FROM BELIEF TO KNOWLEDGE

Tesis presentada por

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Fdo.: David Pearce

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Resumen

La combinación de la creencia y el conocimiento en un formalismo ha sido una problemática en epistemologia. Un formalismo que cumple tal combinación una llamamos ***doxepi***-formalismo. Existen varios ***doxepi***-formalismos basados en lógica modal, donde la combinación razonable de operadores de creencia y conocimiento ocurren. Aunque en todos los existentes, vemos la tendencia a priorizar la lógica modal doxástica bien establecida y construir lógica epistémica de acuerdo con algunos principios que la lógica doxástica sugiere, es decir, el algoritmo es: *tomar una logica doxástica* $\mathbf{L_D}$ *conocida y construir la lógica* $\mathbf{L}$ *que tiene un cierto sabor epistémico y mantiene una buena interrelación con* $\mathbf{L_D}$. En este trabajo tomamos la dirección inversa y damos prioridad a las lógicas modales epistémicas conocidas. Asi que nuestra fórmula, que también motiva el título, es: *tomar una lógica conocida epistémica* $\mathbf{L_E}$ *y encontrar una buena lógica doxástica* $\mathbf{L}$ *que se combina con* $\mathbf{L_E}$ *de una manera suficiente.* En la tesis cumplimos esta tarea para las lógicas **S5** y **S4** y estudiamos varios conceptos y ***doxepi***-formalismos sobre la base de estas lógicas.

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Abstract

Combining belief and knowledge into one formalism has been problematic issue in epistemology. A formalism where such a combination is met we call ***doxepi***-formalism. There exist several modal logic based ***doxepi***-formalisms in the literature where the reasonable combination of belief and knowledge operators occur. Although in all existing ones we see the tendency to prioritise well established doxastic modal logic and construct epistemic logic according to some principles which the doxastic logic suggests, i.e. the algorithm is: *take a well known doxastic logic* $\mathbf{L_D}$ *and construct the logic* $\mathbf{L}$ *which has some epistemic flavor and keeps good interrelation with* $\mathbf{L_D}$. In this work we take converse direction and give the priority to known epistemic modal logics. So our formula, which also motivates the title, is: *take a known epistemic logic* $\mathbf{L_E}$ *and find a good doxastic logic* $\mathbf{L}$ *which combines with* $\mathbf{L_E}$ *in a sufficient way.* In the thesis we accomplish this task for the logics **S5** and **S4** and study several different concepts and ***doxepi***-formalisms based on these logics.

Contents

Chapter 1

Introduction

> *For scientific thinking is nothing but a fixation of the direction of will within the confines of the head, and a scientific truth no more than an infatuation of desire living exclusively in the human head.*
>
> *Every branch of science will therefore run into ever deeper trouble; when it climbs too high it is almost completely shrouded in even greater isolation, where the remembered results of that science take on an independent existence. The "foundation" of this branch of science are investigated, and that soon becomes a new branch of science. One then brings to search for the foundations of science in general and knocks up some "theory of knowledge." (Luitzen Egbertus Jan Brouwer)*

One of the central standing questions in philosophy is to define knowledge in an universal way. Although philosophy has been unable to define knowledge in an unique way. And every definition of knowledge by one author has been criticized by counterexamples from other authors. On the other hand every single definition of knowledge ever met in philosophy has carried truth in it and there are situations when one or the other definition is applicable. The reason for this is that each definition of knowledge is just a projection of an ideal concept to a human (author's) mind. Exactly in the same way as one or the other philosopher can understand knowledge and reflect it as a definition, logic can reflect such a definition in a formal system.

And during this process, although some parts (which logic can not grasp) are lost, other important parts still remain.

1.1 Introduction

Modern epistemic and doxastic logic has its roots in the work from Hintikka [46] which gave rise to a substantial and sustained research programme making significant contributions to philosophy, logic, AI and other fields. Though several specific features of Hintikka's approach to knowledge and belief have, over the years, been questioned and modified by different authors for different purposes, some of the core ideas have stood the test of time and are still relevant today.

One of the central, surviving ideas in this paradigm is that knowledge and belief can be understood as relations between a cognitive agent a and a proposition p, symbolised by $\mathcal{K}_a p$ – "a knows that p" and $\mathcal{B}_a p$ – "a believes that p". The semantic interpretation of these expressions proceeds via possible world frames equipped with an epistemic alternativeness relation, $R_{\mathcal{K}_a}$, interpreted as: $wR_{\mathcal{K}_a}w'$ if w' is compatible with everything a knows in world w. The truth condition for an epistemic operator is then given by:

$$w \models \mathcal{K}_a p \quad \text{iff} \quad w' \models p \quad \text{for all} \quad w' \quad \text{s.t.} \quad wR_{\mathcal{K}_a}w' \tag{1.1}$$

So a knows p in a world w if p is true in all the epistemic alternatives to w; and analogous relations and truth conditions apply to belief sentences: $w \models \mathcal{B}_a p \quad \text{iff} \quad w' \models p \quad \text{for all} \quad w' \quad \text{s.t.} \quad wR_{\mathcal{B}_a}w'$.

Assuming some basic (if idealising) conditions about knowledge, and simplifying by considering a single agent in each system, epistemic/doxastic logics are on this view structurally similar to normal modal logics with a necessity operator $\Box$ interpreted on Kripke frames with a single binary alternativeness or accessibility relation R on worlds. Again, despite many variations, mutations and extensions, this feature remains in many areas concerned with reasoning about knowledge and belief. Let us recall that semantics of $\Box$ coincides with the above discussed semantics of $\mathcal{K}_a$ and $\mathcal{B}_a$.

$$w \models \Box p \quad \text{iff} \quad w' \models p \quad \text{for all} \quad w' \quad \text{s.t.} \quad wRw'. \tag{1.2}$$

Later on, when it is clear from the context weather we are dealing with knowledge or belief, we will use $\Box$ instead of operators $\mathcal{K}_a$ and $\mathcal{B}_a$.

Understood in this way modal language is able to express several interesting philosophical concepts met in epistemology. Such examples are $\Box p \to p$ (knowing p implies p is true) or $\Box p \to \Box\Box p$ (if an agent knows p then he knows that he knows p) which is known as positive introspection. Of course this kind of formalisation can not capture the entire concept of knowledge as it is. In fact it can just mimic on some properties of knowledge which are available to be expressed by such a simple formalism as modal language.

Artificial intelligence is less concerned about the definition of knowledge itself and focuses much more on properties of agents having knowledge (defined anyhow or taken as an abstract undefined notion). In these settings and for the needs of AI taken in an abstract form, $\Box p$ - an agent knows that p, is a perfect formalisation. We have already mentioned that using $\Box$ operator one is able to express some nice epistemic properties of agents. Below we define an agent who is called *perfect reasoner* for the knowledge he has perfectly suits his ability to organise it. The following definition explains what we mean by organising knowledge in a good way.

Definition 1.1.1. *We will say that an agent is a perfect reasoner if he knows all propositional tautologies and additionally his knowledge operator satisfies the following properties:*

- $\Box p \to p$ - *Agent knows p implies p is true;*
- $\Box p \to \Box\Box p$ - *Positive introspection;*
- $\neg\Box p \to \Box\neg\Box p$ - *Negative introspection.*

While different normal systems have been proposed to capture the logic of the knowledge operator, a special place is occupied by **S5** [49]. There are several well-known reasons for this. For one, the logic **S5** satisfies some important principles of introspection that might be considered appropriate for an ideally rational agent (perfect reasoner). For another, it is based on frames whose alternativeness relation

R is in a sense maximal: R is reflexive, symmetric and transitive, i.e. an equivalence relation. Moreover inference in **S5** is fully characterised by the class of *universal* Kripke models, where R is a universal relation (and frames form a so-called *cluster*); so every world counts as an epistemic alternative to a given one. Third, almost all nonmonotonic systems for reasoning about knowledge make use of **S5**-like frames. Furthermore, it is closely related to the important concept of *stable* (belief) set first studied by [**?**] and [64], since these correspond to sets of formulas true on some universal **S5**-frame. Stable sets again play a prominent role in nonmonotonic modal and epistemic logics in general. We will give detaild explanation of stable sets in Chapter 4

The other epistemic logic which we consider in the thesis is obtained by dropping negative introspection on agents knowledge and just keeping positive introspection. Following [81] the agent with such a property is called a normal agent. The resulting modal system is **S4** [47]. Throughout the thesis our main epistemic logics will be **S5** and **S4** although we will consider some intermediate logics such as **S4F** [76] and their nonmonotonic versions for the purpose of capturing concept of minimal knowledge.

1.1.1 Objectives

Combining belief and knowledge into one formalism has been problematic issue in epistemic logic since long ago. Although this is not universally established law, it is common to think, that knowledge is more compound notion and it contains (or implies) belief. Therefore when having both operators, $\mathcal{K}$ - for knowledge and $\mathcal{B}$ - for belief, it is expected that at least the following implication holds: $\mathcal{K}p \rightarrow \mathcal{B}p$. Of course just the implication $\mathcal{K}p \rightarrow \mathcal{B}p$ is not sufficient for the formalism to be acceptable for modeling belief and knowledge simultaneously. The obvious preference is given to formalisms which take roots in philosophy. For example formalisms where one can logically reflect some philosophical foundation of interaction between notions of belief and knowledge. Additionally it is required that one keeps the necessary doxastic and epistemic properties of an agent. To sum up we enumerate the properties which we would like the formalism to have:

1. The knowledge operator $\mathcal{K}$ should reasonably fit the intuition behind some definition of knowledge from epistemology,

2. The belief operator $\mathcal{B}$ is supposed to satisfy properties commonly adopted at least in some philosophical settings,

3. The formal interaction between operators $\mathcal{K}$ and $\mathcal{B}$ should reflect dependency of knowledge on belief (or vice versa) taken from some philosophical foundations.

Let us call a formalism a ***doxepi***-formalism if the above three properties are taken into account. Although as we have already mentioned, coming up with a good ***doxepi***-formalism is not a simple task and deals with trouble of constructing the bimodal system where all three aspects merge in a sufficiently good way. Main motivating example solving this problem at some stage was proposed in [42]. Solution from [42] is stated for the particular case, where the doxastic capabilities of an agent are given by the classical doxastic logic **KD45** and the interconnection between knowledge and belief reflects the idea of knowledge as true belief ($\mathcal{K}p \leftrightarrow p \wedge \mathcal{B}p$). Although the drawback of this case is the first condition of the above pointed three recuirements. In particular the epistemic counterpart of the logic **KD45** [26] turns out not to be the classical epistemic logic **S5** but the logic $\mathbf{S4.4} = \mathbf{S4} + (p \to (\neg\Box p \to \Box\neg\Box p))$. Nevertheless, the axiom $p \to (\neg\Box p \to \Box\neg\Box p)$ is not unintuitive and could be given interpretation of weak version of negative introspection. As a conclusion, [42] gives a clear example of an effective ***doxepi***-formalism although the priority is given to the 2-nd and 3-rd items.

Objective number one: *One of the objective of this thesis is to fill out yet not investigated parts and develop **doxepi**-formalism where the* 1*-st item is prioriterised. In other words the aim is to construct doxastic modal logics, which in combination with the classical epistemic logics, such as* **S5** *and* **S4**, *still keep the interconnection* $\mathcal{K}p \leftrightarrow p \wedge \mathcal{B}p$ *and at the same time carry interesting doxastic properties on their own.*

Minimal Knowledge and Minimal Belief

The paradigm of minimal knowledge derives from the well-known work of Halpern and Moses, especially [40], later extended and modified in works such as [79, 55, 54] and others. Many approaches are based on **S5** Kripke models with a universal accessibility relation and the minimisation of knowledge is represented by maximising the set of possible worlds with respect to inclusion. In general, this has the effect of minimising objective knowledge, ie knowledge of basic facts and propositions. A somewhat different approach was developed by Schwarz and Truszczynski [75] and can be seen as a special case of the very general method of Shoham [79] for obtaining different concepts of minimality by changing the sets of models and preference relations between them.

Schwarz and Truszczynski argue that their approach to minimal knowledge has some important advantages over the method of [40] and they study its properties in depth, in particular showing that while the two-floor models correspond to the modal logic **S4F** first studied by Segerberg [76], minimal knowledge is precisely captured by non-monotonic **S4F**. In [89] they show that non-monotonic **S4F** captures, under some intuitive encodings, several important approaches to knowledge representation. They include disjunctive logic programming under answer set semantics [32], (disjunctive) default logic [72], [33], the logic of grounded knowledge [55], the logic of minimal belief and negation as failure [54] and the logic of minimal knowledge and belief [89]. Recently, Truszczynski [87] and Cabalar [13] have revived the study of **S4F** in the context of a general approach to default reasoning.

Objective number two: *Although minimal knowledge has been extensively studied, the concept of minimal belief has not been explored too much. On the other hand it is clear that the level of importance of the actual concept is as high as of the former one. As one of the objectives we see the formal study of minimal belief and related nonmonotonic formalisms.*

Common Knowledge and Common Belief

Agreement technologies is a newly emerging domain where iterative concepts of belief and knowledge of agents are of special interest. To achieve successful communication and agreement it is important for agents to reason about themselves and what others know or believe. Among the more interesting examples in this direction are the notions of common knowledge and common belief. We will denote the operators for common knowledge and common belief by Ck_G and C_B respectively. We have: $Ck_G\varphi$ iff φ is common knowledge in the group G and $Cb_G\varphi$ iff φ is a common belief in the group G.

Following the analysis of common knowledge as originally defined by Lewis [52], this concept has been extensively studied from various perspectives in philosophy [5], [2], game theory [91], artificial intelligence [45], modal logic [3], [4] etc. Theories of common belief are less well-developed though some approaches can be found in [6, 45, 84]. In the work of van Benthem and Sarenac [7], they showed how a topological semantics for logics of common knowledge may be useful for modeling and distinguishing different concepts. A key idea here is that the knowledge of different agents is represented by different topologies over a set X. Various ways to merge that knowledge can be obtained via different modes of combining logics and topological models. [7] considers for example the fusion logic $\mathbf{S4} \circ \mathbf{S4}$ and product topologies that are complete for the common knowledge logic $\mathbf{S4_2^C}$ of [26].

Objective number three: *The last objective of the thesis is devoted to a study of the common belief of normal agents i.e. agents which have positive introspection on their beliefs.*

1.1.2 Methodology

Probably the most important difference between a normal *doxastic* as opposed to *epistemic* modal logic is that the axiom $\Box p \to p$ (denoted by $\mathbf{T}$) that is appropriate for the latter seems unsuitable for the former. Therefore typically even strong logics of belief, such as **KD45**, do not satisfy this principle. Our aim here is to investigate a doxastic modal logic **KS** [24] as an alternative to system **KD45**. The motivation comes from two main aspects, each of them forming one main part of our study.

The first aspect concerns monotonic doxastic logic. One leading idea is to approach doxastic reasoning by looking for a system having strong analogies to **S5**, but without the **T** axiom. Let us consider then an analogy on the semantical level and take the case of universal **S5** models. An obvious way to avoid principle **T** is to consider irreflexive alternativeness relations and to interpret belief in terms of *other* possible worlds (than the one under consideration). So in place of (**S5**) knowledge where we may assume that *every world is an epistemic alternative*, we would have for belief: *every* **other** *world is a doxastic alternative.* We maintain the analogy to knowledge in that we refer not just to some other worlds but to all of them.

The second aspect of our motivation for this study concerns nonmonotonic reasoning. Nonmonotonic modal logics, also used for reasoning about knowledge, are based on a concept of (theory) expansion. In the standard cases such expansions are stable sets and can be captured by a notion of *minimal* model, again a special kind of **S5**-model that can be thought to represent a concept of minimal knowledge. Evidently each world in such a model satisfies the strong **T** axiom, seemingly appropriate for knowledge but less so for belief. In fact, while **KD45** is usually considered to be a doxastic modal logic, its nonmonotonic variant is precisely autoepistemic logic, ie. a system that seems closer to knowledge rather than belief. To obtain a candidate logic for minimal belief, our strategy is to put **KS** models in place of **S5** models and define a new concept of minimal model. Equivalently, as we show in Chapter 4, we can define a weaker concept of expansion and show that this corresponds to the new definition of minimal model.

The above two aspects mainly give answer to the objective number one and put the grounding to the objective number two. Indeed for capturing minimal belief we make use of the doxastic system **KS** as the base one and observe minimality in a sense of beliefs of **KS**-agent. As we have already mentioned the technique for capturing minimal belief follows the technique [75] for capturing minimal knowledge. For this matter we introduce two modal logics **wK4f** [69] and **wK4Df** [70] which serve as two different doxastic analogs of the modal logic **S4F**. We study nonmonotonic versions of the logic **wK4f** by applying standard methods known from nonmonotonic modal logics while for the logic **wK4Df** we change a methodology by introducing a new concept of expansion which we call weak-expansion. Both methods give us the concept of minimal belief while in the first case we deal with an agent who believes that whatever he believes is true and in the second case the agent does not any more have this property. By studying the two nonmonotonic modal logics we provide the logical foundation of the idea of minimal belief.

It is convenient to mention that all the logics we have considered so far arise from topological spaces. For the reader not familiar with topology it will be made clear later. At this point we just mention that the class of the least topological spaces gives two different logics **S5** and its doxastic companion **KS**. Analogously from the topological spaces that are *minimal*, that is where X has only three open sets. In the first case one arrives at modal logic **S4F**, and the second path yields the logic **wK4f**. Again the class of all topological spaces give two different logics **S4** and **wK4**. The details of connection of logics with topology will be given in Chapter 3, while at the moment we just want to point out that topology can serve as a source for producing good ***doxepi***-formalisms and therefore topological study of many-modal versions of the above logics also will provide interesting outcome.

As stated the third objective of our thesis focuses on the common belief of normal agents, and for ease of exposition we restrict ourselves to the two agent case. We thus consider two agents whose individual beliefs satisfy the axioms of **K4**. In other respects we adopt the main principles of the logic of common knowledge, $\mathbf{S4^C_2}$. This can be seen as a formalization of the idea that common knowledge is equivalent to an infinite conjunction of iterated individual knowledge: $\varphi \wedge \Box_1\varphi \wedge \Box_2\varphi \wedge \Box_1\Box_1\varphi \wedge \Box_1\Box_2\varphi \wedge \Box_2\Box_1\varphi \wedge \Box_2\Box_2\varphi \wedge \Box_1\Box_1\Box_1\varphi \wedge \Box_1\Box_1\Box_2\varphi ...$ Later we shall see that a variation of this formula is 'true' for common belief under the relational semantics. We shall also show that the topological semantics for $\mathbf{K4^C_2}$ is compatible with the idea of common belief as a fixpoint *equilibrium*, a notion used by Barwise [5] to describe common knowledge that can be captured by an expression of the modal μ-calculus [11].

Our approach to providing a topological semantics follows the work of Esakia [24]. Notice that under the topological interpretation of $\Box$ as a knowledge operator, eg.

in [7], $\Box\varphi$ refers to the topological *interior* of the points assigned to φ. In the case of a doxastic logic like **K4** the topological interpretation is different. It is perhaps simpler to state it for the $\Diamond$ operator. Following McKinsey and Tarski [61], the idea is to treat $\Diamond\varphi$ as the *derivative* of the set φ in the topological space. Esakia showed that under this interpretation **wK4** is the modal logic of all topological spaces. **K4** is an extension of **wK4** and is characterized in this semantics by the class of all T_D-spaces [8]. By combining the ideas and results from [7] and [24], we obtain a derived set semantics for the logic of common belief based on bi-topological spaces, where the modality for common belief operates on the intersection of the two topologies. As a main result, we prove that $\mathbf{K4_2^C}$ is sound and complete with respect to the special subclass of all bi-topological T_D-spaces [21].

1.1.3 Structure of the Thesis

The thesis consists of six chapters out of which the first is the introduction. Chapter 2 mainly contains preliminary material about modal epistemic logics such as **S4, S4F, S5** and the following logics of belief **K4, KD45** as well as the study of the modal logic **KS** as a doxastic alternative to the doxastic logic **KD45**. Also we give a study of the monotonic modal logics **wK4f** and **wK4Df**. We mainly touch completeness and finite model property issues. Also we give the description of finite models of some of the mentioned logics in terms of tuples of natural numbers. We conclude the chapter by showing how splitting translation works for combining logics of belief and logics of knowledge into interestings ***doxepi***-foramlisms.

In Chapter 3 we discuss topological roots of the doxastic and epistemic logics mentioned in Chapter 2 and the topological idea of the splitting translation. The technique of forming good ***doxepi***-formalisms via topological methods is also touched. We show how from the class of all topological spaces, by two different semantics, one obtains the two logics **S4** and **wK4**: exactly in the same way the class of all minimal topological spaces give rise to the logics **S4F** and **wK4f** and by the same technique the class of the least topological spaces give logics **S5** and **KS**. Besides these logics are pairwise connected by the splitting translation. The following two papers [**?**] and

[69] contain most of the material underlying the technical part of this chapter

Chapter 4 deals with the concepts of the minimal knowledge and the minimal belief. In the first part of the chapter we discuss the idea of minimal knowledge and the methods of minimisation introduced by Halpern [40] and later modified by Truszczynski and Schwarz [75]. In the second part we propose nonmonotonic modal logics **wK4f** and **wK4Df** as a formalisms for capturing the idea of minimal belief for two different types of agents with distinct properties on their belief sets. The first logic follows standard nonmonotonic reasoning techniques and as a result gives a belief set which does contain the formulas $\Box\varphi \to \varphi$ for an arbitrary formula φ. Belief set having this property hints on conscious belief i.e. agent believes that if he believes φ then φ is true. In the second approach we avoid the study of conscious belief. For this purpose we introduce a not standard method which uses weak-expansion as a generating tool for nonmonotonic inference. By applying this new technique to the logic **wK4Df** we arrive to formal analog of the concept of minimal (not conscious) belief of **KSD**-agent. This is the agent whose belief set satisfies all axioms of the logic **KS** and additionally the axiom (**D**) : $\neg\Box\bot$. In other words the agent who does not believe in falsum. The main results of this chapter are published in [69] and [70].

Chapter 5 discusses multi-agent version of the doxastic/epistemic logics. We just restrict attention to two agent case as far as all the results are carried to arbitrary finite number of agents. First we recall the epistemic modal logic $\mathbf{S4_2^C}$ [26, 7] with common knowledge operator. And in analogy we study the doxastic logic $\mathbf{K4_2^B}$ [71] of two agents extended with common belief operator. In light of [26, 7] and our previous work [69] several natural questions emerge that we answer here. In summary the main contributions of the chapter are:

1. We define a logic $\mathbf{K4_2^C}$ of common belief for normal agents and prove its completeness for a Kripke, relational semantics. We show it has the finite model property and the tree model property.

2. We study the topological semantics for $\mathbf{K4_2^C}$ and show completeness for intersection topologies. Specifically we show that $\mathbf{K4_2^C}$ is the modal logic of all T_D-intersection closed, bi-topological spaces with a derived set interpretation of

modalities.

3. Belief under the topological interpretation of $\mathbf{K4_2^C}$ is understood via colimits and common belief in terms of colimits in the intersection topology. From 2 we derive a topological condition for common belief in terms of colimits that is very similar to the corresponding condition that defines common knowledge in the modal μ-calculus and is discussed at some length in [7].

4. We show how the common knowledge logic $\mathbf{S4_2^C}$ can be embedded in $\mathbf{K4_2^C}$ via the splitting translation that maps $C_K p$ into $p \wedge C_B p$.

All the results of the current chapter can be found in [71]

The last chapter, Chapter 6 relates to the study of trust in settings of two **KS**-agents. For this we consider the modal logic $\mathbf{B}_T^2$ [20] with two unary **KS**-modalities and additional modality T which stands for the trust operator. We touch the completeness issue here also and prove that the logic is sound and complete with respect to the designed neighborhood semantics. The results of this chapter follow the ideas and techniques from [53] and are published in [20].

Chapter 2

Modal Logics of Knowledge and Belief

2.1 Modal Logics of Knowledge

2.1.1 Modal Logic S5

In the 1950s Saul Kripke influenced by Leibniz's philosophy, introduced a relational semantics for modal logic. In Kripke's work knowledge is treated as a necessity understood by Leibniz. More formally a proposition p is known if it is true in all possible worlds.

$$\Box p \text{ holds iff } p \text{ is true in every possible world.}$$

While different normal systems have been proposed to capture the logic of the knowledge operator, a special place is occupied by **S5**. There are several well-known reasons for this. For one, the logic **S5** satisfies some important principles of introspection that might be considered appropriate for an ideally rational agent [93], [39]. For another, it is based on frames whose alternativeness relation R is in a sense maximal: R is reflexive, symmetric and transitive, i.e. an equivalence relation. Moreover inference in **S5** is fully characterised by the class of *universal* Kripke models, where R is a universal relation (and frames form a so-called *cluster*); so every world counts as an epistemic alternative to a given one. [43], [92], [50], [82]. Third, almost all nonmonotonic systems for reasoning about knowledge make use of **S5**-like frames. Furthermore, it is closely related to the important concept of *stable* (belief) set first studied by [**?**] and [64], since these correspond to sets of formulas true on some universal **S5**-frame. Stable sets again play a prominent role in nonmonotonic modal and epistemic logics in general.

Definition 2.1.1 (Syntax of $\mathcal{L}$)**.** *Let* Π *be a countably infinite set of atomic propositions. Formulas of the language* $\mathcal{L}$ *are defined inductively over* Π *by the following grammar:*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box\varphi.$$

The logical symbols '$\top$' and '$\bot$', and additional operators such as '$\wedge$', '$\rightarrow$', '$\leftrightarrow$' and the dual modalities '$\Diamond$' and '$\Diamond^*$' are defined as usual, i.e.: $\top := p \vee \neg p$ for some atomic proposition p; $\bot := \neg\top$; $\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$; $\varphi \rightarrow \psi := \neg\varphi \vee \psi$; $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$; and for modalities we have: $\Diamond p := \neg\Box\neg p$.

The Kripke semantics for the language $\mathcal{L}$ is provided by relational structures where the relation is the equivalence relation. The background understanding of this is the following. A set formally represents the idea of worlds i.e. each point is a possible world and the relation is the dependency between these words. One can think of this in many different ways. For example if one world is related to another then one may say that the first world is preferred over the other, or another interpretation could be that the first world can be reached from the other or the first world is more plausible etc. Formal definition of Kripke frame follows.

Definition 2.1.2 (Semantics of $\mathcal{L}$)**.** *Let* $M = (W, R, V)$ *be a model i.e.* W *is a nonempty set,* $R \subseteq W \times W$ *is a relations and* $V : Prop \rightarrow 2^W$ *is a valuation function defined on* W*. The logical satisfaction relation '* $\Vdash$ *' is defined by induction on the structure of* $\mathcal{L}$*-formulas as follows: For all* $p \in \Pi$ *and all* $\varphi, \psi \in \mathcal{L}$*,*

- $M, w \Vdash p$ *iff* $w \in V(p)$*;*
- $M, w \Vdash \neg\varphi$ *iff* $M, w \nVdash \varphi$*;*
- $M, w \Vdash \varphi \vee \psi$ *iff* $M, w \Vdash \varphi$ *or* $M, w \Vdash \psi$*;*
- $M, w \Vdash \Box\psi$ *iff* wRv *implies* $M, v \Vdash \psi$*;*

We say that a $\mathcal{L}$-formula φ is *satisfiable* if there is a model M and a world w in M such that $M, w \Vdash \varphi$; φ is *valid in* M if $M, w \Vdash \varphi$ for all w in M; φ is *valid* in a frame $\mathcal{F} = (W, R)$ if φ is valid in all models based on the frame; and φ is *valid in a class of frames* C if φ is valid in every frame $\mathcal{F} \in C$.

Now if one takes the properties of a perfect reasoner (Definition 1.1.1) as axioms for the logic, the resulting system will be the modal logic **S5**. More strictly **S5** is the modal logic in a language $\mathcal{L}$ obtained by adding to propositional calculus the following set of axiom schemas:

- (K) $\Box(p \to q) \to (\Box p \to \Box q)$;
- (T) $\Box p \to p$;
- (4) $\Box p \to \Box\Box p$;
- (5) $\neg\Box p \to \Box\neg\Box p$;

and the inference rules for an arbitrary formulas $\varphi, \varphi_1, ..., \varphi_n, \psi_1, ..., \psi_n \in \mathcal{L}$

- modus ponens: $\frac{\vdash\varphi, \vdash\varphi\to\psi}{\vdash\psi}$; substitution $\frac{\vdash\varphi(\varphi_1,..,\varphi_n)}{\vdash\varphi(\psi_1,...,\psi_n)}$; necessitation $\frac{\vdash\varphi}{\vdash\Box\varphi}$;

Definition 2.1.3. *A pair* (C, E) *will be called a cluster if* C *is a set and* $E \subseteq C \times C$ *is a relation satisfying the following propertie:* $(\forall x, y)(xEy)$*. Such relation is called the universal relation. Cl will denote the class of all clusters and* Cl_{fin} *will denote the set of all finite clusters.*

Fact 2.1.4. *[17]* **S5** *is sound and complete w.r.t. the set* Cl_{fin}*.*

It is possible to find a proof of this fact in many different works [17], [11]. One elegant proof by algebraic logic is provided by Paul Halmos in [38].

2.1.2 Modal Logic S4

One of the most well know systems in modal logic with a lot of surprising connections to other branches of mathematics such as intuitsionistic calculus, proof theory, topology etc. is modal logic **S4**. It turns out that **S4** has also relation with epistemic logic. If treated modality as a knowledge operator the axiom **4** represents positive introspection in another words we are dealing with a normal agent [81] who has knowledge about his own knowledge. Modal logic **S4** was first introduced by Lewis [52]. Later it has been studied by several authors. Godel gave a faithful embedding of the intuitionistic calculus into the modal logic **S4** [34], McKinsey and Tarski [61] proved topological completeness of the logic **S4** which is one of the most surprising result in modal logic, Esakia [23] gave algebraic study of the logic and developed duality theory for **S4**, Fagin, Halpern, Moses and Vardi [26] and Benthem and Sarenac [7] dealed with dynamics of **S4** modalities in many agents case, recently Bezhanishvili and Gehrke [9] gave a new proof of topological completeness originally proved by Tarski and McKinsey. There are many other references related with the modal logic. We do not list all of them here. A sufficient number of the related results can be found in the book by Chagrov and Zakharyaschev [16]. Below we list some known facts around this modal system.

S4 is the modal logic in a language $\mathcal{L}$ obtained by adding to propositional calculus the following three axiom schemas:

- (K) $\Box(p \to q) \to (\Box p \to \Box q)$;
- (T) $\Box p \to p$;
- (4) $\Box p \to \Box\Box p$;

As the reader can see the system **S4** can be obtained by dropping axiom **5** in **S5**. The inference rules are standard: modus ponens, substitution and necessitation. Kripke semantics for **S4** is provided by preorders.

Definition 2.1.5. *A pair (W, R) is called a preorder if W is a set and $R \subseteq W \times W$ is a reflexive and transitive relation on W i.e. we have:* $(\forall x)(xRx)$ *and* $(\forall x, y, z)(xRy \wedge$

$yRz \to xRz$). PO will denote the class of all preorders and PO_{fin} will denote the set of all finite preorders.

Fact 2.1.6. *[16]* **S4** *is sound and complete w.r.t. the set PO_{fin}.*

As we have already mentioned there are more interesting results concerning modal logic **S4** in relation with topology. We will return to this topic in forthcoming chapters.

2.1.3 Modal Logic S4F

We will deal with one more logic related to knowledge. This is a normal modal logic **S4F**. The semantics of **S4F**. was first studied by [76]. However in [75, 89] Schwarz and Truszczyński proposed a new approach to minimal knowledge and suggested that this concept is precisely captured by non-monotonic **S4F**. In [89] it was shown that non-monotonic **S4F** captures, under some intuitive encodings, several important approaches to knowledge representation. They include disjunctive logic programming under answer set semantics [32], (disjunctive) default logic [72], [33], the logic of grounded knowledge [55], the logic of minimal belief and negation as failure [54] and the logic of minimal knowledge and belief [89]. Recently, [87] and [13] have revived the study of **S4F** in the context of a general approach to default reasoning. The logic **S4F** is an extension of modal logic **S4** but it is under the logic **S5** i.e. we have the following inclusion: **S4** $\subset$ **S4F** $\subset$ **S5**. Although **S4F** extends the logic **S4** the axiom **F** can not be given any convincing epistemic reading and hence **S4F** in not considered as an epistemic logic. The situation totally changes when we look at nonmonotonic version of **S4F**. In Chapter 4 we will see that it is extremely important logic for modeling concept of minimal knowledge. In this subsection we will just give a very brief introduction to **S4F** and mention important facts around this logic.

