roots $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ should be n^{th}-roots of the unity (in fact, one of them, say α_1, a real number, hence $\alpha_1 = \pm 1$, and the other two, say α_2, α_3, complex conjugate roots of unity) with $\alpha_1 + \alpha_2 + \alpha_3 = -\sqrt{2}$. Since the sum of the two roots of unity α_2 and α_3 cannot be smaller than -2, one deduces that $\alpha_1 = -1$ and $\alpha_2 + \alpha_3 = -\sqrt{2} + 1$, and this automatically gives that

$$p(x) = (x + 1)(x^2 + (\sqrt{2} - 1)x + 1).$$

Next, solving the degree 2 equation $x^2 + (\sqrt{2} - 1)x + 1 = 0$ in $\mathbb{C}$ directly leads us to

$$\alpha_2, \alpha_3 = \frac{1}{2}\left(1 - \sqrt{2} \pm \sqrt{-1 - \sqrt{2}}\right) \in \mathbb{C}.$$

However, the minimal polynomial of α_2 (and also of α_3) over $\mathbb{Q}$ is $x^4 - 2x^3 + x^2 - 2x + 1$, which is definitely not cyclotomic, because its trace equals 2, thus proving that α_2, α_3 cannot be roots of unity, a contradiction, as expected. This ends the example and our considerations after all.

The reason of the above counterexample is that A consists of a single row and column of zeros together with a torsion companion matrix of an irreducible polynomial. When we have at least two rows and columns of zeros, this problem does no longer occur, as can be shown in the following claim.

Proposition 2.10. *Let $\mathbb{F}$ be a field of characteristic zero, and let A be a matrix of the form*

$$A = \left(\begin{array}{c|c} \mathbf{0}_{k,k} & \mathbf{0}_{k,n-k} \\ \hline \mathbf{0}_{n-k,k} & C(p(x)) \end{array}\right) \in \mathbb{M}_n(\mathbb{F}),$$

where $2 \leq k < n - k$ and $C(p(x))$ is a torsion companion matrix of an irreducible polynomial $p(x)$. Then, there exists a square-zero matrix N such that $A + N$ is a torsion matrix.

Proof. Let $r \in \mathbb{N}$ such that $(C(p(x)))^r = \mathrm{Id}$. One observes that the polynomial $p(x)$ is either co-prime with $x^k - 1$ or with $x^k + 1$ because it is irreducible.

If the greatest common divisor of $p(x)$ and $x^k - 1$ is 1, Lemma 2.6 tells us that there exists a square-zero matrix N such that the characteristic polynomial of $A + N$ is $p(x)(x^k - 1)$; in this case, $A + N$ is a torsion matrix with $(A + N)^m = \mathrm{Id}$, where m is the least common multiple of k and r.

If the greatest common divisor of $p(x)$ and $x^k + 1$ is 1, Lemma 2.6 informs us that there exists a square-zero matrix N such that the characteristic polynomial of $A + N$ is $p(x)(x^k + 1)$; in this case, $A + N$ is a torsion matrix with $(A + N)^m = \mathrm{Id}$, where m is the least common multiple of $2k$ and r. ∎

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REMARK 2.11. *This last proposition together with Lemmas 2.4, 2.5, and 2.6 can be used to decompose a periodic matrix A over a field of characteristic zero as the sum of a torsion matrix and a square-zero matrix as soon as the 0-blocks of order one in the decomposition of A given in Remark 2.1 can be distributed in an adequate way among the rest of the blocks.*

We close our work with the following challenging query:

Problem: Find a criterion when a square matrix over an arbitrary field is a sum of a torsion matrix, an idempotent matrix and a square-zero matrix.

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