S4F is the modal logic in a language $\mathcal{L}$ obtained by adding the following axiom schema to the modal logic **S4** (2.1.2):

- (F) $(\Diamond p \wedge \Diamond \Box q) \to \Box(\Diamond p \vee q)$.

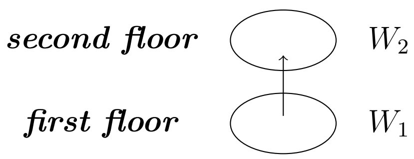

Figure 2.1

It is possible to prove axiom **F** in modal logic **S5**. On the semantic level this fact can be seen much easier. The inference rules are standard: modus ponens, substitution and necessitation. Kripke semantics for **S4F** is provided by one step reflexive frames.

Definition 2.1.7. *A pair* (W, R) *is called a reflexive one step frame if* W *is a nonempty set,* $R \subseteq W \times W$ *is a reflexive relation which satisfies the following first order condition:* $(\forall x, y, z)((xRy \wedge xRz) \to (yRx \vee yRz))$. *OSR will denote the class of all reflexive, one-step relational structures and* OSR_{fin} *will denote the set of all finite, reflexive, one-step Kripke frames.*

Fact 2.1.8. *[76]* **S4F** *is sound and complete w.r.t. the set* OSR_{fin}.

Rooted frames for **S4F** have the form depicted in Figure 2.1 where W_1 and W_2 are clusters, all points are reflexive and every point in W_2 is accessible from every point in W_1. We call W_1, W_2 respectively the *first* and the *second floor* of the model. The former may be empty but the latter not. In these frames the accessibility relation is of course a preorder. It is obvious that finite rooted frames can be uniquely represented by pairs of natural numbers (r_1, r_2) where r_1 stands for the number of points on the first floor while r_2 is a number of points on the second floor.

2.2 Modal Logics of Belief

As a doxastic counterpart to epistemic logic based on **S5** we study the modal logic **KS** that can be viewed as an approach to modelling a kind of objective and fair belief. Modern epistemic and doxastic logic has its roots in the work of [46] which gave rise to a substantial and sustained research programme making significant contributions to philosophy, logic, AI and other fields. Though several specific features of Hintikka's approach to knowledge and belief have, over the years, been questioned and modified by different authors for different purposes, some of the core ideas have stood the test of time and are still relevant today.

Assuming some basic (if idealising) conditions about knowledge, and simplifying by considering a single agent in each system, epistemic/doxastic logics are on this view structurally similar to normal modal logics with a necessity operator $\Box$ interpreted on Kripke frames with a single binary alternativeness or accessibility relation R on worlds. Again, despite many variations, mutations and extensions, this feature survives in many areas concerned with reasoning about knowledge and belief.

2.2.1 Modal Logic KD45

The modal logic **KD45** is probably one of the most studied systems from doxastic perspective. The **KD45**-agents beliefs posses both positive and negative introspection. One of the earliest references where the system **KD45** is studied can be found in is Hintikka [46]; Segerberg [76] studied it as the modal system **DE4**; Nagle [65] and Nagle and Thomason [66] investigated it as a normal extension of the modal system **K5**; Halpern and Moses [40, 41] discussed the completeness and complexity issues for **KD45**, algebraic study of the system were provided by Tokarz [86] and by Bezhanishvili [10]. Also van der Hoek [92], Meyer and van der Hoek [63], and Meyer [62] could be useful references for interested reader.

Modal logic **KD45** is given by the following set of axiom s chemas:

- (K) $\Box(p \to q) \to (\Box p \to \Box q)$;
- (D) $\Box p \to \Diamond p$;
- (4) $\Box p \to \Box\Box p$;
- (5) $\neg\Box p \to \Box\neg\Box p$;

and the inference rules are standard.

Kripke semantics is provided by transitive and serial Kripke frames.

Definition 2.2.1. *A relation $R \subseteq W \times W$ is called a serial relation if the following first order condition takes place:* $(\forall x, y, z)((xRy \wedge xRz) \rightarrow yRz)$.

A Kripke frame (W, R) is serial and transitive if the relation R is both transitive and serial relation. The class of serial and transitive frames will be denoted by ST and the set of finite serial and transitive frames will be denoted by ST_{fin}

Fact 2.2.2. *[40]* **KD45** *is sound and complete w.r.t. the set* ST_{fin}.

One positive point of the system **KD45** can be described as follows. Given an arbitrary rooted **KD45**-model $\mathcal{M}$ i.e. serial, transitive and rooted frame with a fixed valuation one can obtain its belief set. This is the set $Bel_{\mathcal{M}} = \{\varphi | \mathcal{M} \Vdash \Box\varphi\}$. It is not hard to verify that $Bel_{\mathcal{M}}$ is a stable set i.e. if $\varphi \in Bel_{\mathcal{M}}$ then $\Box\varphi \in Bel_{\mathcal{M}}$ and if $\varphi \notin Bel_{\mathcal{M}}$ then $\neg\Box\varphi \in Bel_{\mathcal{M}}$. Moreover $Bel_{\mathcal{M}}$ is closed under propositional consequence. But despite these nice properties **KD45** also has a drawback which could be seen when observing the relation with the epistemic logic **S5**. Unfortunately there is no interesting connection linking these two logics together. More precisely coming up with a reasonable interrelation between the modality of **KD45** and the modality of **S5** linking these two systems into **doxepi** formalism seems to be a problematic issue. This is the main motivation for introducing alternative system which we discuss in the next subsection.

2.2.2 Modal Logic KS

Our aim here is to investigate a doxastic modal logic that we may call **KS** as an alternative to systems like **KD45**. The motivation comes from two main aspects, each of them forming one main part of our study.

The first aspect concerns monotonic doxastic logic and has several features that we now describe briefly. One leading idea is to approach doxastic reasoning by looking for a system having strong analogies to **S5**, but of course without the **T** axiom. Let us consider then an analogy on the semantical level. An obvious way to avoid principle **T** is to consider irreflexive alternativeness relations and to interpret belief in terms of *other* possible worlds (than the one under consideration). So in place of

(**S5**) knowledge where we may assume that *every world is an epistemic alternative*, we would have for belief: *every* **other** *world is a doxastic alternative.* We maintain the analogy to knowledge in that we refer not just to some other worlds but to all of them.

Returning to the models for belief, let us consider further properties. One is simple: if we interpret belief in terms of other worlds we should be sure that at least some such other worlds exist. So in addition to the set W of worlds being non-empty as usual, it should contain at least two elements. Another issue concerns the transitivity of the R relation. While symmetry seems desirable and can safely be maintained given our other concerns, in its presence transitivity may restore reflexivity. Let $R \subseteq W \times W$ be symmetric and transitive and suppose x is not a dead-end (a world seeing no other) so that xRy for some y. Then by symmetry yRx and so by transitivity xRx. To avoid this we can weaken the property of transitivity by just the right amount.

Definition 2.2.3. *We say that a relation $R \subseteq W \times W$ is weakly-transitive if*

$$(\forall x, y, z)(xRy \wedge yRz \wedge x \neq z \Rightarrow xRz).$$

So our doxastic frames should be not transitive like **S5**-frames but weakly-transitive (we will also avoid dead-ends).

The second aspect of our motivation for this study concerns nonmonotonic reasoning and is also by analogy to **S5**. Nonmonotonic modal logics, also used for reasoning about knowledge, are based on a concept of (theory) expansion. In the standard cases such expansions are stable sets and can be captured by a notion of *minimal* model, again a special kind of **S5**-model that can be thought to represent a concept of minimal knowledge. Evidently each world in such a model satisfies the strong **T** axiom, seemingly appropriate for knowledge but less so for belief. In fact, while **KD45** is usually considered to be a doxastic modal logic, its nonmonotonic variant is precisely autoepistemic logic, ie a system for reasoning about knowledge.

To obtain a candidate logic for minimal belief, our strategy is to put **KS** models in place of **S5** models and define a new concept of minimal model. Equivalently, as we show in Section 4, we can define a weaker concept of expansion and show that this corresponds to the new definition of minimal model. However, we first present the main results about **KS** in Section 2 and then turn to the concept of minimal belief in Sections 3 and 4.

The name **KS** does not follow the typical naming convention in modal logic, i.e. it is not modal logic **K** plus axiom **S**. It would rather be called **wK4B** if we followed this style. The system **KS** was named after Krister Segerberg who first introduced this logic in early 1980s as a modal logic of *some other time* [77]. This logic has been extensively studied by several authors, [24, 48]. Later it was rediscovered as a modal logic of *inequality*. In some works it is also known as **DL** [36, 18]. In this section we will see that **KS** can also be seen as a modal logic of inequality.

Definition 2.2.4. *The normal modal logic* **KS** *is defined in a standard modal language (see Definition 2.1.1) with the following set of axiom schemas:*

All propositional tautologies,

K : $\Box(p \to q) \to (\Box p \to \Box q)$,

w4 : $\Box p \wedge p \to \Box\Box p$,

B : $p \to \Box\Diamond p$.

The rules of inference are: Modus Ponens, Substitution and Necessitation.

In this section we will also study one simple extension of the modal logic **KS** by axiom **D** : $\Box p \to \Diamond p$. The axiom **D** can be equivalently formulated as $\neg\Box\bot$ the doxastic reading of which is the following: *'The agent does not believe falsum'*. The modal logic **KSD** is very similar to the modal logic **KS** therefore we will omit the details and just mention the points where the difference is significant.

Let us briefly comment on differences with respect to **KD45**. Obviously the first two axioms are shared by both logics. We have already remarked on how, given our semantic analogy with **S5**, we expect axiom **4** to be weakened in the case of **KS**. In the case of **B** versus **5** there is no direct relation: the latter is expressed by $\Diamond p \to \Box\Diamond p$. However it is easy to see how **B** arises in virtue of our semantics of other doxastic alternative worlds. To establish $\Box\Diamond p$ at w we need to verify $\Diamond p$ at all the alternative worlds. Since each is accessible from w, truth of p in w will suffice for this, while the truth of $\Diamond p$ in w will not. So the implication $p \to \Box\Diamond p$ must be valid.

It is easy to show that if we add the axiom **T** : $\Box p \to p$ to **KS** we obtain the classical epistemic system **S5**. To use an expression of [80] this means that if the **KS**-reasoner is *accurate* (never believes any false proposition) then his beliefs coincide with his knowledge.

From these rather informal sematic considerations we pass now to the formal semantics.

Kripke semantics for KS

Kripke semantics for **KS** is provided by weakly-transitive and symmetric Kripke frames. Weakly-transitive was described in Definition 2.2.3. Clearly, every transitive relation is weakly-transitive as well. Moreover it is immediate to see that weakly-transitive relations differ from transitive ones just by the occurrence of irreflexive points inside the subsets where every two distinct points are related to each other. Such sets will be called *weak-clusters*.

Definition 2.2.5. *A relational structure* (W, R) *is called a weak-cluster if for an arbitrary two points* $w, v \in W$ *if* $w \neq v$ *then* wRv*.*

The reader can notice that every cluster 2.1.3 is a weak-cluster. Moreover weak-clusters are obtained by dropping the reflexivity condition from clusters. As one can see the weak-cluster in the Figure 2.2 is weakly-transitive, but not transitive.

In the study of modal logic the class of rooted frames plays a central role. Recall that a weakly-transitive frame (W, R) is rooted if it contains a point $w \in W$, which can see all other points in W. That is $R(w) \supseteq W - \{w\}$, where $R(w)$ is the set of all successors of w. In some textbooks [11] rooted frames are called *point-generated* frames. The general definition of rooted frames for arbitrary relation is a bit different and involves the notion of generated subframe but as far as we just deal with the classes of frames which are weakly transitive, we will stick to the above special case

The figure represents the diagrammatic view of a Kripke structure, where irreflexive points are coloured grey and reflexive ones are uncoloured. Arrows represent the relation between two distinct points. So, yRx and xRy, but we do not have yRy, which contradicts transitivity, but not weak transitivity as $y = y$.

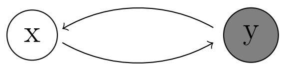

Figure 2.2

of general definition. The following proposition shows that the class of all rooted, weakly-transitive and symmetric frames is in one to one correspondence with the class of all weak-clusters.

Proposition 2.2.6. *A frame* (W, R) *is rooted, weakly-transitive and symmetric iff* (W, R) *is a weak-cluster.*

Proof. It is immediate that every weak-cluster is a rooted, weakly-transitive and symmetric frame. For the other direction let (W, R) be a rooted, weakly-transitive and symmetric frame. Let $w \in W$ be the root. Take two arbitrary distinct points $x, y \in W$. As w is the root, we have: wRx and wRy. Because of symmetry we get xRw. Now as R is weakly-transitive, from $xRw \wedge wRy$ and $x \neq y$ we get xRy. Hence R is a weak-cluster. $\square$

So far we have defined the modal logic **KS** syntactically and given a definition of weak-cluster relation. The following theorem links these two notions:

Theorem 2.2.7. *[24] The modal logic* **KS** *is sound and complete w.r.t. the class of all* ***finite, irreflexive*** *weak-clusters.*

Theorem 2.2.8. *The modal logic* **KSD** *is sound and complete w.r.t. the class of all* ***finite, irreflexive*** *weak-cluster Kripke frames which contain at least two points.*

We do not give the proof here as it is exactly analogous to the prove of Theorem 2.2.7. Now it is easy to see why both modal logics **KS** and **KSD** are modal logics of inequality. As the reader can easily check the interpretation of $\Box$ in irreflexive weak-clusters boils down to the following:

$$w \Vdash \Box\varphi \text{ iff } (\forall v)(w \neq v \Rightarrow v \Vdash \varphi).$$

Consequently our logic is indeed complete for the frames where each world different from the world of evaluation is a doxastic alternative. Moreover it suffices that the set of worlds is finite.

2.2.3 The Class of Finite Weak-cluster Relations and Their Bounded Morphisms.

In the previous section we have seen that the class of finite weak-cluster relations fully captures information about the modal logic **KS**. In this section we characterize finite weak-clusters and their bounded morphisms in terms of pairs of natural numbers. Let $\mathbb{N}$ be the set of natural numbers. The following theorem states that the set of finite weak-clusters can be viewed as the set $\mathcal{N}^2 \equiv \mathbb{N} \times \mathbb{N} - \{(0,0)\}$. Let wCL denote the class of all finite weak-cluster relations considered up to isomorphism.

Theorem 2.2.9. *There is a one-to-one correspondence between the set wCL and the set $\mathcal{N}^2$.*

Proof. For every finite weak-cluster we apart two invariants, the number i of irreflexive points and the number r of reflexive ones see Figure 2.3. The point is that the pair (i, r) represents the frame uniquely up to isomorphism.

Let us describe how we construct the function from wCL to $\mathcal{N}^2$. With every frame $(W, R) \in wCL$ we associate the pair (i, r), where i is a number of irreflexive points in W and r is a number of reflexive ones. We will call the pair (i, r) the ***characterizer***

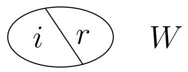

Figure 2.3

of the frame (W, R). It is clear that the correspondence described above defines a function from the set wCL to the set $\mathcal{N}^2$. Let us denote this function by Ch.

Ch is injective. Take any two distinct finite, weak-clusters (W, R) and (W', R'). That they are distinct in wCL means that they are non-isomorphic i.e. either $|W| \neq |W'|$ or $R \not\simeq R'$. In the first case it is immediate that $Ch(W, R) \neq Ch(W', R')$ as far as $|W| = i + r$. In the second case we have that the number of reflexive (irreflexive) points in $|W|$ differs from the number of reflexive (irreflexive) points in $|W'|$. This means that $i \neq i'$ and again $Ch(W, R) \neq Ch(W', R')$.

It is straightforward that if none of these cases above occur i.e. $|W| = |W'|$ and $|\{w \in W | wRw\}| = |\{w' \in W' | w'R'w'\}|$ then (W, R) is isomorphic to $(W'R')$ and hence $(W, R) = (W', R')$ in wCL.

Ch is surjective. Take any pair $(i, r) \in \mathcal{N}^2$. Let us show that the pre-image $Ch^{-1}((i, r))$ is not empty. Take the frame (W, R), where $|W| = i + r$ and W contains i irreflexive and r reflexive points. Then by definition of Ch, we have that $Ch(W, R) = (i, r)$. □

Definition 2.2.10. *A function f between two frames (W, R) and (W', R') is called a bounded morphism if the following two conditions are satisfied:*
(1) $wRv \Rightarrow f(w)R'f(v)$,
(2) $f(w)R'v' \Rightarrow (\exists v \in W)(wRv \wedge f(v) = v')$.

We will say that a frame (W', R') is a bounded morphic image of a frame (W, R) if there is a surjective bounded morphism f defined on a set W with the range in W'.

In the following theorem we characterize the bounded morphisms between two finite weak-cluster relations in terms of conditions on the pairs of natural numbers.

Theorem 2.2.11. *The finite weak-cluster (W', R') with the characterizer (i', r') is a bounded morphic image of the finite weak-cluster (W, R) with the characterizer (i, r)*

iff the following conditions are satisfied:

$r' = 0 \Rightarrow (i, r) = (i', r'),$
$i \geq i',$
$2 \times (r' - r) \leq i - i'.$

Note that the operation minus is defined within the natural numbers i.e. $n - m = 0$ *if* $m > n$*.*

Proof. For the left to right direction assume $f : (W, R) \to (W', R')$ is a surjective bounded morphism. This means that $i + r \geq i' + r'$, as far as f is a surjection. First let us state some general observations which will help us in proving the theorem.

- ***for every irreflexive point*** $w' \in W'$***, we have that*** $f^{-1}(w')$ ***is one irreflexive point.*** Assume not. Then either $f^{-1}(w')$ contains some reflexive point $w \in W$, or it contains at least two irreflexive points $u, v \in W$. In the first case we have wRw while not $f(w)R'f(w)$, so we get a contradiction. In the second case we have $uRv \wedge vRu$ while not $f(v)R'f(u)$ and again this contradicts to f being bounded morphism.

Now we are ready to begin the proof of the theorem. Let us check that all conditions are satisfied.

case 1 Assume $r' = 0$ but $(i, r) \neq (i', r')$. So either $r \neq 0$ or $i \neq i'$. In both cases we get a contradiction by above observation, as reflexive points can not be mapped to irreflexive ones and also two irreflexive point can not be mapped to one irreflexive point.

case 2 Assume $i < i'$. Then there is at least one point $v' \in W'$ such that $f^{-1}(v') = \varnothing$. Because there is not enough irreflexive points in W to cover all irreflexive points in W'. And we know (by above remark) that we can not map reflexive points to irreflexive ones. So we get a contradiction.

case 3 Assume $2 \times (r' - r) > i - i'$. This means that $r' > r$. So there are $r' - r$ reflexive points in W' with the pre-image not containing reflexive point. But then there is at least one reflexive point $w' \in W'$ such that $f^{-1}(w')$ contains less than 2 irreflexive points. This is because by assumption there is not enough pairs of irreflexive points in W to be enough for all reflexive points with pre-image not containing reflexive ones. But this gives a contradiction because either f is not surjective (in case $f^{-1}(w') = \varnothing$) or f is not a bounded morphism (in case $f^{-1}(w') = v$ with v irreflexive).

Now let us prove the converse direction. Let us enumerate points in W in the following way: Let $w_1, ...w_r$ be the reflexive points and $v_1, ...v_i$ irreflexive points. Let us use the same enumeration for points in W' with the difference that we add $'$ to every point. So for example w'_1 is the reflexive and v'_2 is the irreflexive point in W'.

In case $r' = 0$ we know that $(i, r) = (i', r')$ and we can take f to be bijection mapping w_i to w'_i.

In case $r' \neq 0$ we distinguish two subcases.

case 1 When $r > r'$. Let us define $f : W \to W'$ in the following way:

$f(v_j) = v'_j$ for $j \in \{1, .., i'_1\}$,
$f(w_j) = w'_j$ for $j \in \{1, .., r'_1 - 1\}$,
$f(v_{i'+1}) = f(v_{i'+2}) = ... = f(v_i) = f(w_{r'+1}) = .. = f(w_r) = w'_r$.

case 2 When $r \leq r'$. Let us define $f : W \to W'$ in the following way:

$f(v_j) = v'_j$ for $j \in \{1, .., i'\}$,
$f(w_j) = w'_j$ for $j \in \{1, .., r\}$,
$f(v_{i'+2k-1}) = f(v_{i'+2k}) = w'_{r+k}$ for $k \in \{1, .., r' - r - 1\}$,
$f(v_{i'+2(r'-r)-1}) = f(v_{i'+2(r'-r)}) = .. = f(v_i) = w'_r$.

In words we send each reflexive point $w_j \in W$ to the reflexive point $w'_j \in W'$ and each irreflexive point $v_j \in W$ to the irreflexive point $v'_j \in W'$. As far as we have that $i \geq i'$ and $r \leq r'$ there may be left some irreflexive points in W on which we have not yet defined f and also some reflexive points in W' which have no pre-image, so we associate to every pair of such irreflexive points one reflexive point which has

$$
\begin{tikzcd}
 & (0,1) \arrow[r] \arrow[d] \arrow[ddl] \arrow[dddl] & (0,2) \arrow[r] \arrow[d] \arrow[dddl] & (0,3) \arrow[r] \arrow[d] & \cdots \\
(1,0) & (1,1) \arrow[r] \arrow[d] \arrow[ddl] & (1,2) \arrow[r] \arrow[d] \arrow[ddl] & (1,3) \arrow[r] \arrow[d] & \cdots \\
(2,0) & (2,1) \arrow[r] \arrow[d] & (2,2) \arrow[r] \arrow[d] & \ddots & \\
(3,0) & (3,1) \arrow[r] \arrow[d] & \ddots & & \\
\vdots & \vdots & & &
\end{tikzcd}
$$

Figure 2.4

no pre-image. We go on with this process until there is left only one reflexive point without pre-image (we know that such exists by the condition $r' \neq 0$) and associate to it all the rest of irreflexive points on which f was not defined. The condition $2 \times (r - r') \leq i' - i$ guarantees that there are at least two such irreflexive point left.

It is easy to check that f defined in the following way is indeed a bounded morphism. $\square$

We can impose the partial order $\preceq$ on wCL in the following way: We will say that $(W, R) \preceq (W', R')$ iff (W, R) is a bounded morphic image of (W', R'). Figure 2.4 represents the poset $(wCL, \preceq)$ where instead of kripke frames stand their characterizers and $(i, r) \to (i', r')$ stands for $Ch^{-1}(i, r) \preceq Ch^{-1}(i', r')$. Partially ordered set wCL plays a crucial role in the study of logics over **KS**.

2.2.4 Modal Logic K4

In logics for knowledge representation and reasoning, the study of epistemic and doxastic properties of agents with certain, intuitively acceptable, restrictions on their knowledge and belief is a well-developed area. Smullyan [81] discusses various types of agents based on properties of belief. In his terminology, an agent whose belief satisfies the modal axiom $(4) : \Box p \to \Box\Box p$, translated as 'If the agent believes p, then he believes that he believes p', is called a *normal agent.* **K4** is the modal logic which formalizes the belief behavior of normal agents.

Modal logic **K4** is given by adding axiom **4** to the basic modal logic **K** and the inference rules are standard. Kripke semantics is provided by transitive Kripke frames. There are plenty of results in relation with this logic [15] although we do not go into details and just state the completeness result which we make use of in forthcoming chapters.

Fact 2.2.12. *[15]* **K4** *is sound and complete w.r.t. the class of all finite transitive frames.*

2.2.5 Modal Logics wK4f and wKD4f

Following the Tarski/McKinsey suggestion to treat modality as the derivative of the topological space [60], Esakia in [22, 24] introduced $wK4$ as the modal logic of all topological spaces, with the desired (derivative operator) interpretation of the modal $\Diamond$. **wK4f** is a normal modal logic obtained by adding the axiom weak-F to the modal logic **wK4**. **wK4f** is a weaker logic than **S4F** discussed in Segerberg [76] since it doesn't satisfy the axiom T. However since the frames of **wK4f** and **S4F** are closely related, some results about **wK4f** carry over to **S4F**.

Syntax. The normal modal logic **wK4f** is defined in a basic modal language with an infinite set $Prop$ of propositional letters and connectives $\vee, \wedge, \neg, \Box$. The axioms are all classical tautologies plus the axioms listed below. Rules of inference are: modus

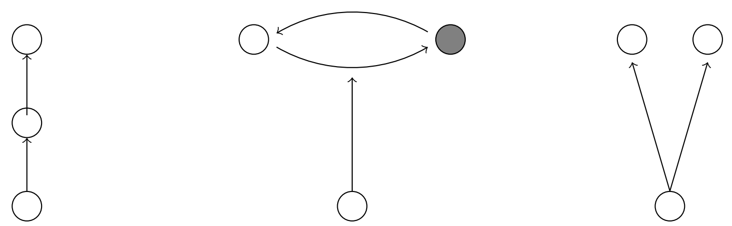

Figure 2.5

ponens, substitution and necessitation.

$$K: \Box(p \to q) \to (\Box p \to \Box q)$$
$$w4: \Box p \wedge p \to \Box\Box p$$
$$f: p \wedge \Diamond(q \wedge \Box\neg p) \to \Box(q \vee \Diamond q)$$

Semantics. Kripke semantics for the modal logic **wK4f** is provided by frames which have in a weak sense height at most two and which do not allow forking. This is made precise in the following two definition.

Definition 2.2.13. *We will say that a relation $R \subseteq W \times W$ is a one-step relation if the following two conditions are satisfied:*

1)$(\forall x, y, z)((xRy \wedge yRz) \Rightarrow (yRx \vee zRy))$,

2)$(\forall x, y, z)((xRy \wedge \neg(yRx) \wedge xRz \wedge y \neq z) \Rightarrow zRy)$.

As the reader can see the first condition restricts the "strict" height of the frame to two. Where informally by "strict" we mean that the steps are not counted within a cluster. So for example we cannot have the situation on the left hand side of the Figure 2.5, while we can have the situations in the middle and on the right hand side.

The second condition is more complicated. Nevertheless it is not too hard to verify that it restricts the "strict" width of the frame to one. So again we cannot have for example the frame on the right hand side of Figure 2.5, while the frame in the middle and on the left are allowed. Here we cheat slightly as indeed the frame is not allowed to fork as in the picture on the right, but condition 2) does not restrict the reversed fork i.e. frame with two points on the bottom and one on the top. So "strict" width does not effect points on the bottom.

Let wOS denote the class of all one-step and weakly-transitive Kripke frames. The following theorem links the logic **wK4f** with the class wOS. The proof uses standard modal logic completeness techniques, so we will not enter into all the details.

Theorem 2.2.14. *The modal logic* **wK4f** *is sound and strongly complete w.r.t. the class wOS.*

We give the soundness proof only for the axiom f. For the proofs for other axioms the reader may consult [24].

Proof. Take an arbitrary, weakly-transitive, one-step model (W, R, V). Assume at some point $w \in W$ it holds that $w \Vdash p \wedge \Diamond(q \wedge \Box\neg p)$. This implies that $w \Vdash p$ and there exists w' such that wRw', $w' \Vdash q$ and it is not the case that $w'Rw$ (as far as $w' \Vdash \Box\neg p$). Now for an arbitrary v with wRv and $v \neq w'$, by the second condition of definition 2.2.13, we have that vRw', which implies that $v \Vdash \Diamond q$ and hence $w \Vdash \Box(q \vee \Diamond q)$.

For strong completeness assume $I \nvdash \varphi$. We will construct the one-step and weakly-transitive model $M^c = \{W^c, R^c, V^c\}$ such that $M^c \Vdash I$ and $M^c \nVdash \varphi$. For M^c we take a standard canonical model ie. $W^c = \{\Gamma | \Gamma \vdash I \text{ and } \Gamma \text{ is a maximal consistent set}\}$. The relation is defined in a standard way $\Gamma R^c \Gamma'$ iff $(\forall\alpha)(\Box\alpha \in \Gamma \Rightarrow \alpha \in \Gamma')$ and $V^c(p, \Gamma) = 1$ iff $p \in \Gamma$.

Lemma 2.2.15 (Truth Lemma)**.** *For any formula φ we have $M^c, \Gamma \Vdash \varphi$ iff $\varphi \in \Gamma$.*

The proof follows a standard pattern found in modal logic textbooks. As $I \nvdash \varphi$ we have that $I \cup \{\neg\varphi\}$ is consistent, so there exists a maximally consistent set $\Gamma_{\neg\varphi}$ containing $I \cup \{\neg\varphi\}$ and by the truth lemma this means that $M^c, \Gamma_{\neg\varphi} \Vdash \neg\varphi$ which completes the proof. The main thing to be checked is that $M^c \in wOS$. For weak transitivity of the relation R^c the reader may consult [24]. Let us show that R^c is a one-step relation.

First let us show that R^c satisfies the first condition of definition 2.2.13. For the contradiction assume there exist three distinct points $\Gamma, \Gamma', \Gamma'' \in W^c$ such that $\Gamma R^c \Gamma' \wedge \Gamma' R^c \Gamma''$ and $\neg(\Gamma' R^c \Gamma) \wedge \neg(\Gamma'' R^c \Gamma')$. This means that there is a formula ψ such that $\Box\psi \in \Gamma'$ and $\neg\psi \in \Gamma$ and there is a formula φ such that $\Box\varphi \in \Gamma''$ and

$\neg\varphi \in \Gamma'$ and as $\Gamma' \neq \Gamma''$ there exists a formula γ with $\gamma \in \Gamma'$ and $\neg\gamma \in \Gamma''$. From these assumptions we have that $(\neg\varphi \wedge \gamma) \wedge \Box\neg\neg\psi \in \Gamma'$. Now as $\Gamma R^c \Gamma'$ we have that $\Diamond((\neg\varphi \wedge \gamma) \wedge \Box\neg\neg\psi) \in \Gamma$ and as $\neg\psi \in \Gamma$ we have that $\Diamond((\neg\varphi \wedge \gamma) \wedge \Box\neg\neg\psi) \wedge \neg\psi \in \Gamma$. Applying axiom f (with $p = \neg\psi, q = \neg\varphi \wedge \gamma$) we get that $\Box((\neg\varphi \wedge \gamma) \vee \Diamond(\neg\varphi \wedge \gamma)) \in \Gamma$. Hence as $\Gamma R^c \Gamma''$ (because of weak transitivity) we have that $(\neg\varphi \wedge \gamma) \vee \Diamond(\neg\varphi \wedge \gamma) \in \Gamma''$. On the other hand $\Diamond(\neg\varphi \wedge \gamma) \notin \Gamma''$ since $\Box\varphi \in \Gamma''$ and $\neg\varphi \wedge \gamma \notin \Gamma''$ because $\neg\gamma \in \Gamma''$. Hence we get a contradiction.

Now let us show that R^c satisfies the second condition of definition 2.2.13. Again the proof is by contradiction. Assume there exist three distinct points $\Gamma, \Gamma', \Gamma'' \in W^c$ such that $\Gamma R^c \Gamma' \wedge \Gamma R^c \Gamma'' \wedge \neg(\Gamma' R^c \Gamma)$ and $\neg(\Gamma'' R^c \Gamma')$. This means that there is a formula ψ such that $\Box\psi \in \Gamma'$ and $\neg\psi \in \Gamma$ and there is a formula φ such that $\Box\varphi \in \Gamma''$ and $\neg\varphi \in \Gamma'$ and as $\Gamma' \neq \Gamma''$ there exists a formula γ with $\gamma \in \Gamma'$ and $\neg\gamma \in \Gamma''$. From these assumptions we have that $(\neg\varphi \wedge \gamma) \wedge \Box\neg\neg\psi \in \Gamma'$. Now as $\Gamma R^c \Gamma'$ we have that $\Diamond((\neg\varphi \wedge \gamma) \wedge \Box\neg\neg\psi) \in \Gamma$ and as $\neg\psi \in \Gamma$ we have that $\Diamond((\neg\varphi \wedge \gamma) \wedge \Box\neg\neg\psi) \wedge \neg\psi \in \Gamma$. Applying axiom f we get that $\Box((\neg\varphi \wedge \gamma) \vee \Diamond(\neg\varphi \wedge \gamma)) \in \Gamma$. Hence as $\Gamma R^c \Gamma''$ we have that $(\neg\varphi \wedge \gamma) \vee \Diamond(\neg\varphi \wedge \gamma) \in \Gamma''$. On the other hand $\Diamond(\neg\varphi \wedge \gamma) \notin \Gamma''$ since $\Box\varphi \in \Gamma''$. Nor can we have $\neg\varphi \wedge \gamma \in \Gamma''$ because $\neg\gamma \in \Gamma''$. Hence we get a contradiction. $\Box$

Recently it was discovered that standard filtration technique does not work for logics over **wK4**. The reason for this is that the weak-transitive closure of a minimal filtration is not filtration any more. To fix this the new technique has been developed by Bezhanishvili, Esakia and Gabelaia [29]. Later Bezhanishvili and Jibladze [30] gave the algebraic generalisation of the technique and they proved that all Kripke complete logics over **wK4** which have subframe property do have a finite model property. As a corollary of the results from [30] and theorem 2.2.21 we get the following theorem:

Theorem 2.2.16. *The modal logic* **wK4f** *is sound and complete wrt the class of all finite one-step and weakly-transitive Kripke frames.*

We do not give the proof of this theorem mainly because it involves material from algebraic logic and duality theory which is far beyond the scope of current thesis. Although we point out that to show that the class of one-step, weakly transitive frames do have a subframe property is quite simple and the reader familiar with the material from [30] can easily check this.

2.2.6 The Class of Finite, Rooted, Weakly-Transitive and One-Step Kripke Frames.

We saw from Theorem 2.2.16 that the class of finite, weakly-transitive and one-step Kripke frames fully captures the modal logic **wK4f**. From general theorems in modal logic it is well known that this class can be reduced to a smaller class of frames which are rooted so that the completeness theorem still holds. In this section we characterise finite, rooted, weakly-transitive and one-step Kripke frames in terms of quadruples of natural numbers.

Definition 2.2.17. *The upper cone (or an upset) of a set $A \subseteq W$ in a weakly-transitive Kripke frame (W, R) is defined as a set $R(A) = \bigcup\{y : x \in A \& xRy\} \cup A$.*

Observe that the general definition of upper cone in an arbitrary Kripke frame is given in terms of the reflexive, transitive closure of a relation, while Definition 2.2.17 is a simplified version for the particular case of weakly transitive frames. In section 2.2.2 we gave a definition of rooted frame. The following theorem gives a characterization of finite, rooted, one-step, weakly-transitive frames.

Let N^4 be the set of all quadruples of natural numbers and let $\mathcal{N}^4 = N^4 - \{(n, m, 0, 0) | n, m \in N\}$. The following theorem states that the set wOS_r of all finite, rooted, one-step, weakly-transitive frames considered up to isomorphism can be seen as the set $\mathcal{N}^4$.

Theorem 2.2.18. *There is a one-to-one correspondence between the set wOS_r and the set $\mathcal{N}^4$.*

Proof. We know that any one-step frame has "strict" width one and "strict" height less than or equal to two (We didn't give the formal definition of "strict" height and width, but it should be clear from the intuitive explanation after the definitions 2.2.13 and 2.2.3 what we mean by this). If additionally we have that the frame is rooted, the case where strict width is greater than one at the bottom is also restricted. It is not difficult to verify that any such frame (W, R) is of the form (W_1, W_2), where $W_1 \cup W_2 = W$, $W_1 \cap W_2 = \varnothing$ and $(\forall u \in W_1, \forall v \in W_2)(uRv)$. Besides because of the weak-transitivity, we have that $(\forall u, u' \in W_1)(u \neq u' \Rightarrow uRu')$ and the same for every two points $v, v' \in W_2$. Pictorially any rooted, weak-transitive and one-step Kripke frame can be represented as in Figure 2. Again we call W_1 the ***first floor*** and W_2 the ***second floor*** of the frame (W, R). Notice that W_1 or W_2 may be equal to $\varnothing$ ie the frame has only one floor. In this case we treat the only floor of the frame as the second floor.

Now let us describe how to construct the function from wOS_r to $\mathcal{N}^4$. With every frame $(W, R) \in \mathcal{N}^4$ we associate the quadruple (i_1, r_1, i_2, r_2), where i_1 is the number of irreflexive points in W_1, r_1 is the number of reflexive points in W_1, i_2 is the number of irreflexive points in W_2 and r_2 is the number of reflexive points in W_2. We will call the quadruple (i_1, r_1, i_2, r_2) the ***characteriser*** of the frame (W, R). In case the frame (W, R) has only one floor, by our earlier remark it is treated as the frame $(\varnothing, W)$. Hence its characteriser has the form $(0, 0, i, r)$. Now it is clear that the correspondence described above defines a function from the set wOS_r to the set $\mathcal{N}^4$. We denote this function by Ch.

Claim 1: Ch is injective. Take any two distinct finite, rooted, weakly-transitive, one-step Kripke frames (W, R) and (W', R'). That they are distinct in wOS_r means that they are non-isomorphic ie either $|W| \neq |W'|$ or $R \not\cong R'$. In the first case it is immediate that $Ch(W, R) \neq Ch(W', R')$ since $|W| = i_1 + i_2 + r_1 + r_2$. In the second case we have three subcases:

1) $|W_1| \neq |W_1'|$. In this subcase $i_1 + r_1 \neq i_1' + r_1'$ and hence $Ch(W, R) \neq Ch(W', R')$.

2) The number of reflexive (irreflexive) points in $|W_1|$ differs from the number of reflexive (irreflexive) points in $|W_1'|$. In this subcase $i_1 \neq i_1'$ and again $Ch(W, R) \neq Ch(W', R')$.

3) The number of reflexive (irreflexive) points in $|W_2|$ differs from the number of reflexive (irreflexive) points in $|W_2'|$. This case is analogous to the previous one.

It is straightforward to see that if none of these cases above occur ie $|W| = |W'|$, $|W_1| = |W_1'|$, $|\{w|w \in W_1 \wedge wRw\}| = |\{w'|w' \in W_1' \wedge w'R'w'\}|$ and $|\{w|w \in W_2 \wedge$

$wRw\}| = |\{w'|w' \in W_2 \wedge w'R'w'\}|$ then (W, R) is isomorphic to $(W'R')$ and hence $(W, R) = (W', R')$ in wOS_r.

Claim 2: Ch is surjective. Take any quadruple $(i_1, r_1, i_2, r_2) \in \mathcal{N}^4$. Let us show that the pre-image $Ch^{-1}((i_1, r_1, i_2, r_2))$ is not empty. Take the frame $(W, R) = (W_1, W_2)$, where $|W_1| = i_1 + r_1$, $|W_2| = i_2 + r_2$, W_1 contains i_1 irreflexive and r_1 reflexive points and $|W_2|$ contains i_2 irreflexive and r_2 reflexive points. Then by the definition of Ch, we have that $Ch(W, R) = (i_1, r_1, i_2, r_2)$. □

The bijection given in theorem 2.2.18 can be extended to bounded morphisms. In the following theorems we characterize bounded morphisms between two finite, rooted, weak-transitive, one-step frames in terms of numeric dependencies of the corresponding quadruples of natural numbers. For simplicity we split the proof in different theorems as it makes the proof much more readable and besides each case is interesting on its own. For the case when both frames have characterizers of the form $(0, 0, i, r)$ and $(0, 0, i', r')$ i.e. for frames with only one floor, the characterization is given in Theorem 2.2.11. We will omit zeroes in the quadruple like $(0, 0, i, r)$ and just represent them shortly as (i, r).

Now let us consider the case, where the first frame has two floors, while the second is one floor frame. Note that the only difference with the conditions in 2.2.11 is that we require that the second frame contains at least one reflexive point. This is because we want to map all first floor points of the first frame to this particular reflexive point.

Theorem 2.2.19. *The finite, rooted, weak-transitive, one-step frame (W', R') with the characterizer (i', r') is a bounded morphic image of the finite, rooted, weak-transitive, one-step frame (W, R) with the characterizer (i_1, r_1, i_2, r_2), where $i_1 + r_1 > 0$ and $i_2 + r_2 > 0$ iff the following conditions are satisfied:*

$r' \neq 0$,
$i_2 \geq i'$,
$2 \times (r' - r_2) \leq i_2 - i'$.

Proof. Assume $r' = 0$. This means that all points in (W', R') are irreflexive. Now as (W, R) has two floors we can represent it as a pair (W_1, W_2) where both sets are non-empty. This implies that there are at least two points $u \in W_1$ and $v \in W_2$, such that $f(u) \in W'$ and $f(v) \in W'$. As far as (W', R') is a cluster, we have that $f(u)R'f(v)$ and $f(v)R'f(u)$. But this gives a contradiction with f being a bounded morphism. This is because $\neg(vRu)$ while $f(v)R'f(u)$ and as $f(u)$ is irreflexive there does not exist a point $w \in W$, with $u \neq w$ and $f(w) = f(v)$ (see the first observation in the proof of Theorem 2.2.11).

Assume $i_2 < i'$. Again by first observation in the proof of Theorem 2.2.11 we know that the pre-image of the irreflexive point can not be reflexive. This means that there exists a point $u' \in W'$, such that, u' is irreflexive and $f^{-1}(u') \cap W_2 = \varnothing$. Different from the analogous case in the proof of Theorem 2.2.11 there is possibility that some irreflexive point $u \in W_1$ is mapped to the point u'. Let us show that this can not happen. Assume $u \in W_1$ and $f(u) = u'$. As u is a first floor point, there exists some $v \in W_2$ with uRv. As f is bounded morphism we have that $f(u)R'f(v)$. As (W', R') is a cluster we have that $f(v)R'f(u)$ as well. Now we get a contradiction as there is no successor of v which is mapped to $f(u) = u'$.

Assume the third condition does not hold. This means that $2 \times (r' - r_2) > i_2 - i'$. In this case we have that there is at least one point $u' \in W'$, such that, u' is reflexive and $f^{-1}(u') \cap W_2$ is either empty set or it contains only one point u with $\neg(uRu)$. This gives a contradiction as far as $u'R'u'$ while there is no successor v of u with $f(u) = u'$.

For the converse direction we construct $f : W \to W$ in the following way: $f_{|W_1} = v'$, where v' is an arbitrary reflexive point in W' (we know that such exists from the first condition of the lema). $f_{|W_2}$ is constructed in exact analogy with the construction in 2.2.11. □

And at last we give the characterization for the frames where both frames have two floors.

Theorem 2.2.20. *The finite, rooted, weak-transitive, one step-frame (W', R') with the characterizer (i'_1, r'_1, i'_2, r'_2), where $i'_1 + r'_1 > 0$ and $i'_2 + r'_2 > 0$ is a bounded morphic image of the finite, rooted, weak-transitive, one step-frame (W, R) with the characterizers (i_1, r_1, i_2, r_2), where $i_1 + r_1 > 0$ and $i_2 + r_2 > 0$ iff the following equalities take* *place:*

$r'_1 = 0 \Rightarrow (i_1, r_1) = (i'_1, r'_1),$
$r'_2 = 0 \Rightarrow (i_2, r_2) = (i'_2, r'_2),$
$i_1 \geq i'_1,$
$i_2 \geq i'_2,$
$2 \times (r'_1 - r_1) \leq i_1 - i'_1,$
$2 \times (r'_2 - r_2) \leq i_2 - i'_2.$

The operation minus is defined within the natural numbers i.e. $n - m = 0$ *if* $m > n$.

Proof. The theorem follows from the previous theorem and the following observation:

• ***If*** (W', R') ***is a two floor frame i.e.*** $i'_1 + r'_1 > 0$ ***and*** $i'_2 + r'_2 > 0$ ***then*** (W, R) ***is also two floor frame.*** Assume not. Then there exist points $w, v \in W$ such that vRw and not $f(v)R'f(w)$ as $f(v)$ is a second floor point while $f(w)$ is a first floor point.

• ***points from the second floor can not be mapped to the points on the first floor.*** For the contradiction assume that the point $f(w)$ is a first floor point while w is a second floor point. This means that there exists $v' \in W'$ such that $f(w)R'v'$ and not $v'R'f(w)$. Then as f is a bounded morphism there exists $v \in W$ such that wRv and $f(v) = v'$. As far as w is a second floor point $wRv \Leftrightarrow vRw$ and we get a contradiction as we have vRw while not $f(v)R'f(w)$.

• ***points from the first floor can not be mapped to the points on the second floor.*** For the contradiction assume that the point $f(w)$ is a second floor point while w is a first floor point. Now either there is another point $w_1 \in W$ on a first floor with $f(w_1)$ also on the first floor or all points including w from the first floor of W are mapped to the second floor of W'. In the first case wRw_1 and not $f(w)Rf(w_1)$ so we get a contradiction. In the second case we get that f is not surjective as far as both frames were supposed to be two floor frames so there is at least one point on the first floor in W' left with empty pre-image. This is because by above observation second floor points can not be mapped to the first floor points.

The converse direction is proved just by repetition of the case for one floor frames for the other floor. $\square$

One simple extension of **wK4f** is the modal logic **wK4Df** which is obtained by adding the axiom D to **wK4f**. As it is suspected axiom D adds the property of seriality $((\forall x)(\exists y)(xRy))$ to the class of one-step frames. Therefore the corresponding class of frames for the logic **wK4Df** are one-step and serial. On the level of finite rooted frames seriality amounts to frames where second floor is not one irreflexive point. In other words those frame which have charateriser (i, r, i', r') where if $r' = 0$ and $i' \neq 0$ then $i' > 1$. The following completeness theorem is a straightforward extension of our previous results for **wK4f** and can be established by standard techniques.

Theorem 2.2.21. *The modal logic* **wK4Df** *is sound and strongly complete wrt the class of all Kripke frames that do not have a dead-end, are one-step and weakly-transitive.*

Notice that rooted, one-step and weakly-transitive Kripke frames without the a dead-end are similar to the two-floor frames in the Figure 2.1, where the ellipses now represent not clusters but weak-clusters and the second floor is not a singleton irreflexive point. For both **S4F** and **wK4Df** we refer to the rooted frames as *two-floor frames.* In Chapter 4 we will make use of the logic **wK4Df** to capture the idea of minimal unconscious belief of an agent.

2.3 Splitting Translation

It is not accidental that we picked out some well known reflexive modal logics (i.e. modal logics over **S4**) and some of the irreflexive (i.e logics ovew **wK4**) ones as a doxastic counterparts. There is a translation which strongly relates the described logics of knowledge with the logics of belief. In particular **S5** to **KS**, the modal logic **S4** to the modal logic **K4** and the modal logic **S4F** to the modal logic **wK4f**. The translation is called splitting translation. We will denote is by Sp. This is a translation from the set of modal formulas to the set of modal formulas that replaces each occurrence of $\Box\varphi$ by $\Box\varphi \wedge \varphi$ other formulas being left unchanged. It appears that the splitting translation was independently discovered by several authors and its first main application is usually attributed to [51], [35] and [12].

As we have already underlined the reflexive logics are interesting from the epistemic perspective. Modal logic **S4** is considered as one of the weakest logic for modeling epistemic reasoning about knowledge as far as the **S4**-agents knowledge satisfies only positive introspection principle. Analogously **K4** (**wK4**) could serve as a candidate to form logics of belief again with only positive (weak) introspection principle. In particular it lacks the strong T axiom $\Box p \to p$. Now if we look at extensions of the logic **S4** many of them give reasonable epistemic interpretations. For example **S4.2** and **S5**. While some of the extensions of **K4** may as well serve as a good doxastic logics. One immediate example is a standard doxastic logic, **KD45**. From this epistemic perspective the splitting translation is also very natural. After interpreting $\Box p$ in extensions of **S4** as "the agent knows p" and in extensions of **wK4** as "the agent believes p", the splitting translation gives us the interpretation of knowledge as 'truth plus belief'. Let us now give the definition of the translation and mention some facts which makes it clear why splitting translation unifies the above considered logics.

Definition 2.3.1. *Let us consider the function from the set of formulas to the set of formulas defined in the following way:*
$Sp(p) = p$ *for every propositional letter* p,
$Sp(\neg\alpha \vee \beta) = \neg Sp(\alpha) \vee Sp(\beta)$,
$Sp(\Box\alpha) = \Box Sp(\alpha) \wedge Sp(\alpha)$.

The following fact builds a bridge between epiestemic and doxastic logics considered so far.

Fact 2.3.2. *[15] Ler* $R \subseteq W \times W$ *be a relation and* $\overline{R}$ *be its reflexive closure i.e.* $\overline{R} = R \cup \{(w, w) : w \in W\}$ *then for an arbitrary valuation* V, *a point* $w \in W$ *and an arbitrary formula* φ *the following condition holds:*

$$(W, \overline{R}, V), w \Vdash \varphi \textit{ iff } (W, R, V), w \Vdash Sp(\varphi).$$

As an immediate consequence of Facts 2.3.2, 2.1.4, 2.1.6, 2.1.8 and theorems 2.2.8, 2.2.12 and 2.2.21 we get the following corollaries:

Corollary 2.3.3. $\vdash_{S4} \varphi$ *iff* $\vdash_{K4} Sp(\varphi)$.

Corollary 2.3.4. $\vdash_{S5} \varphi$ *iff* $\vdash_{KS} Sp(\varphi)$.

Corollary 2.3.5. $\vdash_{S4F} \varphi$ *iff* $\vdash_{wK4f} Sp(\varphi)$.

As we will see in the forthcoming chapter, splitting translation has a very natural topological interpretation which mainly is the expression of closure operator by derivative operator. We will provide further details in the next chapter.

2.4 Conclusions

In the chapter we considered several modal logics of knowledge and belief for the case of single agent. First we recall the well know epistemic modal logics **S5** and **S4** and some known facts around these logics [26]. All important epistemic logics from the literature [11] fall in the interval [**S4**, **S5**] and the two logics are the weakest and the strongest among those. Additionally we recall the logic **S4F** which is also the logic over **S4** and under **S5**. Although the main motivation for presenting **S4F** in the thesis does not arise from the fact that epistemic reading of axiom **F** gives us some important property. In fact **S4F** is interesting from some other perspective as far as non-monotonic version of it captures the idea of minimal knowledge [75]. All the results presented in relation with epistemic logics are well known and could be seen in above cited literature.

Most of the second part of the chapter presents novel material. As an alternative to doxastic modal logic **KD45** we present the logics **KS** and **KSD**. Both logics allow to interpret belief as something true in all alternative worlds. While the logic **KSD** additionally requires that agent does not belief in false statements. At the same time the logics **KS** and **KSD** are related to the epistemic logic **S5** via splitting translation.

The completeness and finite model property for the logic **KS** was known from [24] as for the logic **KSD** things go in exactly the same manner therefore we do not present the details and just mention the results. The main result related to the logics **KS** and **KSD** are given in Subsection 2.2.3 where we characterise the finite rooted frames and their bounded morphisms of these logics.

As a doxastic companions of the logic **S4F** we present the two logics **wK4f** and **wK4Df**. Again none of these logics carry doxastic interest from standard point of view i.e. the axiom **f** can not be given any convincing doxastic reading. Although we will see in Chapter 4 that nonmonotonic versions of each of these logics play an important role for modeling the idea of minimal belief. In this chapter we prove some technical results around these logics. We will make use of most of these results in Chapter 4. The results proved are completeness and finite model property theorems for the logic **wK4f** and we only state the completeness theorem for the logic **wKD4f** (We omit proves for the case of **wKD4f** as far as axiom D does not change anything in technical part of the proofs). We give characterisation of the finite rooted frames and their bounded morphisms in terms of quadruples of natural numbers and we discuss the relation of the logic **wK4f** with the logic **S4F**. This relation is obtained via splitting translation from Section 2.3.

Chapter 3

Logics via Topology

Alfred Tarski [85], together with Chen McKinsey [60, 61], laid the foundations for the algebraic and topological study of intuitionistic and modal logics. The basic idea, recalled and developed by Leo Esakia [25], is that from an arbitrary topological space X we can generate three different algebraic structures each giving rise to different logical systems. By considering the algebra of open sets, $Op(X)$, one is led to the well-known Heyting algebra that forms a semantical basis for intuitionistic logic. By considering the closure algebra, $(\mathcal{P}(X), \mathbf{c})$ one is led to the modal system **S4**.[1]

The third path from topology to logic is via what are known as *derivative algebras*, $(\mathcal{P}(X), der)$. These are Boolean algebras with a unary operation der representing topological derivation: if A is a subset of X then $der(A)$ is the set of all accumulation or limit points of A. The derivative algebra $(P(X), der)$ gives rise to the modal logic **wK4**, a slightly weaker version of the logic **K4**, that was first studied from a topological point of view in [22] (see [25] for a detailed overview).

All three paths to logic are of interest for the modelling of agents' reasoning, their knowledge and beliefs in AI. Intuitionistic logic and its extensions capture different

[1]Recall that a Heyting algebra $(\mathbf{H}, \vee, \wedge, \rightarrow, \bot)$ is a distributive lattice with smallest element $\bot$ containing a binary operation $\rightarrow$ such that $x \leq a \rightarrow b$ iff $a \wedge x \leq b$. $(\mathbf{B}, \vee, \wedge, -, \mathbf{c})$ is a closure algebra if $(\mathbf{B}, \vee, \wedge, -)$ is a Boolean algebra and $\mathbf{c}$ is a closure operator satisfying: $a \leq \mathbf{c}a, \mathbf{cc}a = \mathbf{c}a, \mathbf{c}(a \vee b) = \mathbf{c}a \vee \mathbf{c}b, \mathbf{c}\bot = \bot$.

forms of constructive reasoning, while extensions of **S4**, including **S5**, have formed the basis for epistemic logics of knowledge. On the other hand, extensions of **wK4** are good candidates for doxastic logics of belief. In fact, all the logics **KD45**, **KS**, **wK4f** and **K4** considered in Chapter 2 are extension of **wK4**.

In this chapter we will see how some of the logics considered so far can be unified under two different topological interpretations of modalities. Besides we will see how the relation between epistemic and doxastic logics, so far realised by splitting translation Sp of Section 2.3, does have natural topological roots.

3.1 Topological Preliminaries

In this section we present basics from topology. Such as definition of a topological space, open set, closed set, closure operator, derivative operator etc. In the second subsection we study the lattice of topologies on a fixed set. The reader familiar with topology may skip this part and directly go to section 3.2 where the topological semantics of modal logic are discussed.

3.1.1 Topology

Definition 3.1.1. *A pair (X, τ) is called a topological space if X is a nonempty set and τ is a collection of subsets of X with the following properties:*
1) $X, \varnothing \in \tau$,
2) $U, V \in \tau$ implies $U \cap V \in \tau$,
3) Given an arbitrary set I of indexes, if $U_i \in \tau$ for each $i \in I$ then $\bigcup_{i \in I} U_i \in \tau$.

Elements of τ are called *open sets* or just *opens*. *Closed sets* are defined as complements of opens. Below we list some examples of topological spaces.

Example 3.1.2. *1. The real line $\mathcal{R}$ where τ is the set of all open intervals;*

2. The Sierpinski space is a two element topological space $X = \{x, y\}$ where the set of opens τ contains three elements $\tau = \{\emptyset, X, \{y\}\}$;

3. *The trivial topology on a set X contains only two opens the empty set and the entire space;*

4. *The discret space on a set X is the space where the set of opens τ is a power set of X i.e. every subset $A \subseteq X$ is open..*

A central notion of a topological space is an accumulation point.

Definition 3.1.3. *Assume that (X, τ) is a topological space. Assume that A is an arbitrary subset of X. We say that a point $x \in X$ is an accumulation (limit) point of A if for every open set $U \in \tau$ the following condition takes place: $x \in U$ implies that $U \cap A - \{x\}$ is not empty.*

For an arbitrary topological space (X, τ) one can immediately define a function $d : \mathcal{P}(X) \to \mathcal{P}(X)$, which as an input takes a subset A of X and as an output returns the set $d(A)$ of all accumulation points of A. This operator is called "Cantor-Bendixon derivative" and we will call it shortly "derivative operator". Let us see some examples of how does the operator d act in different cases.

1. Let us consider the first topology from Example 3.1.2. Let $A \subseteq \mathcal{R}$ be the set of all elements of the form $\frac{1}{n}$ where $n \in \omega$. Of course 0 is not an element of A but $0 \in d(A)$ as far as every open interval containing 0 also contains at least one element (distinct from 0) from the set A.

2. Let (X, τ) be Sierpinski space from Example 3.1.2. Let us look at each nonempty subset of X and see how does the operator d act on them. For the singleton $\{x\}$ we have clearly $x \notin d(\{x\})$ and if we look at a point y we know that it has an open neighborhood $\{y\}$ such that $\{x\} \cap \{y\} - \{y\} = \emptyset$ hence $y \notin d(\{x\})$ and hence $d(\{x\}) = \emptyset$. On the other hand the only neighborhood of a point x is the entire space. Therefore $\{y\} \cap X - \{x\} = \{y\}$ and is not empty. Hence $y \in d(\{x\})$. In fact it is easy to check that $y \notin d(\{y\})$ and hence $d(\{y\}) = \{x\}$. In the same way we can see that $d(X) = d(\{x, y\}) = d(\{x\}) \cup d(\{y\}) = \{x\}$.

3. Now let $\tau = \{\emptyset, X\}$ be a trivial topology on X. For an arbitrary point $x \in X$, the set $d(\{x\})$ of all accumulation points of a singleton $\{x\}$ is equal to $X - \{x\}$.

To see this take an arbitrary element $y \in X - \{x\}$ the only neighborhood of y is the entire space X. Now as $X \cap \{x\} - \{y\} = \{x\}$ is not empty we get $y \in d(\{x\})$.

In general it is not difficult to check that a derivative operator satisfies four useful properties listed below.

Remark 3.1.4. *For an arbitrary topological space the corresponding derivative operator satisfies the following four properties:*

$$(1)\ d(\emptyset) = \emptyset, \qquad (2)\ d(A \cup B) = d(A) \cup d(B),$$

$$(3)\ dd(A) \subseteq d(A) \cup A, \qquad (4)\ x \notin d(\{x\}).$$

Converse direction is also true. Given an operator $d : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying the above four properties one can form topology τ on X by claiming the sets of the form $A \cup d(A)$ to be closed sets. More precisely call A closed if $d(A) \subseteq A$ and take $\tau = \{X - A | A \text{ is closed}\}$. It is easy to check that first of all τ is a topology and second that the derivative operator d_τ of this topology is equal to the operator d.

There is another important operator called *Closure operator* defined on the set of subsets of a given topological space. Closure operator was discovered by Kuratowski [21] and it was shown that closure operator also uniquely (up to homeomorphism) defines a topology. While every given topology (X, τ), inherits its closure operator $C : \mathcal{P}(X) \to \mathcal{P}(X)$ by sending every subset $A \subseteq X$ to its topological closure $C(A) \subseteq X$. The closure operator is given by the following properties.

$$(1)\ C(\emptyset) = \emptyset, \qquad (2)\ C(A \cup B) = C(A) \cup C(B),$$

$$(3)\ CC(A) \subseteq C(A), \qquad (4)\ A \subseteq C(A).$$

The derivative operator d is more expressive. For an arbitrary topological space (X, τ) the closure operator C of the topology τ can be expressed by the derivative operator of τ in the following way: $C(A) = d(A) \cup A$ for an arbitrary subset $A \subseteq X$. If we take the dual operators i.e. interior and colimit operators defined in the following way: $I(A) = X - C(X - A)$ and $col(A) = X - d(X - A)$ then we have $I(A) = col(A) \cap A$ which is the topological reflection of the main clause of the splitting translation discussed in Chapter 2. We will see later that in topological semantics splitting translation is exactly translation of this dependency in logical terms under the two different topological semantics.

Observe that for closure operator the third property states that $CC(A) \subseteq C(A)$ while for the derivative operator we have weaker condition. The following example shows that the inclusion $dd(A) \subseteq d(A)$ does not hold in an arbitrary topological space.

Example 3.1.5. *Take $X = \{x, y\}$ and $\tau = \{\emptyset, X\}$ i.e. τ is the trivial topology on X. Then it is easy to check that $d(\{x\}) = \{y\}$ and $d(\{y\}) = \{x\}$. Hence $dd(\{x\}) = \{x\}$ and it is not subset of $d(\{x\})$.*

Some well known properties of topological spaces can be defined in terms of conditions on closure or derivative. Below we give one important example of which we make use later.

Definition 3.1.6. *A topological space (X, τ) is called a T_D-space if every point $x \in X$ can be represented as an intersection of some open set and some closed set.*

Fact 3.1.7. *([24]) A topological space (X, τ) is a T_D-space iff for an arbitrary subset $A \subseteq X$ the following condition is satisfied: $dd(A) \subseteq d(A)$.*

Hence in Example 3.1.5 we gave the example of a space which is not a T_D space. Now, we define the topological analogues of transitive irreflexive relational structures: Alexandroff spaces.

Definition 3.1.8. *A topological space (X, τ) is called an Alexandroff space if an arbitrary intersection of opens is open, that is for an arbitrary set I of indexes if $U_i \in \tau$ for each $i \in I$ then $\bigcap_{i \in I} U_i \in \tau$.*

Alexandroff spaces are also characterised by the following property. A topological space (X, τ) is an Alexandroff space if and only if every point has a minimal open neighborhood i.e. $\forall x \in X$ there is $U_x \in \tau$ such that $x \in U_x$ and for every $V \in \tau$ if

$x \in V$ then $U_x \subseteq V$. The following theorem links irreflexive an transitive relational structures with Alexandroff T_D-spaces.

Fact 3.1.9. *([25]) There is a one-to-one correspondence between Alexandroff, T_D-spaces and transitive, irreflexive relational structures.*

Let us shortly describe the correspondence. For this reason we first introduce the lower cone operator. Assume (X, R) is a transitive and irreflexive relational structure. The lower cone operator R^{-1} is defined in the following way: with every $A \subseteq X$ we take $R^{-1}(A) = \{x|(\exists y)(y \in A \wedge xRy)\}$. Obviously the lower cone operator R^{-1} satisfies the properties of the topological derivative operator for T_D-spaces. Hence we get a T_D-space (X, τ_R), where τ_R is the topology obtained from the derivative operator R^{-1}. Conversely with every topological space (X, τ) one can associate (X, R_τ), where $xR_\tau y$ iff $x \in d_\tau(\{y\})$ (Here d_τ is the derivative in τ). Moreover if (X, τ) is Alexandrof T_D - space then we have that (X, τ_{R_τ}) is homeomorphic to (X, τ) and (X, R_{τ_R}) is order isomorphic to (X, R).

Fact 3.1.10. *[25] The set $A \subseteq X$ is open in the topology (X, τ_R) iff for every element $x \in A$ every R successor is again in A. In other words the implication $xRy \Rightarrow y \in A$ holds for every $y \in X$.*

This correspondence can be directly generalized to Kripke frames with more than one transitive and irreflexive relation. For example if we have a bi-topological space (X, τ_1, τ_2) where τ_1 and τ_2 are Alexandroff spaces satisfying T_D separation axiom then we can by above mentioned procedure construct a birelational frame $(X, R_{\tau_1}, R_{\tau_2})$ where each relation R_{τ_1}, R_{τ_2} is transitive and irreflexive. Moreover the operators $R_{\tau_1}^{-1}$ and $R_{\tau_2}^{-1}$ are equal to the derivative operators d_1 and d_2 of the spaces τ_1 and τ_2 respectively. We will make use of this fact in Chapter 5

3.1.2 Latice of Topologies on a Given Set

A pair $(X, \leq)$ is called a partially ordered set or simply *poset* if X is a set and $\leq$ is a reflexive, transitive and antisymmetric relation on X. For a given poset $(X, \leq)$ and two elements $x, y \in X$, the greatest lower bound $inf(x, y)$ is an element $z \in X$ such that $z \leq x$ and $z \leq y$ and for an arbitrary element $z' \in X$ with the same properties i.e. $z' \leq x$ and $z' \leq y$ we have $z' \leq z$. The least upper bound $sup(x, y)$ of two elements x and y is defined dually. This is an element z such that $x \leq z$ and $y \leq z$ and for any $z' \in X$ if $x \leq z'$ and $y \leq z'$ then $z \leq z'$. A partial order $(X, \leq)$ is called a *lattice* if with every two elements $x, y \in X$ the greatest lower bound $inf(x, y)$ and the least upper bound $sup(x, y)$ exist. Instead of symbols $inf(x, y), sup(x, y)$ usually the notations $x \sqcap y$ and $x \sqcup y$ are used. Observe that the definition of the lattice only requires existence of the infimum and supremum of finite number of element. The lattices where infimum and supremum exists for an arbitrary subsets are called *complete* lattices. Sometimes infimum and supremum are called as meet and join. We may vary between lower bound, meet or infimum and respectively upper bound, join or supremum. While reader should be aware that we mean one and the same thing. Join and meet of an arbitrary subset $B \subseteq X$ will be denoted by $\bigvee B$ and $\bigwedge B$ respectively. A lattice is called *bounded* if it contains the least (bottom) and the greatest (top) elements. They are denoted by 0 and 1. There exists enormous number of examples of lattices but in this work we will present just few of them which are of particular interest for us.

Example 3.1.11. *Given a topological space (X, τ) one may consider the set $L_\tau = \{\tau' \mid \tau \subseteq \tau'\}$ of all topologies on X that are coarser then τ. Let us see that this set forms a complete bounded lattice. The partial order is given by the set theoretic inclusion. Let us describe the meet and the join. It is immediate to check that the set theoretic intersection $\tau_1 \cap \tau_2$ of the two topologies τ_1 and τ_2 from L_τ is again a topology coarser then τ. Hence $\tau_1 \sqcap \tau_2 = \tau_1 \cap \tau_2$. Even more one can take an infinite intersection $\bigcap_{i \in I} \tau_i$ of topologies τ_i on a set X and still it is a topology and if each $\tau_i \supseteq \tau$ the the intersection topology is also coarser then τ. As for the join $\tau_1 \sqcup \tau_2$ we consider the smallest topology which contains both τ_1 and τ_2. That such a topology exists we know because there is at least one topology, which is the discrete topology, satisfying the condition and the intersection of an arbitrary set of topologies coarser then τ is again a topology over τ. On the other hand the join usually is not the set* *theoretic union of τ_1 with τ_2. As a consequence we get that L_τ is a complete and bounded lattice with top - the discrete topology, and the bottom - the topology τ. If we take τ to be the trivial topology L_τ will become the lattice of all topologies over X.*

In subsection 3.1.1 it was mentioned that topologies on a given set are in one-to-one correspondence with the derivative operators on the same set. Based on this observation It is interesting to see what set corresponds to L_τ and if there is any structure on the corresponding set. The answer is provided by the following example.

Example 3.1.12. *Assume that d is a derivative operator on a set X. Let us consider the set $L_d = \{d' \mid d' \leq d\}$ of all derived set operators less or equal to d. The relation $\leq$ on the set of all derivative operators is defined in a standard way: $d_1 \leq d_2$ if for an arbitrary subset $A \subseteq X$ the image $d_1(A)$ is included in the image $d_2(A)$. Therefore the restriction of $\leq$ over the set L_d is a partial order.*

The next proposition establishes an important connection between the poset L_d and the poset L_τ. In fact it turns out that as posets the two are the same when we look at either of them with an order inversed.

Proposition 3.1.13. *The poset $(L_\tau, \subseteq)$ is anti-isomorphic to the poset $(L_d, \leq)$. Here τ is a topology on a set X and d is the derivative operator corresponding to τ.*

Proof. In section 3.1.1 it was already mentioned that as sets L_τ and L_d are in one-to-one correspondence. Now assume that τ_1 and τ_2 are two topologies in L_τ and assume that $\tau_1 \subseteq \tau_2$. This means that if the set B is closed in the first topology τ_1 then it is closed in the second topology τ_2. Let d_1 and d_2 be derivative operators corresponding to τ_1 and τ_2 respectively. Let us prove that $d_2 \leq d_1$. The proof goes by contradiction. Assume that $d_2 \not\leq d_1$. This by definition of $\leq$ means that there is a set $A \subseteq X$ such that $d_2(A) \not\subseteq d_1(A)$. This means that there is a point $x \in X$ with $x \in d_2(A)$ and $x \notin d_1(A)$. Let us take $B = A - \{x\}$. Then we have $x \in d_2(B)$ while $x \notin d_1(B)$. Hence $x \in C_2(B)$ and $x \notin C_1(B)$ where $C_1(B) = d_1(B) \cup B$ and $C_2(B) = d_2(B) \cup B$ are the closure operators of the corresponding topologies. This means that we have a closed set $C_1(B)$ in the topology τ_1 while it is not closed in the second topology

τ_2. This is because $C_2(C_1(B)) \neq C_1(B)$ as far as $x \in C_2(C_1(B))$. And we get a contradiction.

Conversely assume that $d_1 \leq d_2$. Take an arbitrary closed set B in the topology τ_2. By definition this means that $B = d_2(B) \cup B$. Let us show that B is closed in the first topology τ_1. This is to show that $B = d_1(B) \cup B$. From the assumption we have $d_1(B) \cup B \subseteq d_2(B) \cup B$. Hence $d_1(B) \cup B \subseteq B$. □

We have seen that L_τ as a poset is antyisomorphis to L_d. At the same time we have seen in the example 3.1.11 that L_τ has a lattice structure. It follows that the partial order L_d is also a lattice. What about the meet and the join of the two operators d_1 and d_2 in L_d? It turnes out that the meet and the join are not easily representable. The first obvious candidate is to take the meet $d_1 \sqcap d_2$ equal to the intersection of the two operators. This is $d_1 \sqcap d_2(A) = d_1(A) \cap d_2(A)$ for an arbitrary subset $A \subseteq X$. Unfortunately it turns out that $d_1 \sqcap d_2$ is not the derivative operator any longer. In particular the property $d_1 \sqcap d_2(A \cup B) = d_1 \sqcap d_2(A) \cup d_1 \sqcap d_2(B)$ fails to hold. The same problem we have with the join defined as the componentwise union. If we define $d_1 \sqcup d_2(A) = d_1(A) \cup d_2(A)$ then the third property of the derivative operator will fail. This is $d_1 \sqcup d_2(d_1 \sqcup d_2(A)) \not\subseteq d_1 \sqcup d_2(A) \cup A$. In general we may not be able to express the join and the meet of the two operators d_1 and d_2 in L_d just using the finite combinations of the set theoretic operations $(\cap, \cup, \neg)$ and derivatives d_1 and d_2. In some particular cases [2] it happens that we can describe the meet as an infinite sequence of the correlating iteration of the operators d_1 and d_2 but this happens only in the restricted subclass of derivative operators. In general even the theory of complete Boolean algebras (with infinitary operations) is not expressive enough to define the meet of d_1 and d_2. Te best known representation of the meet $d_1 \sqcap d_2$ is given by the fixpoint condition. For an arbitrary subset $A \subseteq X$ the value of the meet operator $d_1 \sqcap d_2(A)$ applied to A is defined as the smallest subset B such that $B = d_1(A) \cap d_2(A) \cap d_1(B) \cap d_2(B)$. As for the join $d_1 \sqcup d_2$ the situation is even more complicated. We do not know any representation of this operator using the

[2]This happens when we consider the sublattice of L_τ which consists of all Allexandroff spaces. In this case $d_1 \sqcap d_2(A) = d_1(A) \cap d_2(A) \cap d_1 d_2(A) \cap d_2 d_1(A) \cap d_1 d_1(A) \cap d_2 d_2(A) \cap ...$

operators d_1, d_2 and possible set theoretic operations of the complete Boolean algebra $\mathcal{P}(X)$ of all subsets of X.

What about the T_D - spaces. We know that the class of all T_D spaces also forms a poset under the set theoretic inclusion and as a poset it is a suborder of the poset of all topological spaces. But as the next example shows the class of all T_D spaces does not form a lattice.

Example 3.1.14. *Take a set $X = \{x, y\}$ consisting of two points. Let $\tau_1 = \{X, \emptyset, \{x\}\}$ be the first topology on X and let $\tau_2 = \{X, \emptyset, \{y\}\}$ be the second topology. Let us see that τ_1 satisfies the T_D separation axiom i.e. we have to check that every point in X is an intersection of an open and closed sets from τ_1. We make use of the fact that the set X is both open and closed in τ_1. To represent the singleton $\{x\}$ as an intersection of open and closed sets we take the closed set X and the open set $\{x\}$. For the singleton $\{y\}$ we take X as an open set and the closed set $\{y\} = X - \{x\}$. In exactly the same way we can show that τ_2 is the T_D space. Now the intersection $\tau_1 \cap \tau_2 = \{X, \emptyset\}$ is a trivial topology and clearly is not T_D space. Hence in the poset $L_\tau^{T_D}$ the greatest lower bound of an arbitrary two elements does not exist.*

Although if we consider the poset $L_\tau^{T_D}$ of all T_D topologies which are coarser then some fixed T_D topology τ then we have a lattice structure. Even more $L_\tau^{T_D}$ is a complete and bounded lattice and it turns out to be equal to L_τ. The lattice $L_\tau^{T_D}$ will be central throughout this work therefore let us describe it carefully. The main observation here is that if we have a topology τ which satisfies the T_D separation axiom then an arbitrary topology which is coarser then τ will also be the T_D space. Because of this as a partially ordered sets $L_\tau^{T_D}$ and L_τ are the same. Moreover if we have two T_D topologies τ_1 and τ_2 coarser then τ their intersection $\tau_1 \cap \tau_2$ is also a T_D space as far as it is again coarser then τ. Therefore the meet in $L_\tau^{T_D}$ coincides with the meet in the lattice L_τ. As for the join obviously the standard join of two topologies preserves the T_D separation property and hence $L_\tau^{T_D} = L_\tau$. Because of this observation from now on we will stick to the notation L_τ.

3.2 Topological Semantics

There are two distinct topological semantics of modal language known from the literature. First one is based on closure operator and the second on the derivative operator. The semantics which uses closure operator for interpreting diamond modality is called C-semantics, while the other, using derivative to interpret diamond, is called d-semantics. To make a parallel with Kripke semantics let us mention that logics in C-semantics always satisfy the T axiom hence in Kripke style formulation these are reflexive logics while logics obtained from the d-semantics, we will call these logics d-logics, never validate the T axiom. More concretely the Kripke counterparts of d-logics (If they exist) are always based on strictly irreflexive relations. Mainly from above motivations the topological semantics intersects with the epistemic and doxastic understandings of modalities. In particular C-logics do have the epistemic character while d-logics stand closer to doxastic intuition. If we follow [19], one possible interpretation of $\Box p$ in a topological model can be formulated as: there exists a piece of evidence for agent (i.e., an open set in topology), which validates the proposition p. In the following subsections we describe the two topological semantics and present the topological semantics of the logics discussed in previous chapter. In both semantics we have the same definition of the topological model although the interpretation of formulas change depending on which semantics we are in.

Definition 3.2.1. *A topological model is a triple $\mathcal{M} = (X, \tau, V)$ where X is a set, τ is a topology on X and $V : Prop \to 2^X$ is a valuation function.*

3.2.1 C-Semantics

C-semantics is based on the closure operator.

Definition 3.2.2. *For a given topological model $\mathcal{M} = (X, \tau, V)$ the satisfaction of a modal formula at a point $w \in X$ is defined in the following way:*
$\mathcal{M}, w \Vdash p$ *iff* $w \in V(p)$,
$\mathcal{M}, w \Vdash \alpha \wedge \beta$ *iff* $\mathcal{M}, w \Vdash \alpha$ *and* $\mathcal{M}, w \Vdash \beta$,

$\mathcal{M}, w \Vdash \neg\alpha$ *iff* $\mathcal{M}, w \nVdash \alpha$,
$\mathcal{M}, w \Vdash \Diamond\varphi$ *iff* $w \in C(V(\varphi))$, *where* C *is the closure operator of* τ.

The correspondence from Fact 3.1.9 links irreflexive, weakly-transitive orders and Alexandroff topological spaces on a given set. In particular irreflexive, weakly-transitive order gives a derivative and vice versa every derivative operator operator and embeds a set with a weakly-transitive irreflexive order. Exactly the same kind of correspondence we have between closure operators and preorders. Moreover this goes in line with the the following connection of closure operator and derivative operator $C(A) = d(A) \cup A$. Indeed if we look at the corresponding to the operator d irreflexive and weakly-transitive relation R_d then the reflexive closure $\overline{R}_d$ of R_d is exactly the reflexive and transitive relation R_C which corresponds to the operator C, ie. the equality $C(A) = d(A) \cup A$ induces the following equality $\overline{R}_d = R_C$. On the other hand if we have the relation R_C corresponding to the operator C, and take the irreflexive part of it, ie. $R_C - \Delta$ where Δ denotes the diagonal of the relation, then we get exactly the relation R_d which corresponds to the derivative operator d.

Fact 3.2.3. *[25] The correspondence mentioned in Fact 3.1.9 preserves the truth of modal formulas, ie.* $(W, R, V), x \Vdash \alpha$ *iff* $(W, \tau_R, V), x \Vdash \alpha$.

Note that in Fact 3.2.12, the symbol $\Vdash$ on the left hand side denotes the satisfaction relation on Kripke models, while on the right hand side it denotes the satisfaction relation on topological frames in C-semantics. Now if we look at the correspondence 3.1.9 it is not hard to verify that corresponding topological space of an arbitrary cluster, does not contain any open sets distinct from the empty set and the entire space. Hence it is the least topology. Taking into account this observation and Facts 3.2.12, 3.1.9 of Chapter 3 and Facts 2.1.4, 2.1.8 and 2.1.6 of Chapter 2, we can easily see the idea of the proof of the following two facts. Although we do not know exact origin of the following fact, we are confident that it was known as a folklore in every group of modal logicians who worked in topological semantics of modal logic.

Fact 3.2.4. *Modal logic* **S5** *is sound and complete w.r.t. the class of all topological spaces with the least topology.*

Before stating the following fact let us recall that a minimal topological space is the space with only three open sets: the whole space the empty set and additionally one open set. In other words minimal topology is obtained by enriching the least topology by one subset distinct from empty set and the whole space. Yet in another words the minimal topologies are the minimal elements in the lattice of all topologies on a given set.

Proposition 3.2.5. *Modal logic* **S4F** *is sound and complete w.r.t. the class of all (finite) trivial topological spaces.*

Proof. The proof is a special case of the proof of Theorem 3.2.16. □

The modal logic **S4** is one of the most interesting modal logics from the viewpoint of topology. The following facts will reveal the whole geometrical beauty of this logic. The original proofs are due to Tarski and MCKinsey [60, 61]. Much simpler modern proofs of the same theorems are due to Bezhanishvili and Gerkhe [9]

Fact 3.2.6. *[60]. Modal logic* **S4** *is sound and complete w.r.t. the class of all (finite) topological spaces.*

Fact 3.2.7. *[60]. Modal logic* **S4** *is sound and complete w.r.t. the real line with the usual topology.*

Fact 3.2.8. *[60]. Modal logic* **S4** *is sound and complete w.r.t. the rational space with the interval topology.*

As a conclusion the C-semantics is more widely used in the literature. A lot of properties of different spaces can be expressed by the closure operator. And due to the properties $A \subseteq C(A)$ and $CC(A) \subseteq C(A)$, which corresponds to the reflexivity and transitivity of the corresponding relation R_C and which are the topological reflection of the axioms $\Box p \rightarrow p$ and $\Box p \rightarrow \Box\Box p$, we can assume that the C-semantics has epistemic roots.

3.2.2 d-Semantics

Definition 3.2.9. *Given a topological space (X, τ) and a set $A \subseteq X$ we will say that $x \in X$ is a colimit point of A if there exists an open neighborhood U_x of x such that $U_x - \{x\} \subseteq A$. The set of all colimit points of A will be denoted by $col(A)$ and will be called the colimit set of A.*

Colimit set serves for giving semantics of box modality, consequently semantics for diamond is provided by the dual of colimit set, which is a derived set. So we have $col(A) = X - d(X - A)$. Below we list some properties of colimit operator.

Fact 3.2.10. *For a given topological space (X, τ) the following properties hold:*
1) $Int(A) = col(A) \cap A \subseteq colcol(A)$, where Int denotes the interior operator,
2) $col(X) = X$ and $col(A \cap B) = col(A) \cap col(B)$,
3) If τ is T_d-space then $col(A) \subseteq colcol(A)$,
4) If $\tau_1 \subseteq \tau_2$ then $col_1(A) \subseteq col_2(A)$ where col_i, $i \in \{1, 2\}$ is a colimit operator of the corresponding topology τ_i.

Now we give a definition of truth of a furmula in a derived set topological semantics.

Definition 3.2.11. *For a given topological model $\mathcal{M} = (X, \tau, V)$ the satisfaction of a modal formula at a point $w \in X$ is defined in the following way:*
$\mathcal{M}, w \Vdash p$ *iff* $w \in V(p)$,
$\mathcal{M}, w \Vdash \alpha \wedge \beta$ *iff* $\mathcal{M}, w \Vdash \alpha$ *and* $\mathcal{M}, w \Vdash \beta$,
$\mathcal{M}, w \Vdash \neg\alpha$ *iff* $\mathcal{M}, w \nVdash \alpha$,
$\mathcal{M}, w \Vdash \Box\varphi$ *iff* $w \in col(V(\varphi))$, *where col is a colimit operator of τ.*

Fact 3.2.12. *[25] The correspondence mentioned in the Fact 3.1.9 preserves the truth of modal formulas, ie. $(W, R, V), x \Vdash \alpha$ iff $(W, \tau_R, V), x \Vdash \alpha$.*

Note that in Fact 3.2.12, the symbol $\Vdash$ on the left hand side denotes the satisfaction relation on Kripke models, while on the right hand side it denotes the satisfaction relation on topological frames in the derived set semantics ie. d-semantics. According to Fact 3.1.9 it is easy to see that irreflexive weak-clusters correspond to the topological space which have opens just the empty set and the entire space. Taking into account this observation and Facts 3.2.12, 3.1.9, 2.2.8 and 2.2.12 from Chapter 2, we can easily see the idea of the proof of the following facts originally proved in [24].

Fact 3.2.13. *[24] Modal logic* **KS** *is sound and complete w.r.t. the class of all trivial topological spaces.*

Another fact also given in [24] which needs slightly more adjustments in the proof then the previous one is given below.

Fact 3.2.14. *[24] Modal logic* **K4** *is sound and complete w.r.t. the class of all (finite)* T_D*-spaces.*

In Chapter 2 we discussed one more important irreflexive logic **wK4f** and its simple extension **wK4Df**. In the following subsection we will see that in derived set semantics the logics **wK4f** is the logic of all minimal topological spaces.

3.2.3 Modal Logic of Minimal Topological Spaces

In this section we show that **wK4f** is the modal logic of minimal topological spaces. A topological space is minimal if it has only three open sets. It is well known that there is a bijection between Alexandrof spaces and weakly-transitive, irreflexive Kripke frames and this bijection preserves modal formulas. In this section we show that the special case of this correspondence for minimal topological spaces gives one-step, irreflexive and weakly-transitive relations as a counterpart. As a corollary of Theorem 2.2.21 and Facts 3.1.9 and 3.2.12 it follows that the logic **wK4f** is sound and complete wrt the class of minimal topological spaces.

Theorem 3.2.15. *There is a one-to-one correspondence between the class of all* ***irreflexive****, weakly-transitive, finite, rooted, one-step Kripke frames and the class of all finite minimal topological spaces.*

Proof. Assume (W, R) is a finite, rooted, irreflexive, weakly-transitive and one-step relational structure. (Note that as the frame is irreflexive its characteriser has the form $(i_1, 0, i_2, 0)$, where $i_1 + i_2 = |W|$.) Let W_1 be the first floor and W_2 the second floor of the frame, then the topology we construct is $\{W, \varnothing, W_2\}$. It is immediate that the space (W, Ω_R), where $\Omega_R = \{W, \varnothing, W_2\}$, is a minimal topological space.

Let us show that the correspondence we described is injective. Take two arbitrary distinct irreflexive, finite, rooted, weakly-transitive frames (W, R) and (W', R'). As they are distinct, either $W \neq W'$ or $R \neq R'$. In the first case it is immediate that $(W, \Omega_R) \neq (W', \Omega_{R'})$. In the second case as both R and R' are irreflexive the second floors are not the same, so $W_2 \neq W_2'$ and hence $\Omega_R \neq \Omega_{R'}$.

For surjectivity take an arbitrary minimal topological space (W, Ω), where $\Omega = \{W, \varnothing, W_0\}$ for some subset $W_0 \subseteq W$. Take the frame (W, R), where $R = (W_0 \times W_0 - \{(w, w)|w \in W_0\}) \cup (-W_0 \times -W_0 - \{(w, w)|w \in -W_0\}) \cup \{(w, w')|w \in -W_0, w' \in W_0\}$. In words every two distinct points are related in W_0 by R and the same in the complement $-W_0 = W - W_0$, besides every point from the $-W_0$ is related to every point from W_0. What we get is the rooted one-step relation which is weakly-transitive, with the second floor equal to W_0. As we didn't allow wRw for any point $w \in W$, the relation R is also irreflexive. $\square$

As we have seen from Definition 5.3.2 the box modality in d-semantics is interpreted as a colimit operator while it is not hard to verify that the interpretation of the diamond is given by derivative operator. $w \Vdash \Diamond\varphi$ iff $w \in der(V(\varphi))$.

Theorem 3.2.16. *The modal logic* **wK4f** *is sound and complete with respect to the class of all minimal topological spaces.*

Proof. Soundness can be checked directly so we do not prove it here. For completeness assume $\nvdash \varphi$. By theorem 2.2.16 there exists a finite, one-step, weakly-transitive frame (W, R) which falsifies φ. Assume that $Ch(W, R) = (i_1, r_1, i_2, r_2)$. It is not difficult to check that (W, R) is a p-morphic image of (W', R'), where $(W', R') = Ch^{-1}(i_1 + 2 \times r_1, 0, i_2 + 2 \times r_2, 0)$. Roughly speaking the main idea here is that two distinct irreflexive points from first floor (second floor) of (W', R') are mapped to one reflexive point of first floor (second floor) of (W, R). So each reflexive point in (W, R) has two irreflexive preimages and each irreflexive point in (W, R) has one irreflexive point as a preimage. Now as you can see on each floor in (W', R') there are enough irreflexive points to cover both reflexive and irreflexive point of the corresponding floor in (W, R). So we have a surjection. To check that the described function satisfies back and forth conditions of the p-morphism is left to the reader. The surjection implies that $(W', R') \nVdash \varphi$. Now as far as (W', R') is irreflexive, the result immediately follows from Theorem 3.2.15, and Fact 3.2.12. □

3.2.4 Conclusions

As a conclusion we present the following sequence of figures to illustrate general situation. Starting with a topological space (X, τ) we immediately obtain two operators: The closure operator $C : P(X) \to P(X)$ and the derivative operator $d : P(X) \to P(X)$. Both operators lead to the two different semantics of modality. And logics obtained by interpreting diamond modality as a closure operator are good candidates for epistemic reasoning while logics obtained from the diamond interpreted as the derivative operator have doxastic flavor.

$$(X, C) \longleftarrow (X, \tau) \longrightarrow (X, d)$$

Figure 3.1

All three pairs on Figure 3.1 are three different representations of the same topological space. Now if we start with (X, C) and form a relational structure by the analogous construction shown in Fact 3.1.9 then we obtain a preorder $R_C \subseteq X \times X$ on a set X. In the same way we can obtain an irreflexive, weakly-transitive relation $R_d \subseteq X \times X$ if we start from the pair $(X.d)$ and follow the procedure shown in Fact 3.1.9. Moreover we have that $R_C = \overline{R_d}$. Where $\overline{R}$ denotes the reflexive closure of the relation R.

Conversely if we starts with the weakly-transitive relation $R \subseteq X$ then we can form two relations $R^i = R - \{(x, x) | x \in X\}$-the irreflexive part of R and $\overline{R}$-the re-

$$\begin{tikzcd}
(X,C) \arrow[d] & (X,\tau) \arrow[l] \arrow[r] & (X,d) \arrow[d] \\
(X,R_C) & R_C = \overline{R_d} & (X,R_d)
\end{tikzcd}$$

Figure 3.2

flexive closure of R. Moving to the lower cone operators each relation gives different representation of the same topology. In particular $(R^i)^{-1}$ is a derivative operator and $(\overline{R})^{-1}$ is a closure operator of one and the same Alexandroff topology on X. Besides the connection is the following: $(\overline{R})^{-1}(A) = (R^i)^{-1}(A) \cup A$ which is a topological representation of the splitting translation Sp.

$$\begin{tikzcd}
(X,\overline{R}) \arrow[d] & (X,R) \arrow[l] \arrow[r] & (X,R^i) \arrow[d] \\
(X,(\overline{R})^{-1}) & Sp & (X,(R^i)^{-1})
\end{tikzcd}$$

Figure 3.3

This technique of forming different variants of operators (relations) out of one fixed topological structure has been studied by Esakia [24] and later used by number of authors like [37], [31], [69]. As a conclusion we want to make emphasis on the fact that the above mentioned technique is also important from epistemic/doxastic logics. In particular we have seen that by the above procedure the trivial topology gives two distinct classes of frames the class of all clusters and the class of all weak clusters and from previous chapter we know that each of these classes play a special role as far as the first (the class of all clusters) give rise to the classical epistemic logic **S5** while the second class studied (the class of all weak-clusters) we have seen that gives the modal logic **KS** the importance of which, from doxastic point of view, has been discussed in Chapter 2. Another case we have discussed and studied was when we start with the minimal topological space. As a main novevel result of this chapter we have shown that starting with the minimal topology we get two important classes of Kripke structures. The first, the class of all reflexive one-step relations and the second, the class of all irreflexive one step relations. IN Chapter 2 we have seen that the valid formulas of the class of all reflexive one-step relations gives the logic **S4F** while the logic **wK4f** os the logic of the class of all irreflexive one-step relations. In the next chapter we will see that both of the classes play extremely important role as far as nonmonotonic versions of the logics **S4F** and **wK4f** stand as formalisms which capture the notions of minimal knowledge and minimal belief.

Chapter 4

Non monotonic Modal Logics

4.1 Non-monotonic modal logics

In the 30 years since their inception, non-monotonic logics have been widely investigated among logicians and computer scientists. Early motivation for these logics included the aim to reason about actions and defaults as well as to model the belief of agents with full introspection [58, 72, 59, 64]. Subsequently, non-monotonic systems based on classical, on modal and on non-classical logics were developed and studied. Logic programming provided another important domain for non-monotonic reasoning (NMR) and led to a rich interaction between NMR and logic-based programming languages. Of particular interest in the area of logic programming are non-monotonic logics based on non-classical logics, including superintuitionistic logics, since they provide a logical foundation for reasoning with stable models or with well-founded models and are therefore applicable to LP paradigms such as *answer set programming* or ASP [68, 67]. The most obvious difference between non-monotonic and monotonic logics concerns the consequence relation. Since non-monotonic consequence, denoted by $\mathrel{|\!\sim}$, does not satisfy the monotonicity axiom, $\Gamma \mathrel{|\!\sim} \varphi$ need not imply $\Gamma' \mathrel{|\!\sim} \varphi$ for a larger set of formulas $\Gamma' \supseteq \Gamma$.

Take an arbitrary monotonic normal modal logic L. The non-monotonic logic $\mathfrak{L}_L$

is based on the same language as L. Formulas are also built in the same way. For axioms and rules of inference the situation is different because the non-monotonic forcing relation is not the same as the standard one. The definition of non-monotonic forcing $\mid\!\sim$ is based on the notion of expansion of a set of formulas. Expansions for non-monotonic modal logics are intuitively analogous to maximal consistent sets for their monotonic modal counterparts. Below we give the fixpoint definition of the expansion of an L-consistent set of formulas I. The original definition of expansion do not necessarily require the set I to be consistent although throughout this thesis we restrict ourselves with expansions over consistent sets of formulas. As we will see this does not reduce generality. First let us recall the definition of the L-consequence relation $\vdash_L$.

Definition 4.1.1. *Let I be a consistent set of formulas in L. We will say that the formula φ is an L-consequence of I ($I \vdash_L \varphi$) if there exists a finite set of formulas $\psi_1, ..\psi_n$ such that for every $k, 1 \leq k \leq n$ holds:*
- *ψ_k is a substitution instance of an axiom of L.*
- *$\varphi_k \in I$.*
- *φ_k is the result of applying modus ponens to formulas ψ_i and ψ_j, for some $i, j \leq k$.*
- *φ_k is the result of applying necessitation to a formula ψ_i, for some $i \leq k$.*

We define $Cn_L(I)$ to be the set of all L-consequences of a set of formulas I i.e. $Cn_L(I) = \{\varphi : \ I \vdash_L \varphi\}$.

Definition 4.1.2. *Let I be a consistent set of formulas in L. A set of formulas E is said to be an L-*expansion *of I if it satisfies the following equation:*

$$E = Cn_L[I \cup \{\neg\Box\varphi : \varphi \notin E\}].$$

If the context makes it clear we may drop reference to L and refer simply to expansion.

From the definition it is clear that for a given L a set I may have several L-expansions. Moreover it may happen that the set of all L-expansions of the set I is uncountable. When L is clear from context we use $Ex(I)$ to denote the set of all expansions of I. Below we state the main property of expansions, which mainly reflects the analogy between the maximally consistent sets containing I and expansions of I.

Fact 4.1.3. *[90] Let L be a normal modal logic and I a consistent set of formulas in L. Then for any expansion E of I, the following holds:* $E = Cn_L(E)$.

The nonmonotonic modal logics are based on monotonic logics. To define a nonmonotonic inference firstly one needs to use corresponding monotonic inference and define expansions and secondly using expansions define nonmonotonic inference. The usual (skeptical) non-monotonic inference relation, $\mid\!\sim$, makes use of all expansions of a given set of formulas I;

Definition 4.1.4. *For a normal modal monotonic logic L and L-consistent set of formulas, I, we define corresponding nonmonotonic inference by the following clause:*

$$I \mid\!\sim \varphi \;\; \textit{iff} \;\; \forall E \in Ex(I), E \vdash \varphi.$$

A nonmonotnic normal modal logic $\mathfrak{L}_L$ *corresponding to the normal monotonic logic L is defined as the set of all nonmonotonic inferences of the form* $I \mid\!\sim \varphi$.

Observe that together with Fact 4.1.3 we have an equivalent definition of nonmonotonic forcing: $I \mid\!\sim \varphi$ iff $\forall E \in Ex(I), \varphi \in E$.

Example 4.1.5. *Let* $\mathfrak{L}_L$ *be the logic* **KD45**. *Then* $\mid\!\sim$ *corresponds to consequence in the well-known system of autoepistemic logic [73].*

4.2 Minimal Model Semantics

In the previous section we have seen the definition of nonmonotonic inference and hence nonmonotonic logic. Whereas to check in concrete cases weather or not the consistent set I of the logic L entails the formula φ nonmonotonicaly i.e. to check $I \mid\!\sim \varphi$ one needs to go through all expansions of I which on practical level is totally impossible to apply except some simple cases. In 1991 Schwarz came up with the method of checking nonmonotonic inference semantically [74]. In fact the minimal model semantics proposed by him can be analogously called Kripke semantics for nonmonotonic logics as far as structures where now the entailment $I \mid\sim \varphi$ is checked are no longer of syntactic nature (like expansions). As we will see the minimal model semantics makes use of standard Kripke semantics of the logic as well. In fact for a given monotonic normal modal logic L to give the minimal model semantics for the corresponding nonmonotonic logic $\mathfrak{L}_L$ one needs to have strong completeness of the logic L with respect to the class of frames C. And moreover it is supposed that C is closed under several conditions which we will discuss in this section. Nevertheless of many requirements minimal model semantics is much more intuitive and flexible to use in practice.

Let us first describe the conditions we want the class of frames to satisfy. The classes of frames which we are going to define are called *cluster closed*. Cluster closed classes are those which provide the general universe where minimal model semantics are defined. To define the cluster closed classes we need the following definitions.

Definition 4.2.1. *Let $\mathcal{N} = (N, S)$ be a Kripke frame. A nonempty set $W \subseteq N$ is called a final cluster if:*
a) W is an upper cone i.e. $x \in W$ and xSy implies $y \in W$,
b) W is a cluster,
c) For every $v \in N - W$ and for every $w \in W$, vRw.

It is immediate that there can not be more then one final clusters for a given frame. Observe that one reflexive point if it is the greatest element of a frame is also a final cluster. Next we define two operations on frames. The intuition behind the first operation is the following: Given a frame from the class take the final cluster (in case it exists) of the frame, delete it, and instead put some other cluster from the same class in place of the deleted cluster.

Definition 4.2.2. *Let $\mathcal{N} = (N, S)$ be a Kripke frame and let N_f be its final cluster. Let $\mathcal{M} = (W, R)$ be a cluster. By cluster substitution of $\mathcal{M}$ in $\mathcal{N}$ we mean the frame $((N - N_f) \cup W, S', V')$, where for each two points $w, v \in (N - N_f) \cup W$ we have $wS'v$ if and only if wSv or $v \in W$. Also one can define cluster substitution for models. In* *fact if we have a valuation U on $\mathcal{N}$ and a valuation V' on $\mathcal{M}$ then in the substituted model V agrees with U on $(N - N_f)$ and agrees with V on W.*

The next operation takes two arbitrary models of the fixed class and puts one on top of the other.

Definition 4.2.3. *By the concatenation of two models (W, R, V) and (N, S, U) where $W \cap N = \varnothing$, we mean the model $(N \cup W, S \cup N \times W \cup R, V')$ where V' agrees with V on W and V agrees with N on U.*

Now at last we are ready to define a cluster closed class. In fact cluster closeness is a property of classes of frames rather then models we do not emphasize this here as far as we will mainly deal with classes of models when defining minimal model semantics.

Definition 4.2.4. *Let C be a class of frames. We say that C is cluster closed if:*

1) C contains all clusters;

2) For each $\mathcal{N} \in C$ at least one of the following two conditions holds: the concatenation of $\mathcal{N}$ and each cluster belongs to C, or $\mathcal{N}$ has a final cluster and for each cluster $\mathcal{M}$, the cluster substitution of $\mathcal{M}$ in $\mathcal{N}$ belongs to C.

The class of all preorders, the class of all universal frames, the class of all reflexive, transitive and one-step frames, The class of all frames do provide fine examples of cluster closed classes. Although there are classes which are not cluster closed. A natural example of such class is the class of all irreflexive one-step frames. Indeed wOS_{irr} from Chapter 3 is not cluster closed as far as it does not contain the class of all clusters (in fact it does not contain even a single cluster) hence the first condition of Definition 4.2.4 fails. Another not cluster closed class of frames, which we consider in section 4.4, is provided by the class wOS_r of all rooted, one-step, weakly transitive frames. For the case of wOS_r the first condition of Definition 4.2.4 is satisfied but the second condition fails. This is because firstly the class of one-step frames is limited in heights therefore it can not be closed under concatenation of two frames and secondly there are frames which do not contain the final cluster (it may for example be that wOS_r frame contains one irreflexive point on the second floor). In Section 4.4 we will see that with small modifications, Schwarz's technique still works for this particular class of all rooted, one-step and weakly transitive frames.

Now we define a preference relation on the class of models. This relation serves as a main tool for defining the minimal model

Definition 4.2.5. *We say that a model* $\mathcal{N} = (N, S, U)$ *is preferred over the universal model (i.e. cluster with valuation)* $\mathcal{M} = (W, R, V)$ *if:*
a) There is a propositional formula ψ *such that* $\mathcal{M} \Vdash \psi$ *and* $\mathcal{N} \nVdash \psi$*,*
b) (W, R) *is the final cluster of* (N, S) *and* V *equals to the restriction of* U *to* W*.*

We next define the notion of minimal model that is central for the semantics of non-monotonic modal logics.

Definition 4.2.6. *An universal model* $\mathcal{M} = (W, R, V)$ *is called a* C*-minimal model for the set of formulas* I *if* $\mathcal{M} \Vdash I$ *and for every preferred model* $\mathcal{N}$ *in the class* C*, we have* $\mathcal{N} \nVdash I$*.*

Now we are ready to state the main fact from [74] which claims that minimal model semantics is adequate semantics for nonmonotonic modal logics.

Fact 4.2.7. *[74] Let a normal modal logic* **L** *which is contained in* **S5** *be sound and strongly complete w.r.t. the cluster closed class of frames* C*. Let* $\mathcal{M} = (W, R, V)$ *be an universal model, and* $T = \{\varphi | \mathcal{M} \Vdash \varphi\}$ *be the theory of* $\mathcal{M}$*. Then* T *is an* L*-expansion of* I *if and only if* $\mathcal{M}$ *is a* C*-minimal model of* I*.*

4.3 Minimal Knowledge

The paradigm of minimal *knowledge* derives from the well-known work of Halpern and Moses, especially [40], later extended and modified by [79, 55, 54] and others. Many approaches are based on Kripke-**S5**-models with a universal accessibility relation and the minimisation of knowledge is represented by maximising the set of possible worlds with respect to inclusion. In general, this has the effect of minimising objective knowledge, ie knowledge of basic facts and propositions. An alternative approach was developed by [75] and can be seen as a special case of the general method of [79] for obtaining different concepts of minimality by changing the sets of models and preference relations between them. The initial models considered by [75] (see also [74]) consist of not one (**S5**) cluster but rather two clusters arranged in such a way that all worlds in one cluster are accessible from all worlds in the other (but not vice versa).

4.3.1 Halpern and Moses method

In [40] Halpern and Moses present a stable set as a knowledge set of perfect reasoner 1.1.1. Stable sets where first introduced in [83]. Let us recall that a set of formulas E is called a *stable set* if E is closed under propositional consequences and if $\varphi \in E$ then $\Box\varphi \in E$ and if $\varphi \notin E$ then $\neg\Box\varphi \in E$. Now the main theorem given in [40] represents stable sets as knowledge sets of perfect reasoner more precisely with every stable set E there is a way to construct an universal model $\mathcal{M}_E$ such that the set $\{\varphi | \mathcal{M}_E \Vdash \Box\varphi\}$ (i.e. the set of facts that are known in $\mathcal{M}_E$ by reasoner) is exactly the stable set E. Also it is easy to check that given an universal model its theory and therefore the set of facts that are known is always a stable set.

Fact 4.3.1. *[40] Every stable set E determines a universal Kripke model $\mathcal{M}_E$ for which $E = \{\varphi | \mathcal{M}_E \Vdash \Box\varphi\}$.*

Now for a given formula φ the question of minimal knowledge is to find the minimal knowledge set of perfect reasoner containing the formula φ. For this it suffices to find the largest universal model validating φ. This easily follows from the following observation: $\mathcal{M} \subseteq \mathcal{M}'$ implies that $\{\varphi | \mathcal{M} \Vdash \Box\varphi\} \supseteq \{\varphi | \mathcal{M}' \Vdash \Box\varphi\}$. Figure 4.1 represents the main idea of the method discussed.

Sometimes (for some formulas) there is a standard procedure to find such largest universal model. The idea is to take the disjoint union $\bigcup_i \mathcal{M}_i$ of all universal models validating the formula and then take the reflexive and transitive closure of the union i.e. extend the relation on $\bigcup_i \mathcal{M}_i$ to universal relation. Unfortunately this is not a solution as far as there are examples of models formulas φ for which the validity is not preserved under the above mentioned operation.

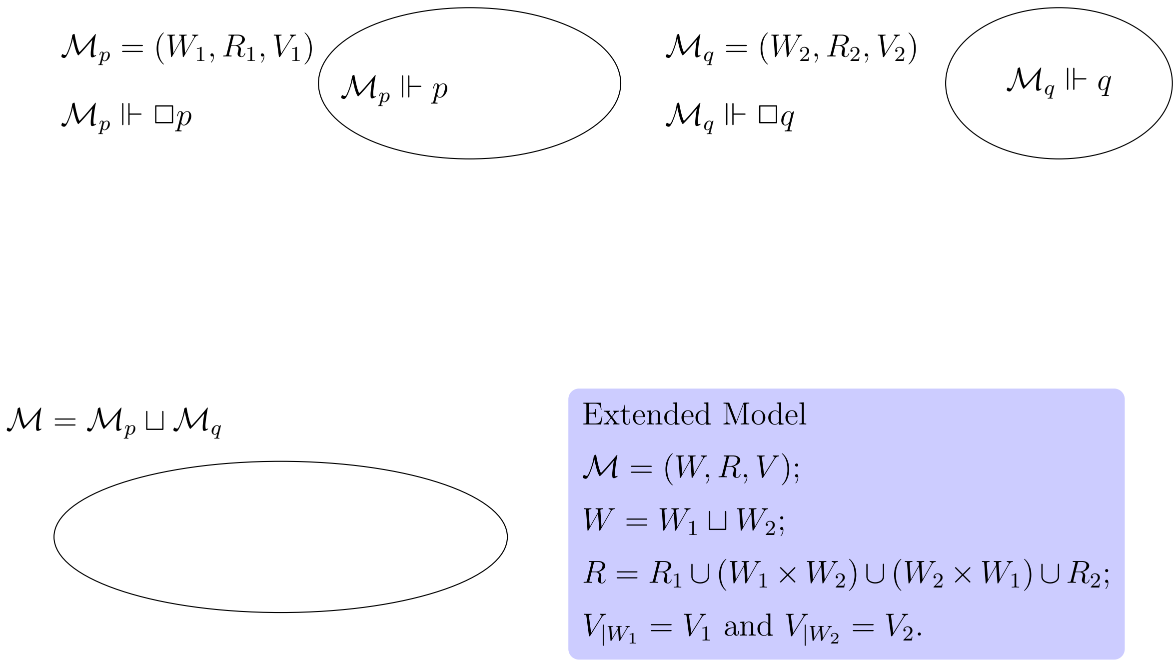

Figure 4.1: Minimisation by extending the model

Example 4.3.2. *Let $\varphi = \Box p \vee \Box q$. In this case if we take the universal model $\mathcal{M}_p$ be the universal model where p is true at every world but for every propositional letter r different from p there exist a world in $\mathcal{M}_p$ where r is falsified. It is clear that $\mathcal{M}_p \Vdash \varphi$ but $\mathcal{M}_p \nVdash \Box q$. Now analogously let $\mathcal{M}_q$ be the universal model where q is true at every world but for every propositional letter r different from q there exist a world in $\mathcal{M}_q$ where r is falsified. Again $\mathcal{M}_q \Vdash \varphi$ but $\mathcal{M}_q \nVdash \Box p$ see Figure 4.1. Now if we take a disjoint union of $\mathcal{M} = \mathcal{M}_p \sqcup \mathcal{M}_q$ with universal relation then $\mathcal{M} \nVdash \varphi$ as far as there exists a world which falsifies p and there exists a world which falsifies q.*

The above discussion lead to the definition of honest formulas which are those for which agent can have the minimal knowledge.

Definition 4.3.3. *A formula φ is honest if from $\mathcal{M} \Vdash \varphi$ and $\mathcal{M}' \Vdash \varphi$ where $\mathcal{M}$ and $\mathcal{M}'$ are universal models, it follows that $\mathcal{M} \sqcup \mathcal{M}' \Vdash \varphi$ where $\mathcal{M} \sqcup \mathcal{M}'$ is the disjoint union of two models where the relation is extended to the universal relation.*

Hence the formula φ from the Example 4.3.2 is an example of dishonest formula.

4.3.2 Truszczynski & Schwarz method

In [75], given a background theory or knowledge set I, minimal knowledge (with respect to I) is captured by an universal **S5**-model of the theory, say $\mathcal{M}$, but now the idea of minimality is that there should be no two-floor model $\mathcal{M}'$ as in Figure 4.2 of the same theory I, where $\mathcal{M}$ coincides with the restriction of $\mathcal{M}'$ to the second floor W_2, and W_1 is smaller in the sense that it fails to verify some objective (non-modal) sentence true in $\mathcal{M}$.

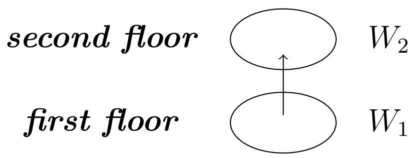

Figure 4.2

In other words the minimisation process starts from an $S5$-model $\mathcal{M}$ where I is a subset of $Th(\mathcal{M})$ (the theory of $\mathcal{M}$) and new worlds are added on the first floor so that I remains subset of the theory of the extended model $\mathcal{M}'$. Of course by construction $\mathcal{M}'$ verifies I but evidently $Th(\mathcal{M}') \subseteq Th(\mathcal{M})$ and hence knowledge set is minimised. Now the minimality says that the model $\mathcal{M}$ is already minimal with respect to I if adding the the new worlds on the first floor (but keeping I true) does not reduce its propositional theory. Schwarz and Truszczyński argue that this approach to minimal knowledge has some important advantages over the method of [40] and they study its properties in depth, in particular showing that while the two-floor models correspond to the modal logic **S4F** first studied by [76], minimal knowledge is precisely captured by non-monotonic **S4F**.

Definition 4.3.4. *We say that a two-floor model $\mathcal{N} = (N, S, U)$ is preferred over $S5$-model $\mathcal{M} = (W, R, V)$ if:*
a) There is a propositional formula ψ such that $\mathcal{M} \Vdash \psi$ and $\mathcal{N} \nVdash \psi$,
b) (W, R) is the second floor of (N, S) and V equals to the restriction of U to the second floor. Briefly, $\mathcal{M}$ is the model which is obtained by deleting the first floor in $\mathcal{N}$.

We next define the notion of minimal knowledge model.

Definition 4.3.5. *An* **S5***-model* $\mathcal{M} = (W, R, V)$ *is called a minimal knowledge model for the set of formulas* I *if* $\mathcal{M} \Vdash I$ *and for every preferred model* $\mathcal{N}$ *we have* $\mathcal{N} \nVdash I$*.*

The following theorem links the idea of minimal knowledge model with the non-monotonic minimal model 4.2.6 in case of the logic **S4F**.

Fact 4.3.6. *[75] An universal Kripke model* $\mathcal{M}$ *is a minimal-knowledge model for* I *if and only if the theory of* $\mathcal{N}$ *is an* **S4F***-expansion of* I*.*

4.4 Minimal Belief

4.4.1 Non-Monotonic wK4f

It is often held that **KD45** represents an adequate logic for belief. One motivation for this is that it allows positive and negative introspection and additionally $\Box p \rightarrow p$ is not derivable in the logic. A Kripke model $\mathcal{M}$ for **KD45** consists of cluster W plus one irreflexive point w so that w is related to every point in W but no point in W is related to w. In other words the first floor of $\mathcal{M}$ is one irreflexive point and the second floor is a cluster. The belief set of an agent is obtained as a theory of the second floor. Indeed $\varphi \in Th(W)$ iff $\mathcal{M} \Vdash \Box\varphi$. In particular minimisation is relative to some base set I of beliefs. The aim is to capture a set which contains I, is closed under positive and negative introspection, is closed under logical deduction and does not contain anything superfluous. One way to obtain such a set is to consider a **KD45** Kripke model $\mathcal{M}$ such that $v \Vdash I$ for every $v \in W$ and extend the cluster W by adding points which still make I true. As a result the set of objective facts true in every world will be reduced while the starting beliefs I will stay unchanged. This approach is applied in [40] to knowledge sets, but as discussed in [89] it has some unintuitive consequences. For this reason we follow the pattern of [89] which relies on the idea that an agent's belief is dependent not only on the objective facts but also on the things that are believed by agent. More concretely we minimise the belief set by adding worlds on the first floor of the model $\mathcal{M}$ leaving the second floor untouched. This form of minimal model semantics provides an alternative way of minimising belief and I-expansions for **wK4f** are exactly minimal belief sets. This is the chief motivation for considering non-monotonic **wK4f** to be a good candidate for the logic of minimal belief.

Formally we want to relate non-monotonic **wK4f** to the idea of *minimal model* introduced and characterised in [74]. However we cannot directly apply the general result (Theorem 3.1) of [74] since that theorem refers to what are called *cluster-closed* logics. Instead we can adapt Schwarz's techniques to our case, starting with the definition of preferred model for a class wOS_r. The preference relation is between one-floor **S5**-models and two-floor models and only two-floor models can be preferred over **S5**-models. For example we can not compare two one-floor models with each other.

Definition 4.4.1. *We say that a two-floor model $\mathcal{N} = (N, S, U)$ is preferred over $S5$-model $\mathcal{M} = (W, R, V)$ if:*
a) There is a propositional formula ψ such that $\mathcal{M} \Vdash \psi$ and $\mathcal{N} \nVdash \psi$,
b) (W, R) is the second floor of (N, S) and V equals to the restriction of U to the second floor. Briefly, $\mathcal{M}$ is the model which is obtained by deleting the first floor in $\mathcal{N}$.

We next define the notion of minimal model that is central for the semantics of non-monotonic modal logics.

Definition 4.4.2. *An* **S5**-*model $\mathcal{M} = (W, R, V)$ is called a wOS_r-minimal model for the set of formulas I if $\mathcal{M} \Vdash I$ and for every preferred model $\mathcal{N} \in wOS_r$ we have $\mathcal{N} \nVdash I$.*

Non-monotonic **wK4f** does not fit the scope of Theorem 3.1 [74] because the class wOS_r which characterises monotonic **wK4f** is not cluster closed. In particular some two-floor models in wOS_r may not have a cluster as a maximum. On the other hand every model in wOS_r has a maximal *weak-cluster* (that is a cluster where irreflexive points are allowed or more precisely it is a rooted, symmetric, weakly-transitive frame). For this reason we need to consider weak-cluster closed classes.

Definition 4.4.3. *Let $\mathcal{N} = (N, S, U)$ be a Kripke model. A nonempty set $W \subseteq N$ is called a final weak-cluster if:*
a) W is an upper cone (def. 2.2.17),
b) W is weak-cluster,
c) For every $v \in N - W$ and for every $w \in W$, vRw.

It is immediate from Definition 4.4.3 and from Theorem 2.2.18 that every rooted, weakly-transitive, one-step frame has a final weak-cluster and it is the second floor (or the only floor) of the frame.

Definition 4.4.4. *Let $\mathcal{N} = (N, S, U)$ be a Kripke model and let N_2 be its final weak-cluster. Let $\mathcal{M} = (W, R, V)$ be a cluster. By cluster substitution of $\mathcal{M}$ in $\mathcal{N}$ we mean the model $((N - N_2) \cup W, S', V')$, where for each $w, v \in (N - N_2) \cup W, wS'v$ if and only if wSv or $v \in W$ and V' agrees with U on $(N - N_2)$ and agrees with V on W. In other words we substitute the cluster W instead of the weak-cluster N_2 into $\mathcal{N}$.*

Definition 4.4.5. *Let C be a class of models. We say that C is weak-cluster closed if C contains all weak-clusters and for each $\mathcal{N} \in C$, at least one of the following two conditions holds: the concatenation of $\mathcal{N}$ and each cluster belongs to C, or $\mathcal{N}$ has a final weak-cluster and for each* **S5***-model $\mathcal{M}$, the cluster substitution of $\mathcal{M}$ in $\mathcal{N}$ belongs to C.*

It is immediate that wOS_r is weak-cluster closed. As usual, non-monotonic modal logics are defined via the notion of expansion 4.1.2. Now we are ready to prove the main theorem of this section.

Theorem 4.4.6. *Let $\mathcal{M} = (W, R, V)$ be an* **S5***-model, and $T = \{\varphi | \mathcal{M} \Vdash \varphi\}$. Then T is an* **wK4f***-expansion of I if and only if $\mathcal{M}$ is a wOS_r-minimal model of I.*

Proof. Assume T is a wK4Df-expansion for I. This means that $T = Cn_L[I \cup \{\neg\Box\varphi | \varphi \notin T\}]$, where L stands for wK4Df. For the contradiction assume $\mathcal{M}$ is not minimal. This means that there is a wK4Df-model $\mathcal{N} = (N, S, U)$ such that $\mathcal{N}$ is preferred over $\mathcal{M}$ and $\mathcal{N} \Vdash I$. That $\mathcal{N}$ is preferred over $\mathcal{M}$ means that there is a propositional formula α such that $\mathcal{M} \Vdash \alpha$ while $\mathcal{N} \nVdash \alpha$. Now take an arbitrary formula $\psi \notin T$. Since T is an expansion we have that $\Diamond\neg\psi \in T$ hence $\mathcal{M} \Vdash \Diamond\neg\psi$. Hence there is at least one point $w \in W$ with $w \Vdash \neg\psi$ and hence for every point y in the first floor of N we have $y \Vdash \Diamond\neg\psi$ which yields that $\mathcal{N} \Vdash \Diamond\neg\psi$. So we get that $\mathcal{N} \Vdash I \cup \{\neg\Box\varphi | \varphi \notin T\}$ and hence $\mathcal{N} \Vdash T$ which is a contradiction since $\alpha \in T$.

For the other direction assume $\mathcal{M}$ is wOS_r-minimal for I. That $Cn_L[I \cup \{\neg\Box\varphi|\varphi \notin T\}] \subseteq T$ follows directly from the fact that $\mathcal{M} \Vdash I$ and if $\mathcal{M} \nVdash \psi$ then there exists at least one point $w \in W$ with $w \Vdash \neg\psi$ and since R is a universal relation, $\mathcal{M} \Vdash \Diamond\neg\psi$.

For the other inclusion we show that for every rooted weakly-transitive and one-step model $\mathcal{N} = (N, S, U)$ the following holds:

$$(*) \quad \mathcal{N} \Vdash Cn_L[I \cup \{\neg\Box\varphi|\varphi \notin T\}] \Rightarrow \mathcal{N} \Vdash T.$$

This by Theorem 2.2.21 will imply that $Cn_L[I \cup \{\neg\Box\varphi|\varphi \notin T\}] \vdash T$ in wK4Dfand, as the left side is closed under consequence, we get that $T \subseteq Cn_L[I \cup \{\neg\Box\varphi|\varphi \notin T\}]$. Now let us prove the star.

Assume $\mathcal{N} \Vdash Cn_L[I \cup \{\neg\Box\varphi|\varphi \notin T\}]$. Note that $\mathcal{N}$ cannot have one irreflexive point as a maximum. This would imply $Ch(N, S) = (i_1, r_1, 1, 0)$, see Theorem 2.2.18. Then the irreflexive point does not satisfy $\neg\Box\bot$, hence $\bot \in T$, which is a contradiction as far a T is the theory of $\mathcal{M}$.

Let us denote the floors of $\mathcal{N}$ by N_1 and N_2 respectively. In case $\mathcal{N}$ is a one-floor frame, $N_2 = \varnothing$. Since wOS_r is weak-cluster closed, there is $\mathcal{N}^* \in wOS_r$ which is either the concatenation of $\mathcal{N}$ and $\mathcal{M}$ or is a cluster substitution of $\mathcal{M}$ in $\mathcal{N}$. We prove by induction on the complexity of a formula that for every point $w \in N_1$, we have $\mathcal{N}^*, w \Vdash \varphi$ iff $\mathcal{N}, w \Vdash \varphi$. The only non-trivial case is for formulas of the form $\Box\varphi$. Assume $\mathcal{N}, w \Vdash \Box\varphi$, then $\varphi \in T$. This means that $\mathcal{M} \Vdash \varphi$. Now for every point $w' \in N_1$ such that wSw' we have $\mathcal{N}, w' \Vdash \varphi$ and hence by the inductive assumption we get that $\mathcal{N}^*, w' \Vdash \varphi$. So $\mathcal{N}^*, w \Vdash \Box\varphi$.

Conversely assume for some point $w \in N_1$ we have $\mathcal{N}^*, w \Vdash \Box\varphi$. By the same argument as in the previous case, for every point $v \in N_1$ such that wSv, $\mathcal{N}, v \Vdash \varphi$. Now if $\mathcal{N}^*$ is a concatenation of $\mathcal{N}$ and $\mathcal{M}$ then $N = N_1$, and hence we have $\mathcal{N}, w \Vdash \Box\varphi$. In case $\mathcal{N}^*$ is cluster substitution we additionally need to show that for every point $v \in N_2$, $\mathcal{N}, v \Vdash \varphi$. From $\mathcal{N}^*, w \Vdash \Box\varphi$ we have that $\mathcal{M} \Vdash \varphi$ and hence

$\mathcal{M} \Vdash \Box\varphi$. This implies that $\neg\Box\varphi \notin T$ and hence $\mathcal{N} \Vdash \Diamond\Box\varphi$. It is not hard to check that this implies that for every point $v \in N_2$ we have $\mathcal{N}, v \Vdash \varphi$. The main point here is that $\mathcal{N}$ cannot have one-irreflexive point as a maximum. Now as $\mathcal{N} \Vdash I$, we have that $\mathcal{N}^* \Vdash I$, hence $\mathcal{N}^*$ is not preferred over $\mathcal{M}$ which implies that $\mathcal{N}^* \Vdash T$. Hence $\mathcal{N} \Vdash T$. □

This yields the promised link between **wK4f** and minimal models in the style of [75, 89]; so non-monotonic **wK4f** may be a promising first step in the search for logics of minimal belief.

4.4.2 The Logic wK4Df

The concept of minimal *belief* was already touched on by [64] in his analysis of autoepistemic logic. It was later shown [88] that autoepistemic logic is in fact equivalent to non-monotonic **KD45**. However, as in the case of models for minimal knowledge, autoepistemic expansions are stable sets and correspond to universal **S5**-models. Our objective here is to arrive at a concept of minimal belief that is based on **KS** (doxastic) models rather than **S5** (knowledge) models. In other respects, however, we want to follow the approach to minimality developed by Schwarz and Truszczyński. In fact we shall see that the minimisation techniques proposed by Halpern are not applicable to the logic **KS**. To accomplish our aim we need a logic that, at the monotonic level, plays a role similar to the of **S4F** in the approach of Schwarz and Truszczyński, namely a logic in which we characterise the idea of minimal model and define a suitable notion of expansion. Evidently this will differ from the usual concept of expansion, since we want it to capture models that are not **S5**. Our base logic will also differ from **KS**, since, like **S4F**, its frames will have a two-floor structure although with irreflexive points allowed. This logic is called **wK4Df**. The monotonic version of it we have studied in Chapter 2.

4.4.3 The Logic of Minimal Belief

We now follow an approach similar to that of [75] and minimise belief sets in the same way as knowledge is minimised. We thereby build a nonmonotonic logic of belief. For the underlying monotonic system we take **wKD4f**. As already remarked, we cannot use the standard method to define a nonmonotonic version of **wKD4f** as this would yield minimal models that are knowledge ie **S5** models. Accordingly we need to alter the usual notions of expansion and nonmonotonic inference. The changed notion is called weak-expansion and serves as a main tool in defining the new inference relation. Let L denote some monotonic modal logic which is contained in **KS**. We first recall the definition of L-consequence relation.

Definition 4.4.7. *Let I be a consistent set of formulas in L. We say that the formula φ is an L-consequence of I ($I \vdash_L \varphi$) if for every L-model $\mathcal{M}$, $\mathcal{M} \Vdash \varphi$ whenever $\mathcal{M} \Vdash I$. We denote by $Cn_L(I)$ the set of all L-consequences of I.*

Definition 4.4.8. *Let L be a (monotonic) modal logic. We say that a set of formulas T is a* weak-expansion *(in L) for the set of formulas I if*

$$T = Cn_L(I \cup \{\neg\Box\varphi \vee \neg\varphi | \varphi \notin T\} \cup \{\neg\Box\varphi | \neg\Box\varphi \in T\}).$$

If L is assumed to be given we usually drop reference to it. Observe that the weak-expansions have properties analogous to normal expansions. In fact if we think of a weak-expansion T as a belief set, we have that agent's beliefs are closed in the following way: (i) If φ is in an agent's belief set then he believes φ. (ii) If φ is not in the agent's belief set then the disjunction $\neg\Box\varphi \vee \neg\varphi$ is present. (iii) If an agent does not believe φ then he believes that he does not believe φ. In the obvious manner weak-expansions can be used to define a nonmonotonic inference relation, $\mid\sim_L$, by setting $I \mid\sim_L \varphi$ if φ is true in all weak-expansions of I.

Now we introduce a preference relation on rooted, one-step and weakly-transitive models in an analogous way to [74]. The preference relation is only between one floor and two floor models and only two floor models can be preferred over one floor models. Recall that here by two floor we mean rooted, weakly-transitive, one-step frames, i.e. we allow weak-clusters on the first and the second floor. See Figure 4.3

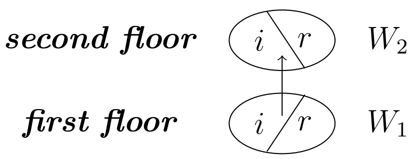

Figure 4.3

Definition 4.4.9. *A two-floor model $\mathcal{M}' = (W', R', V')$ is said to be* preferred *over a one-floor model $\mathcal{M} = (W, R, V)$ (notation: $\mathcal{M}' \succ \mathcal{M}$) if: a) (W, R) is the second floor of (W', R') and V equals to the restriction of V' to the second floor; so $\mathcal{M}$ is the model obtained by deleting the first floor in $\mathcal{M}'$. b) There is a formula ψ such that $\mathcal{M} \Vdash \psi$ and $\mathcal{M}' \nVdash \psi$.*

The standard preference relation proposed by [74] is obtained by requiring the formula ψ in *b*) to be a propositional (modal-free) formula. Nevertheless the following theorem shows that for clusters the two definitions coincide. Let $\mathcal{M}' > \mathcal{M}$ denote the preference relation introduced by Schwarz, applied to the class of all two-floor **wK4Df**-frames.

Theorem 4.4.10. *Let $\mathcal{M}$ be a cluster. Then for every two-floor* **wK4Df***-model $\mathcal{N}$ we have $\mathcal{N} > \mathcal{M}$ iff $\mathcal{N} \succ \mathcal{M}$.*

Proof. We prove the non-trivial direction. Assume $\mathcal{N} \not\succ \mathcal{M}$. Then for every propositional formula α, $\mathcal{M} \Vdash \alpha$ implies $\mathcal{N} \Vdash \alpha$. We show by induction on the modal depth of formula that $\mathcal{M} \Vdash \varphi$ implies $\mathcal{N} \Vdash \varphi$ for every formula φ. The base case is given by assumption. We assume the induction hypothesis holds for every formula with modal depth less than k and show the hypothesis holds for a formula φ with modal depth equal to k. By propositional reasoning we can rewrite φ equivalently by $\varphi \equiv \bigwedge \delta_i$ where each δ_i is of the form $\Box\alpha_1 \vee \Box\alpha_2 \vee .. \vee \Box\alpha_n \vee \neg\Box\beta_1 \vee \neg\Box\beta_2 .. \vee \neg\Box\beta_m \vee \gamma_i$, where each α_j, β_j has modal depth less than k and γ_i is a propositional formula. $\mathcal{M} \Vdash \varphi$ implies that for each i the corresponding disjunct holds in $\mathcal{M}$. So take an arbitrary i from the conjunction and take a disjunct $\Box\alpha_1 \vee \Box\alpha_2 \vee .. \vee \Box\alpha_n \vee \neg\Box\beta_1 \vee \neg\Box\beta_2 .. \vee \neg\Box\beta_m \vee \gamma_i$. If at least at one point v in $\mathcal{M}$, $\neg\Box\beta_j$ holds for some $j \in \{1, ..m\}$ then there is a point v' such that vRv' and $v' \Vdash \neg\beta_j$ hence every point on the first floor of $\mathcal{N}$ can see v' and hence $\neg\Box\beta_j$ holds on the first floor of $\mathcal{N}$. Therefore $\mathcal{N} \Vdash \varphi$. If at least at one point v in $\mathcal{M}$, $\Box\beta_j$ holds for some $j \in \{1, ..n\}$ then every point $w \in \mathcal{M}$ satisfies β_j (here we use that $\mathcal{M}$ is a cluster). Hence $\mathcal{M} \Vdash \beta_j$ and by the induction hypothesis $\mathcal{N} \Vdash \beta_j$. Therefore $\mathcal{N} \Vdash \Box\beta_j$ and hence $\mathcal{N} \Vdash \varphi$. If neither of the above two cases holds then $\mathcal{M} \Vdash \gamma_i$ and again by hypthesis $\mathcal{N} \Vdash \gamma_i$ hence $\mathcal{N} \Vdash \varphi$. So we get $\mathcal{N} \not\succ \mathcal{M}$.⊣

Our aim now is to relate weak-expansions to a notion of *minimal* model. This is exactly like the usual concept ([74]) except that it is based on the new notion of preference that is defined for weak-clusters.

Definition 4.4.11. *A one-floor, rooted* **KS***-model (weak-cluster)* $\mathcal{M} = (W, R, V)$ *is called a* minimal *model for the set of formulas* I *if* $\mathcal{M} \Vdash I$ *and for every preferred model* $\mathcal{M}'$ *we have* $\mathcal{M}' \nVdash I$.

From Theorem 4.4.10 it follows that a *cluster* $\mathcal{M}$ is minimal for the set of formulas I in the usual sense iff $\mathcal{M}$ is minimal for the set of formulas I in a sense of Definition 4.4.11. The following result linking weak-expansions with minimal models is the main theorem of the chapter. Throughout the proof Cn_L will stand for $Cn_{\mathbf{wK4Df}}$.

Theorem 4.4.12. *Let* $\mathcal{M} = (W, R, V)$ *be a weak-cluster and* $T = \{\varphi \mid \mathcal{M} \Vdash \varphi\}$. T *is a weak-expansion for the set of formulas* I *iff* $\mathcal{M}$ *is a minimal model for* I.

Proof. Assume the hypothesis of the theorem and suppose that $\mathcal{M} \Vdash I$. Clearly $\mathcal{M} \Vdash \{\neg\Box\varphi | \neg\Box\varphi \in T\}$. For any $\varphi \notin T$, $\mathcal{M} \nVdash \varphi$, so for some $w \in W, w \Vdash \neg\varphi$ and since $\mathcal{M}$ is a weak-cluster, it follows that $\mathcal{M} \Vdash \neg\Box\varphi \vee \neg\varphi$ and so $\mathcal{M} \Vdash \{\neg\Box\varphi \vee \neg\varphi | \varphi \notin T\}$. Now consider any two-floor model $\mathcal{N} = (N, S, U)$ of I whose second-floor coincides with $\mathcal{M}$, and choose any element x on the first-floor. For any $\neg\Box\varphi \in T$, φ is false at some point v in $\mathcal{M}$ and since xSv, $x \nVdash \Box\varphi$ and so $\mathcal{N} \Vdash \neg\Box\varphi$. For any $\varphi \notin T$, $\neg\varphi$ holds at some point u in $\mathcal{M}$ and, since xSu, $x \Vdash \neg\Box\varphi$. Therefore both models $\mathcal{M}$ and $\mathcal{N}$ satisfy the set of formulas $X = I \cup \{\neg\Box\varphi \vee \neg\varphi | \varphi \notin T\} \cup \{\neg\Box\varphi | \neg\Box\varphi \in T\}$ and therefore also $Cn_L(X)$, and so $Cn_L(X) \subseteq T$. Now if T is a weak-expansion for I, then $T = Cn_L(X)$, consequently $\mathcal{N}$ is not preferred to $\mathcal{M}$ and so $\mathcal{M}$ is a minimal model for I.

For the other direction, let $\mathcal{M}$ be a minimal model for I. With X as above we have already shown that $Cn_L(X) \subseteq T$. We now show that every **wK4Df**-model which validates $Cn_L(X)$ also validates T. This by Theorem 2.2.21 will imply that $Cn_L(X) \vdash T$ in **wK4Df** so $T \subseteq Cn_L(X)$. Now let us prove the assumption. Suppose for some **wK4Df**-model $\mathcal{N} = (N, S, U)$ we have $\mathcal{N} \Vdash Cn_L(X)$. We have two subcases:

Case 1 $\mathcal{N}$ is a one-floor frame. Assume $\varphi \in T$. Then $\Box\varphi \in T$. This implies that $\mathcal{M} \Vdash \Box\varphi$ and since $\mathcal{M}$ is a weak-cluster and $\mathcal{M}$ is not one irreflexive point we have $\mathcal{M} \Vdash \Diamond\Box\varphi$. This implies that $\Diamond\Box\varphi \in T$. Now as $\Diamond\Box\varphi \equiv \neg\Box\neg\Box\varphi$, by the definition of weak-expansions we conclude that $\Diamond\Box\varphi \in Cn_L(X)$. This implies that $\mathcal{N} \Vdash \Diamond\Box\varphi$. Now because $\mathcal{N}$ is a one-floor frame it is symmetric, yielding $N \Vdash \varphi$.

Case 2 $\mathcal{N}$ is a two-floor **wK4Df**-model. Let us denote the floors by N_1 and N_2 respectively. Let $\mathcal{N}_\mathcal{M}$ be the model obtained by substituting $\mathcal{M}$ in $\mathcal{N}$ instead of the second floor N_2. Let us show by induction on the complexity of formulas that for every formula φ and for every point $v \in N_1$ we have $\mathcal{N}_\mathcal{M}, v \Vdash \varphi$ iff $\mathcal{N}, v \Vdash \varphi$. The only non-trivial case is when the formula is of the form $\Box\varphi$. Suppose for some $v \in N_1$, $\mathcal{N}, v \Vdash \Box\varphi$. This means that for every $v' \in \mathcal{N}$ if $v \neq v'$ then $v' \Vdash \varphi$. Besides by the induction hypothesis it means that for every $v' \in N_1$ such that $v \neq v'$, $\mathcal{N}_\mathcal{M}, v' \Vdash \varphi$. Assume for the contradiction that $\mathcal{N}_\mathcal{M}, v \nVdash \Box\varphi$. This means there is a point $w \in \mathcal{M}$ with $w \nVdash \varphi$. That $\mathcal{M} \nVdash \varphi$ means that $\varphi \notin T$ and hence $\Diamond\neg\varphi \vee \neg\varphi \in Cn_L(X)$. So we obtain $\mathcal{N} \Vdash \Diamond\neg\varphi \vee \neg\varphi$. But this yields a contradiction as for every $u \in N_2$ we have that $u \Vdash \varphi$ so no point in N_2 satisfies the formula $\Diamond\neg\varphi \vee \neg\varphi$.

Conversely assume for every $v \in N_1$, $\mathcal{N}_\mathcal{M}, v \Vdash \Box\varphi$. This implies that $\mathcal{M} \Vdash \varphi$ and $\mathcal{M} \Vdash \Box\varphi$ as $\mathcal{M}$ is a final weak-cluster. Therefore $\mathcal{M} \Vdash \Diamond\Box\varphi$ and hence $\Diamond\Box\varphi \in T$. So $\Diamond\Box\varphi \in Cn_L(X)$. Hence $\mathcal{N} \Vdash \Diamond\Box\varphi$. This implies that for every $u \in N_2$, $u \Vdash \Diamond\Box\varphi$ hence $u \Vdash \varphi$. As for the points in N_1, by the induction hypothesis $v' \neq v$ implies that $v' \Vdash \varphi$. So we get that $\mathcal{N}, v \Vdash \Box\varphi$.

From what we have already proved we see that $\mathcal{N}_\mathcal{M} \Vdash I$ and as $\mathcal{M}$ is I-minimal we have that $\mathcal{N}_\mathcal{M}$ is not preferred over $\mathcal{M}$ which by Definition 4.4.9 implies that $\mathcal{N}_\mathcal{M} \Vdash T$. Now applying again the fact that the two models validate the same formulas on the first floors, we get that $\mathcal{N}, v \Vdash T$ for every $v \in N_1$. Now let us take an arbitrary $\varphi \in T$. As T is closed under consequence, $\Box\varphi \in T$ so for every point $v \in N_1$ we have $\mathcal{N}, v \Vdash \Box\varphi$, hence for every $u \in N_2$, $\mathcal{N}, u \Vdash \varphi$. This means that $\mathcal{N} \Vdash \varphi$. Hence $\mathcal{N} \Vdash T$. $\dashv$

Weak-expansions may exist where no corresponding expansion does, as the following example shows.

Example 4.4.13. *Let $I = \{\Box p \vee \Box q\}$. It is easy to check that I has no autoepistemic expansions nor any* **wK4Df** *expansions. However let $\mathcal{M} = (W, R, V)$ be the* **KS***-model where $W = \{w, v\}$, $R = \{(w, v), (v, w)\}$ and $w \Vdash p \wedge \neg q, v \Vdash q \wedge \neg p$. Then $\mathcal{M}$ is minimal for I since $\mathcal{M} \Vdash I$ and for every preferred model at any point on the first floor both $\Box p$ and $\Box q$ (hence $\Box p \vee \Box q$) are falsified.*

This also illustrates how weak-expansions need not contain instances of the **T** axiom $\Box p \to p$. Notice that I is an example (Example 4.3.2) of what [40] calls a dishonest formula.

Though weak-expansions may exist where expansions do not, as a consequence of Theorem 4.4.10 we have the following simple relationship. Let $\mid\!\sim$ be the usual nonmonotonic inference relation for **wK4Df** based on expansions, and let $\mid\!\sim_{wK4Df}$ be the inference relation defined earlier.

Corollary 4.4.14. *For any formula φ and set of formulas I such that $I \mid\!\sim \varphi$ is defined, $I \mid\!\sim_{wK4Df} \varphi \Rightarrow I \mid\!\sim \varphi$.*

We may view this as an indication that reasoning with doxastic (**KS**) models is more skeptical than reasoning with epistemic (**S5**) models.

4.5 Conclusions of Chapter 4

We have presented a new approach to minimal belief and reasoning in nonmonotonic doxastic logic in which **S5** epistemic models are replaced by **KS** models and a weaker notion of expansion is employed. **KS** does not validate the **T** axiom and at least on one reading can be seen as a logic of objective and fair belief. In conclusion we remark that by the so-called splitting translation nonmonotonic **S4F** can be embedded in nonmonotonic **wK4Df**. As we know from the last section of Chapter 2 this translation, replaces each formula of the form $\Box p$ by $(p \wedge \Box p)$ and in non monotonic case yields: $I \mid\!\sim_{S4F} \varphi \Leftrightarrow Sp(I) \mid\!\sim_{wKD4f} Sp(\varphi)$. This is another representation of the idea of minimal knowledge as a form of true minimal belief.

Chapter 5

Logics of Belief and Knowledge for Many Agents

It is a well adopted direction in modal logic to study epistemic and doxastic properties of agents with certain, intuitively acceptable, restrictions on their knowledge and belief. The book by Smullyan [81] where he introduced various types of agents based on properties of belief could serve as a good reference. An agent whose belief satisfies the modal axiom (4) : $\Box p \to \Box\Box p$, translated as 'If the agent believes p, then he believes that he believes p', is called a *normal agent*. **K4** is the modal logic which formalizes the belief behavior of normal agents. The modal logic **K4** generalizes the classical doxastic system **KD45** in the same way as **S4** generalizes the epistemic logic **S5**, by dropping some restrictions on the properties of an agent. There is a tight connection between **S4** and **K4**. This connection is made precise by the splitting translation $Sp(\cdot)$ [25], which maps $\Box p$ to $Sp(\Box p) = \Box p \wedge p$. It is well known that the splitting translation preserves the validity of modal formulas. In particular we have: $\vdash_{S4} \Box\varphi$ iff $\vdash_{K4} Sp(\Box\varphi)$, which reflects the idea of Plato's definition: 'knowledge is justified true belief'.

The main modern interest in epistemic and doxastic logics is related with analyzing iterated concepts of belief and knowledge of agents. For the purpose of communication and agreement it is important for agents to reason about themselves and what others know or believe. Perhaps the most interesting examples in this direction are provided by the notions of common knowledge and common belief. We will denote the operators for common knowledge and common belief by C^G_K and C^G_B respectively. We have: $C^G_K\varphi$ iff φ is common knowledge in the group G and $C^G_B\varphi$ iff φ is common belief in the group G. Common knowledge, originally defined by Lewis [52], has been extensively studied from various perspectives in philosophy [5], [2], game theory [91], artificial intelligence [45], modal logic [3], [4] etc. Theories of common belief can be found in [6], [45], [84]. Here we focus on the common belief of normal agents, i.e. agents whose individual beliefs satisfy the axioms of **K4**. For ease of exposition we restrict ourselves to the two agent case. Let G consist of two agents, agent-1 and agent-2. For this particular case as far as we will deal with the fixed group which consists only of two agents agent-1 and agent-2 we will omit the superscript G in the operators and just write C_K for common knowledge and C_B for common belief in the group $G = \{agent-1, agent-2\}$. Referring to [5], we choose the equilibrium definition of common belief as a basic one. The idea of equilibrium can be formalized in the modal μ-calculus [11] notation in the following way: $C_B\varphi = \nu.p(\Box_1\varphi \wedge \Box_2\varphi \wedge \Box_1 p \wedge \Box_2 p)$. Another existing definition of common belief is given as an infinite conjunction of iterated individual beliefs: $\Box_1\varphi\wedge\Box_2\varphi\wedge\Box_1\Box_1\varphi\wedge\Box_1\Box_2\varphi\wedge\Box_2\Box_1\varphi\wedge\Box_2\Box_2\varphi\wedge\Box_1\Box_1\Box_1\varphi\wedge \Box_1\Box_1\Box_2\varphi$... whereas in [7] it is shown that on Kripke structures the two definitions coincide. In section 2, we describe the syntax and Kripke semantics of the modal logic of common belief for normal agents (we denote this logic by $\mathbf{K4_2^C}$) and prove the completeness and finite model properties with respect to the described semantics. Moreover, we show that $\mathbf{K4_2^C}$ has the tree model property.

The second topic considered is the topological semantics for common belief. The topological semantics for the logics of knowledge and belief always attracted the attention of modal logicians by providing richer examples for modeling philosophical ideas. A good illustration of the preference of topological semantics over Kripke semantics is provided by the result of van Benthem - Sarenac [7], where two "Kripke undistinguishable" definitions of common knowledge are distinguished over topological spaces. In section 3 we give the topological semantics of $\mathbf{K4_2^C}$ based on the work by Esakia [24]. After McKinsey and Tarski [61] suggested to treat modality as the derivative operator of a topological space, Esakia introduced **wK4**, the modal logic of all topological spaces, with the desired (derivative operator) interpretation of the $\Diamond$-modality. **K4** is an extension of **wK4** and is characterized in this semantics by the class of all T_D-spaces [8]. By combining the ideas and results from [7] and [24], we obtain a derived set semantics for the logic of common belief based on bi-topological spaces, where the modality for common belief operates on the intersection of the two topologies. As a main result, we prove that $\mathbf{K4_2^C}$ is sound and complete with respect to the special subclass of all bi-topological T_D-spaces.

In section 4 we discuss the splitting translation and show that Sp is a validity preserving translation from $\mathbf{S4_2^C}$ to $\mathbf{K4_2^C}$. This extends Plato's definition to the following: 'common knowledge is (justified) true common belief'.

5.1 Modal Logic $\mathbf{S4_2^C}$

Common knowledge, originally defined by Lewis [52], has been extensively studied from various perspectives in philosophy [5], [2], game theory [91], artificial intelligence [45], modal logic [3], [4] etc

Definition 5.1.1. *The normal modal logic* $\mathbf{S4_2^C}$ *is defined in a modal language with infinite set of propositional letters* $p, q, r..$ *and connectives* $\vee, \wedge, \neg, \Box_1, \Box_2, C_K$*, where the formulas are constructed in a standard way.*

- *The axioms are all classical tautologies, each box satisfies all* **S4** *axioms and in addition we have equilibrium axiom for common knowledge operator:*

$$(equi) : C_K p \leftrightarrow p \wedge \Box_1 C_K p \wedge \Box_2 C_K p$$

- *The rules of inference are: Modus-ponens, Substitution, Necessitation for* $\Box_1$ *and* $\Box_2$ *and the induction rule:*

$$(ind) : \frac{\vdash \varphi \to \Box_1(\varphi \wedge \psi) \wedge \Box_2(\varphi \wedge \psi)}{\vdash \varphi \to C_K \psi}$$

for an arbitrary formulas φ *and* ψ *of the language.*

The Kripke semantics for the modal logic $\mathbf{S4_2^C}$ is provided by the reflexive and transitive, bi-relational Kripke frames. For interpreting the common knowledge operator $C_{_K}$, the reflexive, transitive closure of a union relation is used.

Definition 5.1.2. *The reflexive, transitive closure $R^\star$ of a relation $R \subseteq W \times W$ is defined in the following way: $R^\star = R^+ \cup \{(w,w) | w \in W\}$.*

The satisfaction of formulas is definition in the following way.

Definition 5.1.3. *For a given bi-relational Kripke model $\mathcal{M} = (W, R_1, R_2, V)$ the satisfaction of a formula at a point $w \in W$ is defined inductively as follows:*
$w \Vdash p$ *iff* $w \in V(p)$,
$w \Vdash \alpha \wedge \beta$ *iff* $w \Vdash \alpha$ *and* $w \Vdash \beta$,
$w \Vdash \neg\alpha$ *iff* $w \nVdash \alpha$,
$w \Vdash \Box_i \varphi$ *iff* $(\forall v)(wR_i v \Rightarrow v \Vdash \varphi)$,
$w \Vdash C_{_K}\varphi$ *iff* $(\forall v)(w(R_1 \cup R_2)^\star v \Rightarrow v \Vdash \varphi)$.

Fact 5.1.4. *[26] The modal logic $\mathbf{S4_2^C}$ is sound and complete with respect to the class of all finite, reflexive, bi-transitive Kripke frames.*

5.2 Modal Logic $\mathbf{K4_2^B}$

We turn to the syntax and Kripke semantics of the logic $\mathbf{K4_2^C}$. The interpretation of common belief operator $C_{_B}$ on bi-relational Kripke frames is similar to the interpretation of the common knowledge operator $C_{_K}$, and is based on the notion of transitive closure of a relation. In this section we show that the logic $\mathbf{K4_2^C}$ is sound and complete with respect to the class of all bi-relational transitive Kripke structures. The proof is a slight modification of the completeness proof for the logic $\mathbf{S4_2^C}$ given in [26] therefore we only sketch the essential parts where the difference shows up. Additionally we show that every non-theorem of $\mathbf{K4_2^C}$ can be falsified on an infinite, irreflexive, bi-transitive tree. Moreover in second part of this section we give a topological study of the logic $\mathbf{K4_2^C}$ and prove the main result of the chapter which is topological completeness of the logic $\mathbf{K4_2^C}$ with respect to the class of all T_D-intersection closed bi-topological spaces.

5.2.1 Iterative Common Belief

There are different notions of common belief [5]. Let us mention common belief as an infinite conjunction of nested beliefs and common belief as an equilibrium. Under the former idea, a proposition p is a common belief of two agents if: agent-1 believes that p and agent-2 two believes that p and agent-1 believes that agent-2 believes that p and agent-2 believes that agent-1 believes that p etc., where all possible finite mixtures occur. If we formalize this idea in a modal language with belief operators $\Box_1$ and $\Box_2$ for each agent respectively, then we arrive at the following concept of a common belief operator C^{ω}_{B}.

$$C^{0}_{B}p = \Box_1 p \wedge \Box_2 p;$$
$$C^{n+1}_{B}p = \Box_1 C^{n}_{B}p \wedge \Box_2 C^{n}_{B}p;$$
$$C^{\omega}_{B}p = \textstyle\bigwedge_{n\in\omega} C^{n}_{B}p.$$

C^{ω}_{B} exactly formalizes the intuition behind the former idea of common belief. However, since C^{ω}_{B} is an infinite intersection, it cannot be expressed as an ordinary formula of modal logic and hence studied in the usual approaches to standard modal logic. Nevertheless it turns out that we can capture the infinitary behavior of C^{ω}_{B} in a finitary sense. This idea is made more precise via the modal logic $\mathbf{K4^{C}_{2}}$.

5.2.2 Syntax

Throughout we work in the modal language $\mathcal{L}_{\mathcal{C}}$ with an infinite set $Prop$ of propositional letters and symbols $\wedge, \neg, \Box_1, \Box_2, C_B$. The set of formulas $Form$ is constructed in a standard way: $Prop \subseteq Form$. If $\alpha, \beta \in Form$ then $\neg\alpha, \alpha \wedge \beta, \Box_1\alpha, \Box_2\alpha, C_B\alpha \in Form$. We will use standard abbreviations for disjunction and implication, $\alpha \vee \beta \equiv \neg(\neg\alpha \wedge \neg\beta)$ and $\alpha \to \beta \equiv \neg\alpha \vee \beta$.

• The axioms of the logic $\mathbf{K4^{C}_{2}}$ are all classical tautologies, each box satisfies all **K4** axioms, ie. we have: (K) $\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$, (4) $\Box_i p \to \Box_i\Box_i p$, for each $i \in \{1, 2\}$ and in addition we have the equilibrium axiom for the common belief operator:

$$(equi) : C_B p \leftrightarrow \Box_1 p \wedge \Box_2 p \wedge \Box_1 C_B p \wedge \Box_2 C_B p.$$

• The rules of inference are: Modus-Ponens, Substitution, Necessitation for $\Box_1$ and $\Box_2$ and the induction rule for the common belief operator:

$$(ind): \frac{\vdash \varphi \to \Box_1(\varphi \wedge \psi) \wedge \Box_2(\varphi \wedge \psi)}{\vdash \varphi \to C_B \psi}$$

where φ and ψ are arbitrary formulas of the language.

5.2.3 Kripke Semantics

The Kripke semantics for the modal logic $\mathbf{K4_2^C}$ is provided by transitive, bi-relational Kripke frames. The triple (W, R_1, R_2), with W an arbitrary set and $R_i \subseteq W \times W$ where $i \in \{1, 2\}$, is a *bi-transitive Kripke frame* if both R_1 and R_2 are transitive relations. A quadruple (W, R_1, R_2, V) is a bi-transitive Kripke model if (W, R_1, R_2) is a bi-transitive Kripke frame and $V : Prop \to P(W)$ is a valuation function. Observe that we only have two relations, which give a semantics for $\Box_1$ and $\Box_2$. To interpret the common belief operator, C_B, we construct a new relation, which is a transitive closure of the union of R_1 and R_2.

Definition 5.2.1. *The transitive closure R^+ of a relation R is defined as the least transitive relation containing the relation R.*

Two points x and y are related by the transitive closure of the relation if there exists a finite path $\langle x_1, .., x_n \rangle$ starting at x and ending at y.

Definition 5.2.2. *For a given bi-relational Kripke model $\mathcal{M} = (W, R_1, R_2, V)$ the satisfaction of a formula at a point $w \in W$ is defined inductively as follows:*
$w \Vdash p$ *iff* $w \in V(p)$,
$w \Vdash \alpha \wedge \beta$ *iff* $w \Vdash \alpha$ *and* $w \Vdash \beta$,
$w \Vdash \neg\alpha$ *iff* $w \nVdash \alpha$,
$w \Vdash \Box_i \varphi$ *iff* $(\forall v)(w R_i v \Rightarrow v \Vdash \varphi)$,
$w \Vdash C_B \varphi$ *iff* $(\forall v)(w (R_1 \cup R_2)^+ v \Rightarrow v \Vdash \varphi)$.
A formula α is valid in a model $\mathcal{M}$, in symbols $\mathcal{M} \Vdash \alpha$, if for every point $w \in W$ we have $w \Vdash \alpha$. α is valid in a bi-relational frame $\mathcal{F} = (W, R_1, R_2)$, in symbols $\mathcal{F} \Vdash \alpha$, iff α is valid in every model $\mathcal{M} = (\mathcal{F}, V)$ based on the frame. α is valid in a class of bi-relational frames K if for every frame $\mathcal{F} \in K$ we have $\mathcal{F} \Vdash \alpha$.

Proposition 5.2.3. *(Completeness) Modal logic* $\mathbf{K4_2^C}$ *is sound and complete with respect to the class of all finite, bi-transitive Kripke frames.*

Proof. Let us first show siundness. The only non-trivial cases are to show that equilibrium axiom and the induction rule hold in the class of all bi-transitive models. Let $\mathcal{M} = (W, R_1, R_2, V)$ be an arbitrary bi-transitive Kripke model. And let $w \in W$. Assume $w \Vdash C_B\varphi$. Let us first show that $w \Vdash \Box_1\varphi$. Take arbitrary $v \in W$ such that wR_1v. This implies that $w(R_1 \cup R_2)^*v$ hence $v \Vdash \varphi$. Let us show that $w \Vdash \Box_1 C_B\varphi$. Take arbitrary v and v' such that wR_1v and $v(R_1 \cup R_2)^*v'$. By definition 5.2.1 this means that there exists a finite path $\langle v_1, .., v_n\rangle$ such that each $v_i(R_1 \cup R_2)v_{i+1}$ and $v_1 = v$ and $v_n = v'$. Then the new path $\langle w, v_1, .., v_n\rangle$ is also finite going from w to v'. Hence $w(R_1 \cup R_2)^*v$ which implies that $v \Vdash \varphi$. In the same way we prove that $w \Vdash \Box_2\varphi \wedge \Box_2 C_B\varphi$.

For the other direction assume $w \nVdash C_B\varphi$. By definition 5.2.2 this means that there is a finite path $\langle v_1, .., v_n\rangle$ such that each $v_i(R_1 \cup R_2)v_{i+1}$ and $v_1 = w$ and $v_n \nVdash \varphi$. Without loss of generality we may assume that $v_1R_1v_2$. In case $n = 2$ we have that $w \nVdash \Box_1\varphi$. In case $n > 2$ we have that $v_2 \nVdash C_B\varphi$ hence $w \nVdash \Box_1 C_B\varphi$.

Now let us show that induction rule preserves validity of formulas in a model. We show this by contraposition. Assume for some $\mathcal{M} = (W, R_1, R_2, V)$ we have $\mathcal{M} \nVdash p \to C_Bq$. This means that there is a point $w \in W$ with $w \Vdash p$ and $w \nVdash C_Bq$. This implies that there is a finite path $\langle w, v_1, .., v_n\rangle$ starting from w with $v_n \nVdash q$. Now we look at v_{n-1}. Since $v_{n-1}(R_1 \cup R_2)v_n$ we have that $v_{n-1} \nVdash \Box_1(p \wedge q) \wedge \Box_2(p \wedge q)$. Now in case $v_{n-1} \Vdash p$ we get that $v_{n-1} \nVdash p \to \Box_1(p \wedge q) \wedge \Box_2(p \wedge q)$ hence $\mathcal{M} \nVdash p \to \Box_1(p \wedge q) \wedge \Box_2(p \wedge q)$. In case $v_{n-1} \nVdash p$ we repeat the procedure and move to v_{n-2}. By repeating this at most $n - 1$ times, either we find the point which falsifies $p \to \Box_1(p \wedge q) \wedge \Box_2(p \wedge q)$ or get that $v_1 \nVdash p$. The last implies that $w \nVdash p \to \Box_1(p \wedge q) \wedge \Box_2(p \wedge q)$.

Before starting the completeness proof we introduce the special closure of a set of subformulas of a given formula. This set will serve as the carrier set for the Kripke model we construct to falsify a formula which is not a theorem of $\mathbf{K4_2^C}$. Assume φ is an arbitrary formula. Let $Sub(\varphi)$ be the set of all sub-formulas of φ. Let $Sub^+(\varphi)$ denote the closure of $Sub(\varphi)$ in the following way: if $C_B\alpha \in Sub^+(\varphi)$ then the formulas $\Box_1\alpha$, $\Box_2\alpha$, $\Box_1 C_B\alpha$ and $\Box_2 C_B\alpha$ are also in $Sub^+(\varphi)$. Let $\sim Sub^+(\varphi)$ denote the closure of $Sub^+(\varphi)$ under single negation. From readability motivations let us denote this set by $FL(\varphi)$ (Another motivation for $FL(\varphi)$ is that this construction is very much alike to Fisher-Ladner closure used in completeness proof for propositional dynamic logic PDL [28]).

Now we are ready to prove completeness.

Assume $\mathbf{K4_2^C} \nvdash \varphi$. Let W be the set of all maximally consistent subsets of $FL(\varphi)$. Let us define the relations R_1 and R_2 on W in the following way: For every $\Gamma, \Gamma' \in W$ we define $\Gamma R_x \Gamma'$ iff $(\forall\alpha)(\Box_x\alpha \in \Gamma \Rightarrow \Gamma' \vdash \alpha \wedge \Box_x\alpha)$, where $x \in \{1, 2\}$.

Claim 5.2.4. *Each R_x is transitive.*

Proof. Assume $\Gamma R_x \Gamma' \wedge \Gamma' R_x \Gamma''$ and $\Box_x\alpha \in \Gamma$. This implies that both $\Box_x\alpha$ and α are in $FL(\varphi)$. By definition of $\Gamma R_x \Gamma'$ we have $\Gamma' \vdash \alpha \wedge \Box_x\alpha$, which implies $\Gamma' \vdash \Box_x\alpha$. As $\Box_x\alpha \in FL(\varphi)$ and Γ' is maximally consistent set, we get $\Box_x\alpha \in \Gamma'$. Now we use again the definition of $\Gamma' R_x \Gamma''$ and we get that $\Gamma'' \vdash \alpha \wedge \Box_x\alpha$. Hence $\Gamma R_x \Gamma''$ □

So far we defined a finite set W with two transitive relations R_1, R_2 on it. Let $R_{1\vee 2}$ denote the transitive closure of the union of relations R_1 and R_2 ie., $R_{1\vee 2} = (R_1 \cup R_2)^*$. At this point we are able to prove the truth lemma with respect to the model $\mathcal{M} = (W, R_1, R_2, R_{1\vee 2}, V)$, where $\Gamma \Vdash p$ iff $p \in \Gamma$. The proof goes in analogy with the proof for common knowledge operator given in [26].

Lemma 5.2.5. *(Truth) For every formula $\alpha \in FL(\varphi)$ and every point $\Gamma \in W$ of the model $\mathcal{M}$, the following equivalence holds: $\Gamma \Vdash \alpha$ iff $\alpha \in \Gamma$.*

The proof goes by induction on the length of formula. Base case follows immediately from the definition of valuation. Assume for all $\alpha \in FL(\varphi)$ with length less then k it holds that: $\Gamma \Vdash \alpha$ iff $\alpha \in \Gamma$.

Let us prove the claim for $\alpha \in FL(\varphi)$ with length equal to k. If α is conjunction or negation of two formulas then the result easily follows from the definition of satisfaction relation and the properties of maximal consistent sets, so we skip the proofs. Assume $\alpha = \Box_1\beta$ and assume $\Gamma \Vdash \alpha$. Take a set $B = \{\gamma : \Box_1\gamma \in \Gamma\} \cup \{\Box_1\gamma : \Box_1\gamma \in \Gamma\} \cup \{\neg\beta\}$. The sub claim is that B is inconsistent. Assume not, then there exists $\Gamma' \in W$ such that $\Gamma' \supseteq B$. This by definition of the relation R_a means that $\Gamma R_1 \Gamma'$. This is because for every α if $\Box_1\alpha \in \Gamma$ then $\Gamma' \vdash \alpha$ and $\Gamma' \vdash \Box_1\alpha$ and hence $\Gamma' \vdash \alpha \wedge \Box_1\alpha$. Now as $\neg\beta \in \Gamma'$, by inductive assumption we get $\Gamma' \Vdash \neg\beta$. Hence we get a contradiction with our assumption that $\Gamma \Vdash \Box_1\beta$. So B is inconsistent. This means that there exists $\gamma_{i_1}, \gamma_{i_2}, ...\gamma_{i_n}, \Box_1\gamma_{j_1}, ..\Box_1\gamma_{j_m} \in B$ such that $\vdash \gamma_{i_1} \wedge \gamma_{i_2} \wedge .. \wedge \gamma_{i_n} \wedge \Box_1\gamma_{j_1} \wedge .. \wedge \Box_1\gamma_{j_m} \to \beta$. Now we take the bigger conjunct, in particular we add $\Box\gamma_i$ for every γ_i occurring in the conjunction, so we get: $\vdash (\gamma_{i_1} \wedge \Box_1\gamma_{i_1}) \wedge (\gamma_{i_2} \wedge \Box_1\gamma_{i_2}) \wedge .. \wedge (\gamma_{i_n} \wedge \Box_1\gamma_{i_n}) \wedge \Box_1\gamma_{j_1} \wedge .. \wedge \Box_1\gamma_{j_m} \to \beta$. Applying necessitation rule for $\Box_1$ and using axiom 4 we get $\vdash \Box_1\gamma_{i_1} \wedge .. \wedge \Box_1\gamma_{i_n} \wedge \Box_1\gamma_{j_1} \wedge .. \wedge \Box_1\gamma_{j_m} \to \Box_1\beta$ so $\Gamma \vdash \Box_1\beta$, hence as $\Box_1\beta \in FL(\varphi)$ we get that $\Box_1\beta \in \Gamma$.

We just showed the left-to-right direction of our claim for $\alpha = \Box_1\beta$. For the right-to-left implication assume $\Box_1\beta \in \Gamma$. By definition of R_1 for every Γ' with $\Gamma R_1 \Gamma'$ we have $\Gamma' \vdash \beta \wedge \Box_1\beta$. From this it follows that $\Gamma' \vdash \beta$. As $\beta \in FL(\varphi)$ we imply $\beta \in \Gamma'$ so by inductive assumption $\Gamma' \Vdash \beta$.

The most important case is when α is of the form $C_B\beta$. Assume $\Gamma \Vdash \alpha$. Let $D = \{\Gamma \in W : \Gamma \Vdash C_B\beta\}$ and let $\delta = \bigvee_{\Gamma \in D} \hat{\Gamma}$, where $\hat{\Gamma}$ is the conjunction of all formulas inside Γ. Observe that as W is finite $\hat{\Gamma}$ is a formula in our language. We want to show that $\vdash \delta \to \Box_1(\delta \wedge \beta) \wedge \Box_2(\delta \wedge \beta)$. We do it part by part.

First we show $\vdash \delta \to \Box_1\beta$. This follows by an analogous argument as in previous claim. So let us take $B = \{\gamma : \Box_1\gamma \in \Gamma\} \cup \{\Box_1\gamma : \Box_1\gamma \in \Gamma\} \cup \{\neg\beta\}$. This set is inconsistent, otherwise there would exist $\Gamma' \in W$ with $\Gamma R_1 \Gamma'$ and $\Gamma' \nVdash \beta$, which contradicts $\Gamma \Vdash C_B\beta$. From the inconsistency of B by the same argument as in the first claim, follows that $\vdash \hat{\Gamma} \to \Box_1\beta$. As Γ was chosen arbitrarily we have $\vdash \delta \to \Box_1\beta$. Analogously we get $\vdash \delta \to \Box_2\beta$.

Now let us show that $\delta \to \Box_1\delta$. For this we take arbitrary $\Gamma \in D$ and arbitrary $\Gamma' \notin D$ and show $\vdash \hat{\Gamma} \to \Box_1\neg\hat{\Gamma}'$. As $\Gamma \in D$ we have that $\Gamma \Vdash C_B\beta$ while for Γ' we have $\Gamma' \nVdash C_B\beta$. This implies that $\Gamma \not R_1 \Gamma'$, so by definition of R_1, there is a formula ψ, such that $\Box_1\psi \in \Gamma$, while $\Gamma' \nvdash \Box_1\psi \wedge \psi$. From $\Gamma' \nvdash \Box_1\psi \wedge \psi$ we conclude that $\Box_1\psi \notin \Gamma'$ or $\psi \notin \Gamma'$. Now as both ψ and $\Box_1\psi$ are in $FL(\varphi)$ we have $\neg\Box_1\psi \in \Gamma'$ or $\neg\psi \in \Gamma'$. This means that $\hat{\Gamma}'$ has the form either $\neg\Box_1\psi \wedge \psi \wedge \bigwedge \gamma_i$ or $\neg\Box_1\psi \wedge \neg\psi \wedge \bigwedge \gamma_i$ or $\Box_1\psi \wedge \neg\psi \wedge \bigwedge \gamma_i$. Then $\neg\hat{\Gamma}'$ is of the form $\Box_1\psi \vee \neg\psi \vee \bigvee \neg\gamma_i$ or $\Box_1\psi \vee \psi \vee \bigvee \neg\gamma_i$ or $\neg\Box_1\psi \vee \psi \vee \bigvee \neg\gamma_i$. In each case $\vdash \Box_1\psi \wedge \psi \rightarrow \neg\hat{\Gamma}'$. By applying necessitation rule we get: $\vdash \Box_1\Box_1\psi \wedge \Box_1\psi \rightarrow \Box_1\neg\hat{\Gamma}'$ and by axiom 4 for $\Box_1$ we conclude $\vdash \Box_1\psi \rightarrow \Box_1\neg\hat{\Gamma}'$. Now as $\Box_1\psi \in \Gamma$, we have $\vdash \hat{\Gamma} \rightarrow \Box_1\neg\hat{\Gamma}'$ and as Γ and Γ' were taken arbitrarily we get $\vdash \bigvee_{\Gamma\in D}\hat{\Gamma} \rightarrow \bigwedge_{\Gamma'\notin D}\Box_1\neg\hat{\Gamma}'$. It is not difficult to prove that $\vdash \bigwedge_{\Gamma'\notin D}\Box_1\neg\hat{\Gamma} \leftrightarrow \Box_1\bigvee_{\Gamma\in D}\hat{\Gamma}$, so we get a desired result $\vdash \delta \rightarrow \Box_1\delta$. Analogously we can prove $\vdash \delta \rightarrow \Box_2\delta$.

Now combining $\vdash \delta \rightarrow \Box_1\beta$ and $\vdash \delta \rightarrow \Box_1\delta$ we get $\vdash \delta \rightarrow \Box_1(\delta\wedge\beta)$ and analogously $\vdash \delta \rightarrow \Box_2(\delta\wedge\beta)$. So we have $\vdash \delta \rightarrow \Box_1(\delta\wedge\beta)\wedge\Box_2(\delta\wedge\beta)$. Now we apply the induction rule and get $\vdash \delta \rightarrow C_B\beta$. In particular we have $\vdash \hat{\Gamma} \rightarrow C_B\beta$. The last validity implies that $C_B\beta \in \Gamma$. So we proved the left-to-right direction of the of the truth lemma in case of $\alpha = C_B\beta$.

For the other direction assume $C_B\beta \in \Gamma$. Let us show by induction on k that if Γ' is reachable from Γ in k steps then both $C_B\beta$ and β are in Γ'.

Case for $k = 1$: Without loss of generality we can assume that $\Gamma R_1 \Gamma'$. By the axiom (*Equi*) we have $\vdash C_B\beta \rightarrow \Box_1\beta \wedge \Box_1 C_B\beta$. Now by construction both $\Box_1\beta, \Box_1 C_B\beta \in FL(\varphi)$. This implies that $\Box_1 C_B\beta \in \Gamma$ and $\Box_1\beta \in \Gamma$. By definition of R_1 we get $\Gamma' \vdash \Box_1\beta \wedge \beta$ and $\Gamma' \vdash \Box_1 C_B\beta \wedge C_B\beta$. This implies that $\Gamma' \vdash \beta$ and $\Gamma' \vdash C_B\beta$ and as β and $C_B\beta$ are in $FL(\varphi)$ we get $\beta \in \Gamma'$ and $C_B\beta \in \Gamma'$.

Assume the induction hypotheses holds for $k \leq n$ and let us show for the case $k = n$. So we have $\Gamma R_x \Gamma_1 R_x ... R_x \Gamma_{n-1} R_x \Gamma'$, where $x \in \{1, 2\}$. By induction hypotheses both $C_B\beta$ and β are in Γ_{n-1}, so by the same argument as in the case of $k = 1$ we get that $\beta \in \Gamma'$, hence $\Gamma \Vdash C_B\beta$. This finishes the truth lemma.

Now if we take $\Gamma_{\neg\varphi}$ to be maximally consistent set containing $\neg\varphi$, by truth lemma we get that $\mathcal{M}, \Gamma_{\neg\varphi} \nVdash \varphi$.This finishes the completeness proof. $\Box$

According to proposition 5.2.3 every non-theorem of $\mathbf{K4_2^C}$ is falsified on a finite, bi-transitive frame. The following theorem shows that every non-theorem of $\mathbf{K4_2^C}$ can be falsified on a frame (W^t, R_1^t, R_2^t, V^t), where for each $k \in \{1,2\}$ the pair (W^t, R_k^t) is a transitive tree.

Definition 5.2.6. *A frame (W, R) is called a* tree *if:*
1) it is rooted, ie. there is a unique point (the root) $r \in W$ such that for every $v \in W$ it holds that $v \neq r \Rightarrow rR^+v$,
2) every element distinct from r has a unique immediate predecessor; that is, for every $v \neq r$ there is a unique v' such that $v'Rv$ and for every v'' we have that $v''Rv \Rightarrow v''Rv'$,
3) R is acyclic; that is, for every $v \in W$ it is not the case that vR^+v.

If in addition R is transitive, ie. $R = R^+$, then (W, R) is called a transitive tree.

Theorem 5.2.7. *The modal logic $\mathbf{K4_2^C}$ has tree model property.*

Proof. Assume $\nvdash \varphi$. From theorem 5.2.3 we know that φ can be falsified on a finite, transitive, bi-modal Kripke model. Moreover, we can assume that this model is rooted. Let $\mathcal{M} = (W, R_1, R_2, V)$ be the model and w be the root where φ is falsified. Let us unravel the frame (W, R_1, R_2) around w. As a result we get a frame (W^t, R_1', R_2') where both (W^t, R_1') and (W^t, R_2') are trees. This is a standard technique in modal logic [11]. The underlying set W^t consists of all finite strings of the form $\langle w, w_1, .., w_n\rangle$, where each $w_i \in W$ and $w(R_1 \cup R_2)w_1 \wedge w_i(R_1 \cup R_2)w_{i+1}$ for every $i \leq n-1$. The relation R_k' ($k \in \{1,2\}$) is defined in the following way: $\langle w, w_1, .., w_n\rangle R_k' \langle w, w_1', .., w_m'\rangle$ iff $m = n+1$, $w_i = w_i'$ for every $i \leq n$ and $w_n R_k w_m$. To say in words, one sequence is in R_k' relation with the other if the second sequence is taking first sequence and adding as a tail an element which is R_k-successor of the tail of first sequence. The relation R_k^t is defined as a transitive closures of R_k' ie., $R_k^t = (R_k')^*$ for each $k \in \{1,2\}$. We define model $\mathcal{M}^t = (W^t, R_1^t, R_2^t, V^t)$, where the valuation V^t is defined by reflecting the valuation V, so $\langle w_1, .., w_n\rangle \Vdash p$ iff $w_n \Vdash p$. It is easy to see that the function $f : W^t \to W$ which sends each element $\langle w_1, .., w_n\rangle$ of W^t to its tail w_n, is a bounded morphism from the model $\mathcal{M}^t = (W^t, R_1^t, R_2^t, V^t)$ to the model $\mathcal{M} = (W, R_1, R_2, V)$. At this point we can say that if φ does not contain common belief operator C_B then $\mathcal{M}^t, w \nVdash \varphi$. This is because bounded morphism preserves satisfaction of formulas. But we can not yet say that defined bounded morphism f preserves formulas containing C_B. In fact it does. We can easily show that the function f defined above is a bounded morphism between extended models $\mathcal{M}^t = (W^t, R_1^t, R_2^t, (R_1^t \cup R_2^t)^*, V^t)$ and $\mathcal{M} = (W, R_1, R_2, (R_1 \cup R_2)^*, V)$. $\square$

Note 5.2.8. *Observe that the relation* $(R_1^t \cup R_2^t)^+$ *does not contain cycles and in particular it is irreflexive.*

The mains reason for introducing $\mathbf{K4_2^C}$ was to mimic the infinitary operator C_B^ω by finitary C_B. Though we cannot claim that on a logical level C_B and C_B^ω are equivalent, we can establish a semantical equivalence, in particular on Kripke structures.

Theorem 5.2.9. *For any transitive bi-relational Kripke model* $\mathcal{M} = (W, R_1, R_2, V)$ *and point* w*:* $\mathcal{M}, w \Vdash C_B\varphi$ *iff* $\mathcal{M}, w \Vdash C_B^\omega$.

Proof. The proof follows easily from Definitions 5.2.1 and 5.2.2. $\square$

5.2.4 Common Belief as Equilibrium

We mentioned that common belief can also be understood as an equilibrium concept[1]. On Kripke structures the equilibrium conception coincides with common belief by infinite iteration, while in general the equilibrium conception has a much closer connection to the logic $\mathbf{K4_2^C}$. It can be formalized in the modal μ-calculus in the following way:

$$C_\nu\varphi = \nu.p(\Box_1\varphi \wedge \Box_2\varphi \wedge \Box_1 p \wedge \Box_2 p).$$

The greatest fixpoint ν is defined as the fixpoint of a descending approximation sequence defined over the ordinals. Denote by $|\varphi|$ the truth set of φ in the appropriate model $\mathcal{M}$ where evaluation occurs:

$$|C_\nu^0\varphi| = |\Box_1\varphi \wedge \Box_2\varphi|;$$

$$|C_\nu^{k+1}\varphi| = |\Box_1\varphi \wedge \Box_2\varphi \wedge \Box_1 C_\nu^k\varphi \wedge \Box_2 C_\nu^k\varphi|;$$

$$|C_\nu^\lambda\varphi| = |\textstyle\bigcap_{k<\lambda} C_\nu^k\varphi|, \text{ for } \lambda \text{ a limit ordinal.}$$

[1]For the remainder of this section and later on for Theorem 9 we assume some familiarity with the modal μ-calculus. Lack of space hinders a fuller treatment, however for more details on the modal μ-calculus we refer to [11][part 3, chapter 4]; see also the discussion in [7].

We obtain $|C_\nu\varphi| = |C^\gamma_\nu\varphi|$, where γ is a least ordinal for which the approximation procedure halts: ie. $|C^\gamma_\nu\varphi| = |C^{\gamma+1}_\nu\varphi|$. Halting is guaranteed because the occurrence of the propositional variable p in operator $F(p)$, where $F(p) = \Box_1\varphi \wedge \Box_2\varphi \wedge \Box_1 p \wedge \Box_2 p$, is positive. Hence by the Knaster-Tarski theorem the sequence will always reach a greatest fixpoint. Then the semantics of the operator C_ν is defined in the following way:

$$\mathcal{M}, w \Vdash C_\nu\varphi \text{ iff } w \in |C^\gamma_\nu\varphi|$$

In general this procedure may take more than ω steps, but in case of Kripke structures the situation is simpler. The following property relates the different operators on Kripke models.

Theorem 5.2.10. *For every bi-relational Kripke model $\mathcal{M} = (W, R_1, R_2, V)$ and a point $w \in W$ the following condition holds: $\mathcal{M}, w \Vdash C^\omega_B\varphi$ iff $\mathcal{M}, w \Vdash C_\nu\varphi$.*

Proof. Observe that we can rewrite $C^\omega_B\varphi = \Box_1\varphi \wedge \Box_2\varphi \wedge \Box_1\Box_1\varphi \wedge \Box_1\Box_2\varphi \wedge \Box_2\Box_1\varphi \wedge \Box_2\Box_2\varphi \wedge \Box_1\Box_1\Box_1\varphi \wedge \Box_1\Box_1\Box_2\varphi ...$ in the following way: $\Box_1\varphi \wedge \Box_2\varphi \wedge \Box_1(\Box_1\varphi \wedge \Box_2\varphi) \wedge \Box_2(\Box_1\varphi \wedge \Box_2\varphi) \wedge ...$ Hence $|C^\omega_B\varphi| = |C^\omega_\nu\varphi|$. It is known that on Kripke structures stabilization process does not need more than ω steps [7] i.e. $|C_\nu\varphi| = |C^\omega_\nu\varphi|$. Hence $w \Vdash C_\nu\varphi$ iff $w \Vdash C^\omega_B\varphi$ $\Box$

It follows that on transitive bi-relational Kripke structures the three operators C_B, C^ω_B and C_ν coincide.

5.3 Topological Semantics

The idea of a derived set topological semantics originates with the McKinsey-Tarski paper [61]. This idea was taken further in [24]. The following works contain some important results in this direction: [8], [78], [57], [31]. The derived set topological semantics for $\mathbf{K4^C_2}$ is provided by the class of all bi-topological spaces. In the same way, as it is done in [7] for the common knowledge operator, we interpret the common belief operator on the intersection topology. On the other hand, different from C_K, for which the semantics is given using interior of the intersection of the two topologies, we provide the semantics of $C_B\varphi$ as a set of all colimits of $|\varphi|$ in the intersection topology. As a main result we prove the soundness and completeness of the logic $\mathbf{K4_2^C}$ with respect to the class of all T_D-*intersection closed*, bi-topological spaces where each topology satisfies the T_D separation axiom. The basic definitions about topology and topological semantics in derived set setting can be found in Chapter 3.

Besides In Chapter 3 we mentioned that the correspondence from Fact 3.1.10 can be directly generalized to Kripke frames with more than one transitive and irreflexive relation. Of course then we will have one Alexandroff T_D-space for each irreflexive and transitive order. Below we prove the proposition which builds a bridge between Kripke and topological semantics for $\mathbf{K4_2^C}$.

Proposition 5.3.1. *If R_1 and R_2 are two irreflexive and transitive orders on X and $(R_1 \cup R_2)^+$ is also irreflexive and transitive, then $\Omega_{(R_1 \cup R_2)^+} \cong \Omega_{R_1} \cap \Omega_{R_2}$.*

Before starting the proof, observe that $(R_1 \cup R_2)^+$ may not be irreflexive even if both R_1 and R_2 are. For example: Let $X = \{x, y\}$ and $R_1 = \{(x, y)\}$ and $R_2 = \{(y, x)\}$ then $(R_1 \cup R_2)^+ = \{(x, y), (y, x), (x, x), (y, y)\}$. On the topological side this example shows that T_D-spaces do not form a lattice. That is why in Proposition 5.3.1 we require $(R_1 \cup R_2)^+$ to be a irreflexive and transitive.

Proof. Assume that $A \in \Omega_{(R_1 \cup R_2)^+}$. By Fact 3.1.10 this means that if $x \in A$ then for every y such that $x(R_1 \cup R_2)^+y$ it holds that $y \in A$. Since $R_i \subseteq (R_1 \cup R_2)^+$ for each $i \in \{1, 2\}$, it holds that $xR_1y \Rightarrow y \in A$ and $xR_2y \Rightarrow y \in A$ for every $y \in X$. Hence $A \in \Omega_1 \cap \Omega_2$ according to Fact 3.1.10.

Conversely assume $A \in \Omega_1 \cap \Omega_2$. This means that $x \in A \Rightarrow (x(R_1 \cup R_2)y \Rightarrow y \in A)$. Now take arbitrary y such that $x(R_1 \cup R_2)^+y$. By definition this means that there is a $(R_1 \cup R_2)$-path $\langle x_1, x_2, ..x_n\rangle$ starting at x going to y. But this means that each member of this path is in A because A is open in the intersection of the two topologies. Hence $y \in A$ and hence $A \in \Omega_{(R_1 \cup R_2)^+}$ $\square$

Now we extend the satisfaction relation to the language with the common belief operator.

Definition 5.3.2. *The satisfaction of a modal formula on a bi-topological model* $\mathcal{M} = (W, \Omega_1, \Omega_2, V)$ *at a point* $w \in W$ *is defined in the following way:*
$\mathcal{M}, w \Vdash p$ *iff* $w \in V(p)$,
$\mathcal{M}, w \Vdash \alpha \wedge \beta$ *iff* $\mathcal{M}, w \Vdash \alpha$ *and* $\mathcal{M}, w \Vdash \beta$,
$\mathcal{M}, w \Vdash \neg\alpha$ *iff* $\mathcal{M}, w \nVdash \alpha$,
$\mathcal{M}, w \Vdash \Box_i\varphi$ *iff* $w \in \tau_i(V(\varphi))$, *where* τ_i *is a colimit operator of* Ω_i, $i \in \{1, 2\}$,
$\mathcal{M}, w \Vdash C_{B}\varphi$ *iff* $w \in \tau_{1\wedge 2}(V(\varphi))$, *where* $\tau_{1\wedge 2}$ *is a colimit operator in* $\Omega_1 \cap \Omega_2$.

As an immediate corollary of the proposition 5.3.1 and a many-modal version of the Fact 3.2.12, we get the following proposition.

Proposition 5.3.3. *If* R_1 *and* R_2 *are two irreflexive and transitive orders and* $(R_1 \cup R_2)^+$ *is also topological then for every formula* α *in* $\mathbf{K4_2^C}$ *the following holds:*

$$(W, R_1, R_2, V), x \Vdash \alpha \text{ iff } (W, \Omega_{R_1}, \Omega_{R_2}, V), x \Vdash \alpha.$$

Note that in Fact 5.3.3, the symbol $\Vdash$ on the left hand side denotes the satisfaction relation on Kripke models, while on the right hand side it denotes the satisfaction relation on topological frames in the derived set semantics. Now it is clear that we can reduce the topological completeness problem to Kripke completeness if for every non-theorem $\mathbf{K4_2^C} \nvdash \varphi$ we can find a bi-relational topological counter-model (W, R_1, R_2, V) with $(R_1 \cup R_2)^+$ being also a topological relation.

Definition 5.3.4. *The triple* (X, Ω_1, Ω_2) *is a* T_D*-intersection closed bi-topological space if each of the topologies* Ω_1, Ω_2 *and* $\Omega_1 \cap \Omega_2$, *satisfies the* T_D*-separation axiom.*

Theorem 5.3.5. $\mathbf{K4_2^C}$ *is sound and complete with respect to the class of all* T_D*-intersection closed, bi-topological, Alexandroff spaces.*

Proof. (Soundness) Take an arbitrary T_D-intersection closed, bi-topological model $\mathcal{M} = (X, \Omega_1, \Omega_2, V)$. From 2) and 3) of Fact 3.2.10 it follows that $K4$-axioms are valid for each box. Let us show that at each point $x \in X$, the equilibrium axiom is satisfied. Assume that $\mathcal{M}, x \Vdash C_{B}p$. Hence by Definition 5.3.2 we have $x \in \tau_{1\wedge 2}|p|$. By 4) of Fact 3.2.10 we get $x \in \tau_1|p|$ and $x \in \tau_2|p|$. By 3) we have $\tau_{1\wedge 2}|p| \subseteq$

$\tau_{1\wedge 2}\tau_{1\wedge 2}|p| \subseteq \tau_1\tau_{1\wedge 2}|p|$. Analogously $\tau_{1\wedge 2}|p| \subseteq \tau_2\tau_{1\wedge 2}|p|$. Hence we have $x \Vdash \Box_1 p \wedge \Box_2 p \wedge \Box_1 C_B p \wedge \Box_2 C_B p$.

For the other direction assume that $x \in \tau_1\tau_{1\wedge 2}|p| \cap \tau_1|p| \cap \tau_2\tau_{1\wedge 2}|p| \cap \tau_2|p|$. By 2) of Fact 3.2.10 we get $x \in \tau_1(\tau_{1\wedge 2}|p| \cap |p|) \cap \tau_2(\tau_{1\wedge 2}|p| \cap |p|)$. By 1) of Fact 3.2.10 we conclude $x \in \tau_1(Int_{1\wedge 2}|p|) \cap \tau_2(Int_{1\wedge 2}|p|)$, where $Int_{1\wedge 2}$ denotes the interior operator in the intersection topology. By definition of colimit there exists $U^1_x \in \Omega_1$ such that $x \in U^1_x$ and $U^1_x - \{x\} \subseteq Int_{1\wedge 2}|p|$ and there exists $U^2_x \in \Omega_2$ such that $x \in U^2_x$ and $U^2_x - \{x\} \subseteq Int_{1\wedge 2}|p|$. Hence $(U^1_x \cup U^2_x) - \{x\} \subseteq Int_{1\wedge 2}|p|$. Let us show that $Int_{1\wedge 2}|p| \cup \{x\}$ is open in $\Omega_1 \cap \Omega_2$. Since $U^1_x \in \Omega_1$ and $Int_{1\wedge 2}|p| \in \Omega_1$ we have $U^1_x \cup Int_{1\wedge 2}|p| = Int_{1\wedge 2}|p| \cup \{x\} \in \Omega_1$. Analogously we show that $Int_{1\wedge 2}|p| \cup \{x\} \in \Omega_2$. Hence $x \in \tau_{1\wedge 2}|p|$.

Let us show that the induction rule is valid in the class of all T_D-intersection closed bi-topological spaces. The proof goes by contraposition. Assume not $\vdash p \to C_B q$. This means that for some T_D-intersection closed, bi-topological model $\mathcal{M} = (X, \Omega_1, \Omega_2, V)$ and a point $x \in X$ it holds that: $x \Vdash p$ while $x \nVdash C_B q$. We want to show that not $\vdash p \to \Box_1(p \wedge q) \wedge \Box_2(p \wedge q)$. It suffices to find a T_D-intersection closed bi-topological model which falsifies the formula. For such a model one could take $\mathcal{M}' = (X, \Omega_1 \cap \Omega_2, \Omega_1 \cap \Omega_2, V)$. Indeed as $(X, \Omega_1, \Omega_2, V)$ is T_D-intersection closed, the topology $\Omega_1 \cap \Omega_2$ satisfies the T_D-separation axiom. Besides since in $\mathcal{M}'$ both topologies are the same, their intersection is also $\Omega_1 \cap \Omega_2$ and hence again is a T_D-space. Now it is immediate that $\mathcal{M}', x \nVdash p \to \Box_1(p \wedge q) \wedge \Box_2(p \wedge q)$. This is because by construction of $\mathcal{M}'$ we have $\mathcal{M}', x \nVdash \Box_i q$ iff $\mathcal{M}, x \nVdash C_B q$ for every $x \in X$ and $i \in \{1, 2\}$.

(Completeness) Assume $\mathbf{K4_2^C} \nvdash \varphi$. According to Theorem 5.2.7 there exist a tree model $M^t = (W^t, R^t_1, R^t_2, V)$ which falsifies φ. We know that $(R_1 \cup R_2)^+$ is irreflexive and transitive order (see Note 5.2.8). By applying Proposition 5.3.3 we get that the formula φ is falsified in the corresponding bi-topological model $(W^t, \Omega_{R^t_1}, \Omega_{R^t_2}, V)$, which is T_D-intersection closed because of Fact 3.1.9, Proposition 5.3.1 and Note 5.2.8. $\Box$

We can now show how the semantical definition of common belief $C_B\varphi$ as a colimit of the intersection topology meshes with the general equilibrium concept: on topological models the two operators C_B and C_ν coincide.

Theorem 5.3.6. *For every bi-topological model $\mathcal{M} = (X, \Omega_1, \Omega_2, V)$ and an arbitrary formula φ the following equality holds: $\nu.p(\tau_1(|\varphi|) \cap \tau_2(|\varphi|) \cap \tau_1(p) \cap \tau_2(p)) = \tau_{1\wedge 2}(|\varphi|)$.*

Proof. That $\tau_{1\wedge 2}(|\varphi|)$ is a fixpoint of the operator $F(p) = \tau_1(|\varphi|) \cap \tau_2(|\varphi|) \cap \tau_1(p) \cap \tau_2(p)$ follows from the soundness proof of the equilibrium axiom, see Theorem 5.3.5. Now let us show that $\tau_{1\wedge 2}(|\varphi|)$ is the greatest fixpoint of $F(p)$. Take an arbitrary fixpoint B of the operator $F(p)$. That B is a fixpoint immediately implies that $B \subseteq \tau_1(|\varphi|) \cap \tau_2(|\varphi|) \cap \tau_1(B) \cap \tau_2(B)$. By 1) of Fact 3.2.10 we have $B \subseteq Int_i(B) = \tau_i(B) \cap B$ for each $i \in \{1, 2\}$. Hence $B = Int_{1\wedge 2}(B)$ where $Int_{1\wedge 2}$ is the interior operator in the intersection topology of the two topologies. Now let us show that for every $x \in B$ the set $\{x\} \cup (B \cap |\varphi|)$ is open in the intersection of the two topologies. Take an arbitrary point $y \in \{x\} \cup (B \cap |\varphi|)$. Since $y \in B \subseteq \tau_1(|\varphi|)$ we know that there exists an open neighborhood $U^1_y \in \Omega_1$ of y such that $U^1_y - \{y\} \subseteq |\varphi|$. This means that $B \cap U^1_y \in \Omega_1$ and $B \cap U^1_y \subseteq \{x\} \cup (B \cap |\varphi|)$. This means that for every point $y \in \{x\} \cup (B \cap |\varphi|)$ there is an open neighborhood $B \cap U^1_y \in \Omega_1$ of y such that $B \cap U^1_y \subseteq \{x\} \cup (B \cap |\varphi|)$ hence $\{x\} \cup (B \cap |\varphi|) \in \Omega_1$. In exactly the same way we show that $\{x\} \cup (B \cap |\varphi|) \in \Omega_2$. Hence $\{x\} \cup (B \cap |\varphi|) \in \Omega_1 \cap \Omega_2$. This means that $x \in \tau_{1\wedge 2}(|\varphi|)$ since there exists an open neighborhood $U_{1\wedge 2} = \{x\} \cup (B \cap |\varphi|) \in \Omega_1 \cap \Omega_2$ with $U_{1\wedge 2} - \{x\} \in |\varphi|$. $\square$

5.4 From Belief to Knowledge

In this section we discuss the connection between the logics of common knowledge $\mathbf{S4^C_2}$ and common belief $\mathbf{K4^C_2}$. This connection generalizes the existing splitting translation between **S4**-logics and **K4**-logics. As a result we obtain a validity preserving translation from $\mathbf{S4^C_2}$ formulas to $\mathbf{K4^C_2}$ formulas in which common knowledge is expressed in terms of common belief.

Definition 5.4.1. *Consider the following function from the set of formulas in $\mathbf{S4^C_2}$ to the set of formulas in $\mathbf{K4^C_2}$.*

$Sp(p) = p$ *for every propositional letter* p,
$Sp(\neg\alpha \vee \beta) = \neg Sp(\alpha) \vee Sp(\beta)$,
$Sp(\Box_i\alpha) = \Box_i Sp(\alpha) \wedge Sp(\alpha)$,
$Sp(C_K\alpha) = C_B Sp(\alpha) \wedge Sp(\alpha)$.

Theorem 5.4.2. $\vdash_{\mathbf{S4_2^C}} \varphi$ *iff* $\vdash_{\mathbf{K4_2^C}} Sp(\varphi)$.

Proof. We prove the theorem by a semantical argument using the Kripke completeness results, see Proposition 5.2.3 and Fact 5.1.4. Let us first show by induction on the length of formula that for every bi-relational Kripke model $\mathcal{M} = (W, R_1, R_2, V)$ and every $w \in W$ the following holds:

$$(a) \quad \mathcal{M}^\star = (W, R_1^\star, R_2^\star, V), w \Vdash \varphi \text{ iff } \mathcal{M}^+ = (W, R_1^+, R_2^+, V), w \Vdash Sp(\varphi).$$

The only nonstandard case is when $\varphi = C_K\psi$. Assume $\mathcal{M}^\star, w \Vdash C_K\psi$. By the definition of $(R_1 \cup R_2)^\star$ this means that $\mathcal{M}^\star, w \Vdash \psi$ and for every w' such that $w(R_1 \cup R_2)^\star w'$, we have $\mathcal{M}^\star, w' \Vdash \psi$. Now by the induction hypotheses we have that $\mathcal{M}^+, w \Vdash \psi$ and $\mathcal{M}^+, w' \Vdash \psi$. Since w' was arbitrary $(R_1 \cup R_2)^\star$ successor of w we have $\mathcal{M}^+, w \Vdash C_B\psi$. This is because $(R_1 \cup R_2)^\star \supseteq (R_1 \cup R_2)^+$. Hence we get $\mathcal{M}^+, w \Vdash C_B\psi \wedge \psi$. The converse direction follows by the same argument.

Now assume $\vdash_{\mathbf{S4_2^C}} \varphi$. By fact 5.1.4 this means that φ is valid in every reflexive and transitive, bi-relational model. Take arbitrary transitive, bi-relational model $\mathcal{M}$. Then by assumption we have $\mathcal{M}^\star \Vdash \varphi$. Hence by (a) we have that $\mathcal{M} \Vdash Sp(\varphi)$. As $\mathcal{M}$ was arbitrary transitive, bi-relational model from Proposition 5.2.3 we get that $\vdash_{\mathbf{K4_2^C}} Sp(\varphi)$. Conversely assume $\vdash_{\mathbf{K4_2^C}} Sp(\varphi)$. Then by Proposition 5.2.3, $Sp(\varphi)$ is valid in the class of all transitive, bi-relational models. Take arbitrary reflexive and transitive, bi-relational model $\mathcal{N}$. Then $\mathcal{N} \Vdash Sp(\varphi)$ because $\mathcal{N} = \mathcal{N}^+$. So by (a) we have that $\mathcal{N}^\star \Vdash \varphi$. Now as $\mathcal{N}$ was reflexive and transitive, $\mathcal{N}^\star = \mathcal{N}$, hence $\mathcal{N} \Vdash \varphi$. Sinse $\mathcal{N}$ was arbitrary reflexive and transitive, bi-relational model, by Fact 5.1.4 we have $\vdash_{\mathbf{S4_2^C}} \varphi$. $\square$

5.5 Conclusions of Chapter 5

Our main aim in this chapter has been to extend the work of [7] on topological semantics for common knowledge by interpreting a common belief operator on the intersection of two topologies in a bi-topological model. In particular we considered a logic $\mathbf{K4_2^C}$ of common belief for normal agents, first under a Kripke, relational semantics, showing it to have the finite model property and the tree model property. We then showed that $\mathbf{K4_2^C}$ is the modal logic of all T_D-intersection closed, bi-topological spaces with a derived set interpretation of modalities and we saw how the common knowledge logic $\mathbf{S4_2^C}$ can be embedded in $\mathbf{K4_2^C}$ via the splitting translation that maps $C_K p$ into $p \wedge C_B p$.

Chapter 6

Trust and Belief, Interrelation

6.1 Logic for Trust

Different types of trust have been proposed and studied in the disciplines like philosophy, economics, computer science, etc [1, 14, 27, 44]. We focus on the interrelation of trust and belief in the two agents case. On the one hand, we follow the ideas introduced in [53], but on the other hand, we simplify the language in a sense that we leave only two types of modalities: $\Box_i$ as a belief operator of agent i and $T_{i,j}$ for trust of agent-i in agent-j. So the logic $\mathbf{B}_T^2$ we introduce is especially designed to talk about belief and trust and their interrelation. As for belief operators, as we already mentioned, they satisfy the axioms of **KS** and this makes the main difference with the logic **BA** discussed in [53].

In this section we extend the language for multi-agent case and in addition we add modalities for trust. We restrict the language for the case with two agents as far as the other cases (for finite agents) follow as an easy generalizations of two agent case. We take the ideas from [53] and introduce modal logic which has enough expressive power to talk about trust and belief. We do not consider the same language as in [53], but just its fragment, since we are only interested in interrelation between trust and belief. We give the semantics for this logic and as a main result of this chapter we prove the completeness of the described logic with respect to the semantics. Main crucial difference from [53] lies in the fact that the doxastic properties of agents follow **KS** axioms not **KD45** axioms as is classically adopted.

6.1.1 Syntax

The language consists of infinite set of propositional letters $p, q, r, ...$, connectives $\vee, \wedge, \neg, \rightarrow$, and modalities $\Box_1, \Box_2$ $T_{1,2}, T_{2,1}$

Axioms:Each $\Box_i$ satisfies **KS** axioms,

$\vdash T_{1,2}p \leftrightarrow \Box_1 T_{1,2}p$,

$\vdash T_{2,1}p \leftrightarrow \Box_2 T_{2,1}p$.

Rules of inference are: Modus ponens and substitution for each modality, necessitation for $\Box_i$ where $i \in \{1, 2\}$ and the following rule $\frac{\vdash p \leftrightarrow q}{\vdash T_{i,j}p \leftrightarrow T_{i,j}q}$ for each $T_{i,j}$ with $i, j \in \{1, 2\}$ and $i \neq j$.

The desired interpretation of $T_{i,j}p$ carries the following idea: Agent-i trusts agent-j about the claim p. In these settings the last two axioms have very intuitive meaning, mainly: Agent-i trusts agent-j about the claim p iff Agent-i believes that he trusts agent-j about p. Hence trust does not contradict one's beliefs. These are the only (very natural) restrictions we have on the interrelation between trust and belief.

6.1.2 Semantics

Kripke semantics is provided by bi-relational Kripke frames together with two neighborhood functions. More formally

Definition 6.1.1. *A* $\mathbf{B}^2_T$*-frame* $\mathfrak{F}$ *is a tuple* $(W, R_1, R_2, u_{1,2}, u_{2,1})$*, where:*

W is a set of possible worlds,

$R_1, R_2 \subseteq W \times W$ are weakly transitive and symmetric relations,

$u_1 : W \to PP(W), u_2 : W \to PP(W)$ are functions (neighborhood maps), such that the following equalities take place: $u_{i,j}(w) = \bigcap_{v \in R_i(w)} u_{i,j}(v)$, for every $i, j \in \{1, 2\}$ where $i \neq j$.

$\mathbf{B}^2_T$-model is a pair $M = (\mathfrak{F}, V)$, where $\mathfrak{F}$ is a $\mathbf{B}^2_T$-frame and $V : Prop \to P(W)$ is a valuation.

Definition 6.1.2. *A satisfaction of a formula in a given $\mathbf{B}^2_T$-model $M = (\mathfrak{F}, V)$ and a point $w \in W$ is defined inductively as follows:*
$w \Vdash p$ iff $w \in V(p)$,
$w \Vdash \neg\alpha$ iff $w \nVdash \alpha$,
$w \Vdash \alpha \wedge \beta$ iff $w \Vdash \alpha$ and $w \Vdash \beta$,
$w \Vdash \Box_i\alpha$ iff $(\forall w')(wR_iw' \Rightarrow w' \Vdash \alpha)$,
$w \Vdash T_{i,j}\alpha$ iff $|\alpha| \in u_{i,j}(w)$. Here $|\alpha|$ denotes the set $\{v | v \Vdash \varphi\}$.

A formula is valid in in a given $\mathbf{B}^2_T$-model if it is true at every point of the model. A formula is valid in a $\mathbf{B}^2_T$-frame if it is valid in every model based on the frame. A formula is valid in a class of $\mathbf{B}^2_T$-frames if it is valid in every frame in the class.

Theorem 6.1.3. *The logic $\mathbf{B}^2_T$ is sound and complete with respect to the class of all $\mathbf{B}^2_T$-frames.*

Proof. The soundness easily follows by direct check as for completeness, the proof is standard and therefore we just give a sketch.

Let W be the set of all maximally consistent subsets of formulas in a logic $\mathbf{B}^2_T$. Let us define the relations R_1 and R_2 on W in the following way: For every $\Gamma, \Gamma' \in W$ we define $\Gamma R_i \Gamma'$ iff $(\forall\alpha)(\Box_i\alpha \in \Gamma \Rightarrow \alpha \in \Gamma')$, where $i \in \{1, 2\}$. The following lemma is proved in [24] when proving completeness of the logic KS. It also directly follows from Sahlqvist theorem and the observation that KS-axioms characterize the class of all weakly-transitive and symmetric frames.

Lemma 6.1.4. *[24] Each R_i is weakly-transitive and symmetric.*

So far we defined a set W with two weakly-transitive and symmetric relations R_1, R_2 on it. Now we define functions $u_{1,2}$ and $u_{2,1}$ in the following way:

$$u_{i,j}(\Gamma) = \{\{\Gamma' | \varphi \in \Gamma'\} | T_{i,j}\varphi \in \Gamma\}.$$

It immediately follows that $u_{i,j}$ are functions defined from W to $PP(W)$. Before we show that $u_{i,j}(\Gamma) = \bigcap_{\Gamma' \in R_i(\Gamma)} u_{i,j}(\Gamma')$, for every $i, j \in \{1, 2\}$, where $i \neq j$ and every $\Gamma \in W$, let us define the valuation and prove the truth lemma.

The valuation V is defined in the following way: $V(\Gamma) = \{p | p \in \Gamma\}$.

Lemma 6.1.5 (*Truth*)**.** *For every formula $\alpha \in \mathbf{B}^2_T$ and every point $\Gamma \in W$ of the canonical model, the following equivalence holds: $\Gamma \Vdash \alpha$ iff $\alpha \in \Gamma$.*

Proof. The proof goes by induction on the length of formula. Base case follows immediately from the definition of valuation. Assume for all $\alpha \in \mathbf{B}^2_T$ with length less then k holds: $\Gamma \Vdash \alpha$ iff $\alpha \in \Gamma$.

Let us prove the claim for α with length equal to k. If α is conjunction or negation of two formulas then the result easily follows from the definition of satisfaction relation and the properties of maximal consistent sets, so we skip the proofs. Assume $\alpha = \Box_i\beta$ and assume $\Gamma \Vdash \alpha$. Take a set $B = \{\gamma | \Box_i\gamma \in \Gamma\} \cup \{\neg\beta\}$. The sub claim is that B is inconsistent. Assume not, then there exists $\Gamma' \in W$ such that $\Gamma' \supseteq B$. This by definition of the relation R_i means that $\Gamma R_i \Gamma'$. Now as $\neg\beta \in \Gamma'$, by inductive assumption we get $\Gamma' \Vdash \neg\beta$. Hence we get a contradiction with our assumption that $\Gamma \Vdash \Box_i\beta$. So B is inconsistent. This means that there exists $\gamma_1, \gamma_2, ...\gamma_n \in B$ such that $\vdash \gamma_1 \wedge \gamma_2 \wedge .. \wedge \gamma_n \to \beta$. Applying necessitation rule for $\Box_i$ we get $\vdash \Box_i\gamma_1 \wedge .. \wedge \Box_i\gamma_n \to \Box_i\beta$ so $\Gamma \vdash \Box_i\beta$, hence we get that $\Box_i\beta \in \Gamma$.

We just showed the left-to-right direction of our claim for $\alpha = \Box_i\beta$. For the right-to-left implication assume that $\Box_i\beta \in \Gamma$. By definition of R_i for every Γ' with $\Gamma R_i \Gamma'$ we have $\beta \in \Gamma'$. From this by inductive assumption it follows that $\Gamma' \vdash \beta$. So we imply that $\Gamma \Vdash \Box_i\beta$.

Now assume $\alpha = T_{i,j}\varphi$. Assume $\Gamma \Vdash T_{i,j}\varphi$. By definition this means that $|\varphi| \in u_{i,j}(\Gamma)$. Hence there exists β such that $\{\Gamma''|\beta \in \Gamma''\} = |\varphi|$ with $T_{i,j}\beta \in \Gamma$. This means that we have $\vdash \beta \leftrightarrow \varphi$ in $\mathbf{B}^2_T$. Hence by the rule for trust modality we have $\vdash T_{i,j}\beta \leftrightarrow T_{i,j}\varphi$. But the last implies that $T_{i,j}\varphi \in \Gamma$.

Conversely assume that $T_{i,j}\varphi \in \Gamma$ this implies that $\{\Gamma''|\varphi \in \Gamma''\} \in u_{i,j}(\Gamma)$. Now by inductive assumption we know that $\Gamma'' \Vdash \varphi$ iff $\varphi \in \Gamma''$ hence $|\varphi| \in u_{i,j}(\Gamma)$. Hence $\Gamma \Vdash T_{i,j}\varphi$. $\square$

Now let us show that the model we constructed falls into the class of $\mathbf{B}^2_T$-models. The only thing left to show is the following equality:

$$u_{i,j}(\Gamma) = \bigcap_{\Gamma' \in R_i(\Gamma)} u_{i,j}(\Gamma').$$

Assume $X \in u_{i,j}(\Gamma)$. This means that X is of the form $\{\Gamma''|\varphi \in \Gamma''\}$ for some φ with $T_{i,j}\varphi \in \Gamma$. Because of the $\mathbf{B}^2_T$ axioms and because Γ is maximally consistent set, we imply that $\Box_i T_{i,j}\varphi \in \Gamma$. From this we imply that $T_{i,j}\varphi \in \Gamma'$ for every $\Gamma' \in R_i(\Gamma)$. Now by definition of $u_{i,j}$ this means that $\{\Gamma''|\varphi \in \Gamma''\} \in u_{i,j}(\Gamma')$ and as Γ' was arbitrary member of $R_i(\Gamma)$, we get that $X = \{\Gamma''|\varphi \in \Gamma''\} \in \bigcap_{\Gamma' \in R_i(\Gamma)} u_{i,j}(\Gamma')$.

Conversely assume some set $X \subseteq W$ belongs to $\bigcap_{\Gamma' \in R_i(\Gamma)} u_{i,j}(\Gamma')$. By definition this means that there exists a formula φ such that $T_{i,j}\varphi \in \bigcap_{\Gamma' \in R_i(\Gamma)} \Gamma'$ and $X = \{\Gamma''|\varphi \in \Gamma''\}$. Now as far as $(\forall \Gamma')(\Gamma R_i \Gamma' \Rightarrow T_{i,j}\varphi \in \Gamma')$ by truth lemma we get that $(\forall \Gamma')(\Gamma R_i \Gamma' \Rightarrow \Gamma' \Vdash T_{i,j}\varphi)$. Hence $\Gamma \Vdash \Box_i T_{i,j}\varphi$. Now applying axioms for trust modality we get that $\Gamma \Vdash T_{i,j}\varphi$ and hence $X \in u_{i,j}(\Gamma)$. This completes the proof.

$\square$

Chapter 7

Summary and Future Work

In the thesis we have studied several ***doxepi***-formalisms which were unified under the understanding of knowledge as true belief. The main direction, which also motivated the title of the thesis, was to fix the already established epistemic logics **S5** and **S4** and adjust to them logics of belief at the same time keeping the interrelation between knowledge and belief operators. We have discussed a general method of achieving this, by using topology as a source for generating pairs of logics with the desired properties. As a result for the epistemic logics **S5** we have obtained it's doxastic counterpart, the logic **KS**, and to the epistemic logic **S4** we adjusted the doxastic modal logic **K4**. Taking the ***doxepi***-formalism $(\mathbf{S5}, \mathbf{KS})$ as a foundational ground we have studied the concept of minimal belief. And in analogy to the techniques for formalising minimal knowledge we have developed a new non-monotonic formalism which captures the idea of minimal belief of **KS**-agent. Based on ***doxepi***-formalism $(\mathbf{S4}, \mathbf{K4})$ we have studied the concept of common belief and it's logical formalisation the logic $\mathbf{K4_2^B}$. We took analogies from the logic $\mathbf{S4_2^C}$ of common knowledge and have studied several important properties of the logics apart as well as the newly born ***doxepi***-formalism $(\mathbf{S4_2^C}, \mathbf{K4_2^B})$. As a small corollary to the developed ***doxepi***-formalisms we have studied the logic of trust of the **KS**-agents.

We conclude with giving several possible topics for the future work. Chapter 4 discusses the concept of minimal belief which is formalised in nonmonotonic logic **wK4Df**. It is shown how the logic captures the idea of minimal belief although the nonmonotonic logic **wK4Df** itself is not studied much. One question to answer is to find out the place of nonmonotonic **wK4Df** in the class of nonmonotonic modal logics. In particular find what is the relation between nonmonotonic **wK4Df** and other closely related nonmonotonic logics **KD45** and **S4F**.

While preparing the final draft of this work, we came across the article [56] by Lismont and Mongin. This paper treats several logics of common belief including one that is equivalent to $\mathbf{K4_2^C}$ considered in Chapter 5. Besides a relational semantics, the authors also consider a more general neighborhood semantics and discuss the equilibrium conception of common belief in this setting. While the semantics and methods of [56] are formally different to ours, there are obvious similarities. A detailed comparison of our topological approach with the neighborhood systems of [56] would be a worthwhile exercise for the future. Another direction for future work is to look for concrete topological structures which would fully capture the behavior of the logic $\mathbf{K4_2^C}$ or some of it's extensions.

The logic of trust discussed in the thesis is constructed and explored in analogy to the logics from [53]. The trust operator $T_{i,j}$ has a neighborhood semantics. One question is to try to find topological semantics for the trust operator so that all the properties are kept unchanged.

Chapter 8

Resumen en Castellano

8.1 Antecedentes

Moderno lógica epistémica y doxástica tiene sus raíces en el trabajo de Hintikka [46], que dio un programa de investigación haciendo importantes contribuciones a la filosofía, la lógica, la AI y otros campos. Una de las ideas centrales, los supervivientes de enfoque Hintikka es que el conocimiento y las creencias pueden ser entendidas como relaciones entre un agente cognitivo y una proposición p, simbolizado por $\mathcal{K}_a p$ - "a sabe que p" y $\mathcal{B}_a p$ - "a cree que p". La interpretación semántica de estos ingresos a través de expresiones posibles marcos mundiales equipados con una relación de alternatividad epistémica, R_A, interpretado como: $wR_a w'$ si w' es compatible con todo lo que sabe el agente en el mundo w. La condición de verdad para un operador epistémico viene dado por:

$$w \models \mathcal{K}_a p \quad \text{iff} \quad w' \models p \quad \text{por todos} \quad w' \quad \text{s.t.} \quad wR_{\mathcal{K}_a} w' \tag{8.1}$$

Así que una sabe que p en un mundo w si p es verdadero en todas las alternativas epistémicas w', y relaciones análogas y las condiciones de verdad se aplica a las operaciones de creencia:$w \models \mathcal{B}_a p \quad \text{iff} \quad w' \models p \quad \text{por todos} \quad w' \quad \text{s.t.} \quad wR_{\mathcal{B}_a} w'$. Asumiendo algunas condiciones básicas sobre el conocimiento de un solo agente en cada sistema, epistémicos/doxásticas lógicas son estructuralmente similares a las lógicas modales normales con un operador $\Box$ necesidad interpretado en los marcos de Kripke con una relación binaria R la accesibilidad en los mundos. A pesar de muchas variaciones y ampliaciones, esta característica se mantiene en muchas áreas relacionadas con el razonamiento sobre conocimiento y creencia. Recordemos que la semántica de $\Box$ coincide con la semántica anteriores tratados de $\mathcal{K}_a$ y el $\mathcal{B}_a$.

$$w \models \Box p \quad \text{iff} \quad w' \models p \quad \text{por todos} \quad w' \quad \text{s.t.} \quad wRw'. \tag{8.2}$$

Entendido de esta forma modal lenguaje tiene capaz para expresar varios conceptos filosóficos interesantes se reunieron en la epistemologa. Tales ejemplos son $\Box p \to p$ (El agente sabe p implica que p es verdadero) o$\Box p \to \Box\Box p$ (si el agente sabe p, entonces él sabe que sabe p), que se conoce como la introspección positiva. Por supuesto, este tipo de formalización no puede capturar todo el concepto de conocimiento como es. De hecho, sólo puede imitar en algunas propiedades de los conocimientos que están disponibles para ser expresada por un formalismo simple como lenguaje modal. La inteligencia artificial es menos preocupados acerca de la definición del propio conocimiento y se centra mucho más en las propiedades de los agentes que tengan conocimiento (definida de cualquier manera o tomado como una noción abstracta). En estos contextos en una forma abstracta, $\Box p$ - un agente sabe que p, es una formalización perfecto.

8.2 Objectivos

La combinación de la creencia y el conocimiento en un formalismo ha sido cuestión problemática en la lógica epistémica. Es común pensar que el conocimiento contiene (o implica) la creencia. Por lo tanto, al tener ambos operadores, $\mathcal{K}$ - para el conocimiento y $\mathcal{B}$ - de la creencia, se espera que por lo menos tiene la siguiente implicación: $\mathcal{K}p \to \mathcal{B}p$.

Por supuesto, sólo la implicación $\mathcal{K}p \to \mathcal{B}p$ no es suficiente para el formalismo que sea aceptable. La preferencia obvia es dado a formalismos que tienen raíces en la filosofía. Adicionalmente, se requiere que uno mantiene las propiedades necesarias doxásticos y epistémicas de un agente. En resumen podemos enumerar las propiedades que nos gustaría que el formalismo de tener:

1. El operador $\mathcal{K}$ de conocimiento razonablemente debe ajustarse a la intuición detrás de una definición de conocimiento a partir de la epistemología,

2. El operador $\mathcal{B}$ de creencia se supone que satisface las propiedades comúnmente adoptado por lo menos en algunos contextos filosóficos,

3. La interacción formal entre operadores $\mathcal{K}$ y $\mathcal{B}$ deben reflejar la dependencia de la conocimiento sobre la creencia (o viceversa) tomadas de algunos fundamentos filosóficos.

Vamos a llamar a un formalismo ***doxepi***-formalismo si las tres propiedades se tienen en cuenta. El subir con un buen ***doxepi***-formalismo no es una tarea simple. Ejemplo principal de la solución de este problema en algún momento se propuso en [42]. Solución de [42] se especifica para un caso particular, donde las capacidades doxásticos de un agente están dados por la lógica clásica doxástica **KD45** y la interconexión entre conocimiento y creencia refleja la idea del conocimiento como creencia verdadera ($\mathcal{K}p \leftrightarrow \mathcal{B}p \wedge p$). Aunque el inconveniente de este caso es la primera condición de los anteriores tres requisitos. En particular, la contraparte epistémica de la lógica **KD45** no ser la lógica clásica epistémica **S5** pero la lógica $\mathbf{S4.4} = \mathbf{S4} + (p \to (\neg\Box p \to \Box\neg\Box p))$. Como conclusión, [42] ofrece un claro ejemplo de un efectivo ***doxepi***-formalismo aunque la prioridad se da a la articulos secunda y tercera.

El objetivo número uno: Uno de los objetivos de este tesis consiste en rellenar las partes aún no investigados y desarrollar ***doxepi***-formalismo donde el artculo 1 tiene prioridad. En otras palabras, el objetivo es construir doxásticas lógicas modales, que en combinación con las lógicas clásicas epistémicas, como son **S5** y **S4**, conservan el interconexión ($\mathcal{K}p \leftrightarrow \mathcal{B}p \wedge p$) y al mismo tiempo tienen interesantes propiedades doxásticos por su propia cuenta.

El paradigma de un mínimo conocimiento deriva de la obra de Halpern y Moses, en especial [40], prorrogada y modificada posteriormente en obras como [79, 55, 54] y otras. Muchos enfoques se basan en modelos de Kripke **S5** con una relación de accesibilidad universal y la reducción al conocimiento minimal está representado por la maximización del conjunto de mundos posibles con respecto a la inclusión. Un enfoque diferente ha sido desarrollado por Schwarz y Truszczynski [75] y puede ser visto como un caso especial del método muy general de Shoham [79] para la obtención de diferentes conceptos de minimalidad cambiando los conjuntos de modelos y las relaciones de preferencia entre ellos.

Objetivo número dos: Si bien el conocimiento mínimo ha sido ampliamente estudiado, el concepto de creencia mínimo no se ha explorado lo suficiente. Por otra parte, es evidente que el nivel de importancia del concepto real es tan alta como de la anterior. Como uno de los objetivos nos vemos a estudiar la creencia mínima y formalismos relacionados no monotónicos.

Tecnologías acuerdo es un dominio emergente donde conceptos iterativos de creencia y el conocimiento de los agentes son de especial interés. Entre los ejemplos más interesantes en este sentido son las nociones de conocimiento común y la creencia común.

El conocimiento común, definida originalmente por Lewis [52], ha sido ampliamente estudiada desde diversas perspectivas en la filosofía [5], [2], la teoría de juegos [91], la inteligencia artificial [45], la lógica modal [3], [4] etc . Las teorías de la creencia común, no están tan bien desarrollado, aunque algunos enfoques se pueden encontrar en [6, 45, 84].

El objetivo número tres: El último objetivo de la tesis está dedicada al estudio de la creencia común de los agentes normales, es decir agentes que tienen introspección positiva en sus creencias.

8.3 Metodología

Nuestro objetivo es investigar una lógica modal doxástica **KS** [24] como una alternativa al sistema **KD45**. La motivación viene de dos aspectos principales, cada uno de ellos formando una parte principal de nuestro estudio. El primer aspecto se refiere a la lógica doxástica monótona. Una idea principal es acercar razonamiento doxástica mediante la búsqueda de un sistema que tiene fuertes analogías con **S5**, pero sin el axioma T. Una manera obvia de evitar T principio es el siguente: considerar las relaciones de alternatividad irreflexivas e interpretar las creencias en términos de otros mundos posibles. Así que en lugar de conocimiento donde se puede suponer que cada mundo es una alternativa episémica, tendríamos para creer: cada otro mundo es una alternativa doxástica.

El segundo aspecto de la motivación para este estudio se refiere a razonamiento no monotónico. No monótona lógica modal, también se utiliza para el razonamiento acerca del conocimiento, se basan en un concepto de la (teoría) de expansión. En los casos normales tales expansiones son conjuntos estables y pueden ser capturados por una noción de modelo mnimo, de nuevo un tipo especial de **S5**-modelo que puede ser considerado para representar un concepto de conocimiento mínimo. Evidentemente cada mundo de tal modelo satisface el axioma T, apropiado para el conocimiento, pero no para creencia. La lógica **KD45** es generalmente considerado como una lógica modal doxástica, su variante lógica no monotónica es precisamente lógica autoepistemica. Un sistema que parece más cerca de conocimientos en lugar de la creencia. Para obtener una lógica candidato a la creencia mínimo, nuestra estrategia es la siguiente: poner los modelos **KS** en lugar de los modelos **S5** y definir un nuevo concepto de modelo mínimo. De manera equivalente, como veremos en el Captulo 4, se puede definir un concepto de expansión débil y demostrar que lo que corresponde a la nueva definición del modelo mnimo. Estos dos aspectos principalmente dar respuesta al objetivo número uno y poner el fundamento al objetivo número dos. De hecho, para la captura de creencia mínima hacemos uso de la sistema **KS** como una base y observar minimalidad en sentido de las creencias de **KS**-agentes. Como ya hemos mencionado la técnica para capturar creencia mínima sigue la técnica de [75] para la capturar de conocimientos mínimos. Para esta materia se introducen dos lógicas modales **wK4f** [69] y **wK4Df** [70], que sirven como dos diferentes análogos doxásticos del la lógica modal **S4F**. Estudiamos no monotonicos versiones de la lógica **wK4f** mediante la aplicación de métodos convencionales conocidos de no monotonicas lógicas modales. Para el lógica **wK4Df** cambiamos una metodología y introducimos de un nuevo concepto de expansión que llamamos débil expansión. Ambos métodos nos dan el concepto de creencia mínimo, mientras que en el primer caso nos ocupamos de un agente que cree que todo lo que él cree es verdad y en el segundo caso el agente no nada más tiene esta propiedad. Mediante el estudio de las dos lógicas modales no monotónicas nos proporcionan la lógica basada de la idea de la creencia mínimal.

Es conveniente mencionar que todas las lógicas que hemos considerado hasta ahora surgen de los espacios topológicos. Los mnimos espacios topológicos da dos lógicas diferentes **S5** y su acompanantes **KS** doxásticos. Análogamente a partir de los espacios topológicos que son mínimos, en el primer caso se llega a lógica modal **S4F**, y en el segundo caso en lógica modal **wK4f**. La clase de todos los espacios topológicos dan dos lógicas diferentes **S4** y **wK4**. Los detalles de conexión de las lógicas con topología se presentan en el Capítulo 3. Por el momento sólo queremos senalar que la topología puede servir como fuente para producir buenos ***doxepi***-formalismos y por lo tanto el estudio topológico de muchos-modales versiones de los anteriores lógicas también proporcionar resultados interesantes.

Como se ha indicado el tercer objetivo de nuestra tesis se centra en la creencia común de los agentes normales, y para facilitar la exposición nos restringimos al caso de dos agentes. En consecuencia, consideramos dos agentes normales. Esos son agentes con creencias que satisfacen los axiomas de **K4**. Por lo demás, adoptamos los principios fundamentales de la lógica del conocimiento común, $\mathbf{S4_2^C}$. Esto puede ser visto como una formalización de la idea de que el conocimiento común es equivalente a un conjunto infinito de conocimientos individuales iterada: $\varphi \wedge \Box_1\varphi \wedge \Box_2\varphi \wedge \Box_1\Box_1\varphi \wedge \Box_1\Box_2\varphi \wedge \Box_2\Box_1\varphi \wedge \Box_2\Box_2\varphi \wedge \Box_1\Box_1\Box_1\varphi \wedge \Box_1\Box_1\Box_2\varphi$... Más adelante veremos que una variación de esta fórmula es la creencia común "verdadera" en la semantica Kripke. También vamos a mostrar que la semántica topológica para $\mathbf{K4_2^C}$ es compatible con la idea de la creencia común como un equilibrio, un concepto utilizado por Barwise [5] para describir conocida común que puedan ser capturados por una expresión modal de la μ-cálculo [11].

Nuestro enfoque para proporcionar una semántica topológica sigue el trabajo de Esakia [24]. Observe que la interpretación topológica de $\Box$ como un operador del conocimiento, por ejemplo. en [7], se refiere al interior topológica. En el caso de una lógica doxástica, como es **K4** la interpretación topológica es diferente y se refiere al co-derivativa topológica [21]. Mediante la combinación de las ideas y resultados a partir de [24] y [7] se obtiene una semántica conjunto derivado de la lógica de la creencia común, basada en bi-topológicos espacios, donde la modalidad de la creencia común opera en la intersección de las dos topologías. Como resultado principal, se prueba que $\mathbf{K4_2^C}$ es sólida y completa con respecto a la subclase especial de todas las T_D-espacios bi-topológicos [21].

8.4 Conclusiones

En la tesis nosotros hemos estudiado varias ***doxepi***-formalismos que se unificaron en la comprensión del conocimiento como creencia verdadera. La dirección principal, que también motivó el título de la tesis, era siguente: fijar las lógicas epistémicas ya establecidos **S5** y **S4** y ajustar a ellos la lógica doxastica, al mismo tiempo manteniendo la relación entre los operadores del conocimiento y creencia. Hemos hablado de un método general de lograr esto, mediante el uso de topología como una fuente para generar pares de lógicas con las propiedades deseadas. Como resultado de las lógicas epistémicas **S5** hemos obtenido su doxastic contraparte, el lógica **KS**, y al lógica epistémica nos ajustamos a la lógica modal doxástica **K4**. Tomando el ***doxepi***-formalismo $(\mathbf{S5}, \mathbf{KS})$ como motivo fundamental hemos estudiado el concepto de creencia mínimo. Y de modo similar a las técnicas de mínimo conocimiento hemos desarrollado una nueva nonmonotónica formalismo que captura la idea de mínima creencia para **KS** agentes. Sobre la base de ***doxepi***-formalismo $(\mathbf{S4}, \mathbf{K4})$ hemos estudiado el concepto de creencia común y su formalización lógico que resulta en lógica $\mathbf{K4_2^B}$. Tomamos analogas de la lógica $\mathbf{S4_2^C}$ del conocimiento común y hemos estudiado varias propiedades importantes de estas lógicas. Como corolario pequeo para los desarrollados ***doxepi***-formalismos hemos estudiado la lógica de la confianza de los **KS**-agentes.

Llegamos a la conclusión de dar varios temas posibles para el trabajo futuro. El Captulo 4 analiza el concepto de creencia mínima que se formaliza en la lógica nonmonotonica **wK4Df**. Una de las preguntas a responder es averiguar el lugar de nonmonotonica **wK4Df** en la clase de lógicas modal no monotónicas. En particular, encontrar cuál es la relación entre **wK4Df** no monótonica y otras lógicas no monotónicas estrechamente relacionados **KD45** y **S4F**.

Durante la preparación de la versión final de este trabajo, nos encontramos con el artículo [56] por Lismont y Mongin. En este trabajo se trata a varias lógicas de la creencia común entre ellos uno que es equivalente a $\mathbf{K4_2^C}$ considerado en el Captulo 5. Además de una semántica relacional, los autores también consideran una semántica barrio más generales y discutir el concepto de equilibrio de la creencia común en este contexto. Mientras que la semántica y los métodos de [56] son formalmente diferente a la nuestra, hay similitudes obviosos. Una comparación detallada de nuestro enfoque topológico con los sistemas locales de [56] sería un ejercicio útil para el futuro. Otra dirección para el trabajo futuro es la búsqueda de estructuras de hormigón topológicos que permitirá reflejar plenamente el comportamiento de la lógica $\mathbf{K4_2^C}$ o parte de él extensiones. La lógica de confianza discutido en la tesis se construye y se exploró en analoga a las lógicas de [53]. La confianza operador $T_{i,j}$ tiene una semántica barrio. Una cuestión es encontrar la semántica topológica para el operador de confianza, de modo que todas las propiedades se mantienen sin cambios.

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